# The Banzhaf value in TU games with restricted cooperation


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## Abstract

A survey of the existing results for TU games with restricted cooperation is provided. Different approaches have been proposed, some of them are based on the affinities among the players and others on the incompatibilities. New solution concepts based on the Banzhaf value are proposed and characterized for TU games with incompatibilities and for TU games with graph restricted communication and a priori unions. Finally, the studied models and solution concepts are illustrated by real world examples coming from the political field.



**Keywords:** Cooperative games, a priori unions, graph restricted cooperation, incompatibilities, Banzhaf value.


## Introduction

Situations in which a set of agents cooperates in order to reach agreements can be studied using cooperative transferable utility games. In such models there are no restrictions to cooperation. Consequently, it is assumed that any group of players can reach binding agreements with the same probability. In many real situations, however, there exists a priori information concerning the behavior of the individuals. Using this external information, it is possible to consider games in which the cooperation is restricted in some way. In this work we consider different models of games with restricted cooperation.

## Basic concepts and notation

- A TU game is a pair $(N,v)$ where,
- $N$, is the set of agents and
- $v: 2^N \to \mathbb{R}$, is the characteristic function.
- $G(N)$, is the set of all TU games with player set $N$.
- $g(N)$, is the set of all undirected graphs without loops on $N$.
- A TU game with graph restricted communication is a triple $(N,v,B)$, where $(N,v)\in G(N)$ and $B\in g(N)$ is the communication graph.
- $C(N)$, is the set of all TU games with graph restricted communication.
- A TU game with incompatibilities is a triple $(N,v,I)$, where $(N,v)\in G(N)$ and $I\in g(N)$ is the incompatibility graph.
- $I(N)$, is the set of all TU games with incompatibilities.
- $P(N)$, is the set of all partitions of $N$.
- A TU game with a priori unions is a triple $(N,v,P)$, where $(N,v)\in G(N)$ and $P\in P(N)$.
- $U(N)$, is the set of all TU games with incompatibilities.
- A TU game with graph restricted communication and a priori unions is a quadruple $(N,v,B,P)$, where $(N,v)\in G(N)$, $B\in g(N)$, and $P\in P(N)$.
- $CU$, is the set of all TU games with graph restricted communication and a priori unions.
- A value on $H$ is a map that assigns a vector in $\mathbb{R}^n$ to every element of $H$, where $H\in\{G(N),C(N),I(N),U(N),CU\}$.

## TU games

Two of the most well known one-point solutions are the Shapley and Banzhaf values. The values considered in this work are extensions of them. The amount allotted to each $i\in N$ by these solutions is,

Shapley (1953) $\quad \varphi_i(N,v)=\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}[v(S\cup i)-v(S)].$

Banzhaf (1965) $\quad \beta_i(N,v)=\frac{1}{2^{n-1}}\sum_{S\subseteq N\setminus i}[v(S\cup i)-v(S)].$

For a comparison between the properties that each solution concept satisfies see Feltkamp (1995).

## TU games with graph restricted communication

We may assume that players can only communicate, and hence cooperate, if there is a link (or path inside a coalition) joining them. From this idea of communication we obtain an equivalence relation on $N$.

Associated to every game with graph restricted communication $(N,v,B)\in C(N)$ we have the communication game $(N,v^B)\in G(N)$, defined by, $v^B(S)=\sum_{T\in S/B}v(T)$. The well known Myerson and Banzhaf graph values arise from the natural extension of $\varphi$ and $\beta$ to this setting.

Myerson (1977) $\quad \varphi^c(N,v,B)=\varphi(N,v^B).$

Owen (1986) $\quad \beta^c(N,v,B)=\beta(N,v^B).$

## TU games with incompatibilities

In this case we are given a graph that describes incompatible players. Given a TU game with incompatibilities $(N,v,I)\in I(N)$, we denote by $(N,v^I)\in G(N)$ the *I-restricted game* whose characteristic function is given by, $v^I(S)=\max_{P\in P(S,I)}\sum_{T\in P}v(T)$, where $P(S,I)$ is the set of partitions of $S$ whose classes are I-admissible coalitions (coalitions not containing incompatible players). Using this game we can build values on $I(N)$ as follows,

Bergantiños (1993) $\quad \varphi^I(N,v,I)=\varphi(N,v^I).$

New value on $I(N)$ $\quad \beta^I(N,v,I)=\beta(N,v^B).$

The new value on $I(N)$ can be characterized using I-Total power. A value on $I(N)$ $f$, satisfies **I-Total power** if for all $S\in N/I^c$,

$$\sum_{i\in S}f_i(N,v,I)=\frac{1}{2^{s-1}}\sum_{i\in S}\sum_{L\subseteq S\setminus i}\left[v^I(L\cup i)-v^I(L)\right],$$

where $I^c=\{(i:j)\in g(N)|(i:j)\notin I\}$ is the dual graph of $I$.

In the next table we present the properties that each value satisfies. Properties in red mean that the value is characterized by them.

| | $\varphi^I$ | $\beta^I$ |
|---|---|---|
| I-Efficiency | ✓ | |
| I-Fairness | ✓ | ✓ |
| I-Balanced contributions | ✓ | ✓ |
| I-Isolation | ✓ | ✓ |
| I-Total power | | ✓ |

## Example

Let us consider the Parliament of the Basque Country emerging from elections in November 1986.

The situation can be studied as a simple game that assigns 1, to any coalition with 38 seats or more and 0, to the rest.

We may consider the incompatibility graph compound by the following links: (PSE:HB), (PSE:CP), (HB:EE), (HB:CP), and (HB:CDS).

| Party | Seats | $\varphi$ | $\beta$ | $\bar{\beta}$ | $\varphi^I$ | $\beta^I$ | $\bar{\beta}^I$ |
|---|---|---|---|---|---|---|---|
| PSE/PSOE | 19 | .2524 | .4688 | .2542 | .2333 | .4063 | .2241 |
| EAJ/PNV | 17 | .2524 | .4688 | .2542 | .3167 | .5313 | .2931 |
| EA | 13 | .1524 | .2813 | .1525 | .2333 | .4063 | .2241 |
| HB | 13 | .1524 | .2813 | .1525 | .0333 | .0938 | .0517 |
| EE | 9 | .1524 | .2813 | .1525 | .1500 | .2813 | .1552 |
| AP-PL | 2 | .0190 | .0313 | .0169 | 0 | 0 | 0 |
| CDS | 2 | .0190 | .0313 | .0169 | .0333 | .0938 | .0517 |

Table: The distribution of the power of the simple game and the game with incompatibilities

## TU games with a priori unions

The games with a priori unions model has been studied in-depth in the literature. There are three values based on $\varphi$ and $\beta$, namely $\phi$ (Owen (1977)), $\psi$ (Owen (1982)), and $\pi$ (Alonso-Meijide et al. (2002)). All of them are built following a two steps procedure, in the first the worth of the grand coalition is divided among the unions and in the second the amount allotted to each union is shared among the members of the union.

| Sharing out | $\phi$ | $\psi$ | $\pi$ |
|---|---|---|---|
| among the unions | Shapley | Banzhaf | Banzhaf |
| within each union | Shapley | Banzhaf | Shapley |

In Alonso-Meijide et al. (2007) parallel axiomatic characterizations were provided making the comparison among $\phi$, $\psi$, and $\pi$ easier.

## TU games with graph restricted communication and a priori unions

We can consider both restrictions to the cooperation (graph restricted communication and a priori unions) jointly. In this way we are able to enrich the model with different sorts of external information. As it will be done in the next example, we may use the a priori unions to describe the coalitions that have been formed in the past, and the communication graph to describe the relations among the political parties. This model was introduced in Vázquez-Brage et al. (1996). $\phi^c$, a value on $CU$ was proposed and characterized by means of the following properties;

**CE.** $f$ satisfies **Component efficiency** if for all $(N,v,B,P)\in CU$ and $T\in N/B$, $\sum_{i\in T}f_i(N,v,B,P)=v(T)$.

**FQ.** $f$ satisfies **Fairness in the quotient** if for all $(N,v,B,P)\in CU$ and all $P_k,P_s\in P$,

$$\sum_{i\in P_k}f_i(N,v,B,P)-\sum_{i\in P_k}f_i(N,v,B^{-(P_k,P_s)},P)=\sum_{i\in P_s}f_i(N,v,B,P)-\sum_{i\in P_s}f_i(N,v,B^{-(P_k,P_s)},P),$$

where $B^{-(P_k,P_s)}$ is the graph obtained from $B$ when the links between members of $P_k$ and $P_s$ are deleted.

**BCU.** $f$ satisfies **Balanced contributions for the unions** if for all $(N,v,B,P)\in CU$, $P_k\in P$, and $i,j\in P_k$,

$$f_i(N,v,B,P)-f_i(N,v,B,P_{-j})=f_j(N,v,B,P)-f_j(N,v,B,P_{-i}),$$

where $P_{-j}$ is the partition obtained from $P$ when player $j$ decides to leve his union to form a new union by himself.

### Theorem Vázquez-Brage et al. (1996)

There is a unique value on $CU$ satisfying **CE**, **FQ**, and **BCU**. Moreover, it is defined by $\phi^c(N,v,B,P)=\phi(N,v^B,P)$.

Following this idea, two new values on $CU$, $\psi^c$ and $\pi^c$ are proposed and characterized by means of the following properties.

**GI.** $f$ satisfies **Graph isolation** if for all $(N,v,B)\in C(N)$ and any $\{i\}\in N/B$, it holds that, $f_i(N,v,B,P^n)=v(i)$, where $P^n=\{\{1\},\{2\},\dots,\{n\}\}$ is the trivial singleton coalition structure.

**PM.** $f$ satisfies **Pairwise merging** if for all $(N,v,B)\in C(N)$ and every $i,j\in N$ such that $(i:j)\in B$,

$$f_i(N,v,B,P^n)+f_j(N,v,B,P^n)=f_p(N^{ij},v^{ij},B^{ij},P^{n-1}),$$

where $(N^{ij},v^{ij},B^{ij},P^{n-1})$ is the game obtained from $(N,v,B,P^n)$ when player $i$ and $j$ merge into a new player $p$.

**FG.** $f$ satisfies **Fairness in the graph** if for all $(N,v,B)\in C(N)$ and all $i,j\in N$ such that $(i:j)\in B$,

$$f_i(N,v,B,P^n)-f_i(N,v,B^{-ij},P^n)=f_j(N,v,B,P^n)-f_j(N,v,B^{-ij},P^n),$$

where $B^{-ij}$ is the graph obtained from $B$ when the link $(i:j)$ is deleted.

**NID.** $f$ satisfies **Neutrality under individual desertion** if for all $(N,v,B,P)\in CU$ and $i,j\in N$ such that $\{i,j\}\in P_k\in P$,

$$f_i(N,v,B,P)=f_i(N,v,B,P_{-j}).$$

**1-QG.** $f$ satisfies **1-Quotient game** property if for all $(N,v,B,P)\in CU$ and $i\in N$ such that $\{i\}=P_k\in P$,

$$f_i(N,v,B,P)=f_k(M,v^{BP},B^M,P^m),$$

where $B^M$ is the complete graph on $M$ and $v^{BP}(R)=\sum_{L\in(\cup_{r\in R}P_r)/B}v(L)$, for all $R\subseteq M$.

**QG.** $f$ satisfies **Quotient game** property if for all $(N,v,B,P)\in CU$ and $P_k\in P$,

$$\sum_{i\in P_k}f_i(N,v,B,P)=f_k(M,v^{BP},B^M,P^m).$$

### Theorem Alonso-Meijide et al. (2009)

There is a unique value on $CU$ satisfying **CE** (for $P^n$), **FG**, **BCU**, and **QG**. It is defined by $\phi^c(N,v,B,P)=\phi(N,v^B,P)$.

There is a unique value on $CU$ which satisfies **GI**, **PM**, **FQ**, **NID**, and **1-QG**. It is defined by $\psi^c(N,v,B,P)=\psi(N,v^B,P)$.

There is a unique value on $CU$ which satisfies **GI**, **PM**, **FQ**, **BCU**, and **QG**. It is defined by $\pi^c(N,v,B,P)=\pi(N,v^B,P)$.

## Example

Let us consider the Parliament of the Basque Country emerging from elections in April 2005.

As we said, we propose a communication graph based on the ideology and relations among the parties and a system of a priori unions taking into account the composition of the government before the elections.

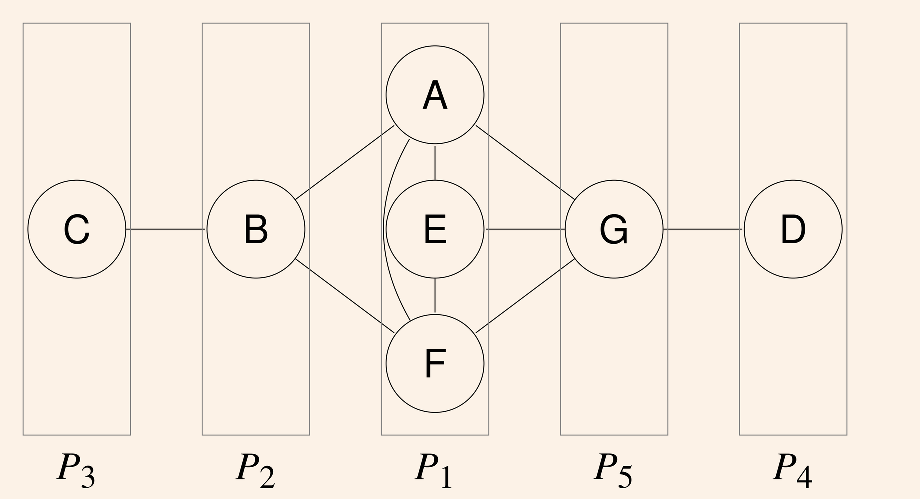


Figure: A: EAJ/PNV, B: PSE-EE/PSOE, C: PP, D: EHAK/PCTV, E: EA, F: EB/IU, G: Aralar

| Party | Seats | $\varphi$ | $\beta$ | $\phi^c$ | $\psi^c$ | $\pi^c$ |
|---|---|---|---|---|---|---|
| EAJ/PNV | 22 | .3524 | .5938 | .3806 | .4688 | .4479 |
| PSE-EE/PSOE | 18 | .2524 | .4062 | .2500 | .3750 | .3750 |
| PP | 15 | .1857 | .3438 | 0 | 0 | 0 |
| EHAK/PCTV | 9 | .0857 | .1562 | .0833 | .1250 | .1250 |
| EA | 7 | .0857 | .1562 | .0722 | .1250 | .1042 |
| EB/IU | 3 | .0190 | .0312 | .1306 | .0938 | .0938 |
| Aralar | 1 | .0190 | .0312 | .0833 | .1250 | .1250 |

Table: The distribution of power of the simple game and the game with graph restricted communication and a priori unions

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