A Unifying Model of Winner-takes-all Contests

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Northwestern University
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Universidad de Vigo

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Outline

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Motivation

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- Contests:

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  Well Known!

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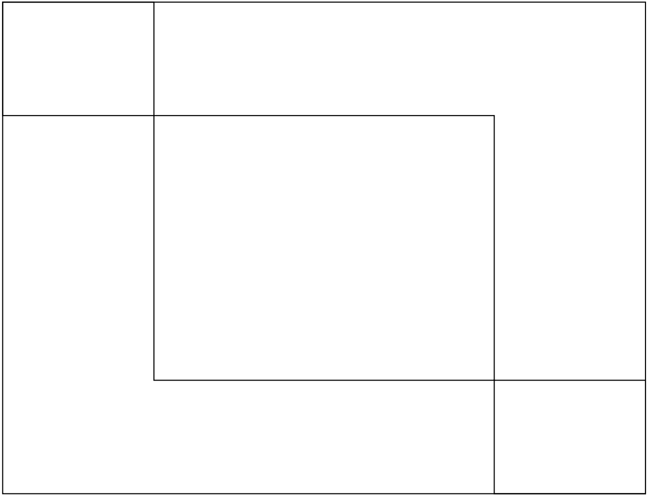
Complete information

First Example: Sharing a Cake



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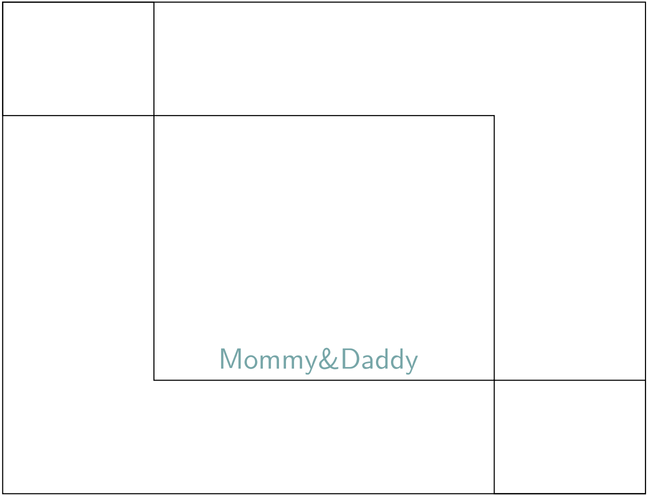

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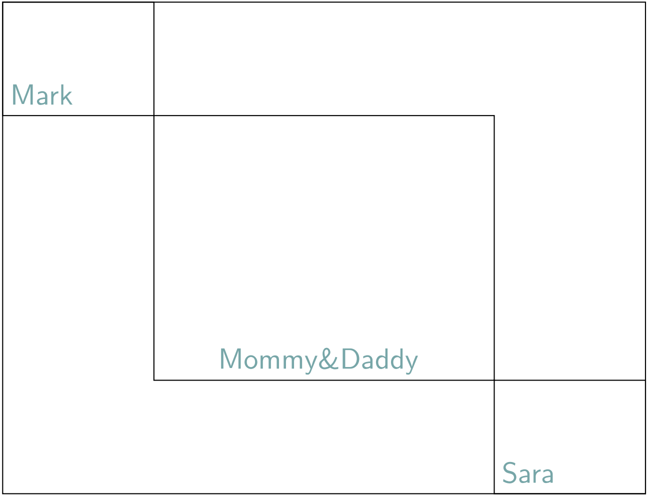

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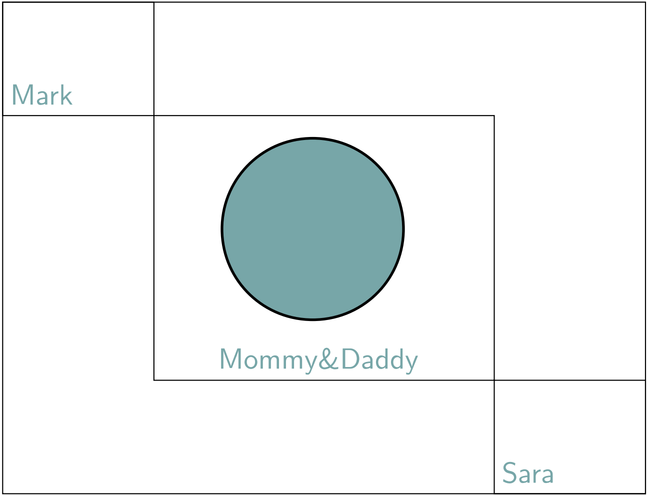

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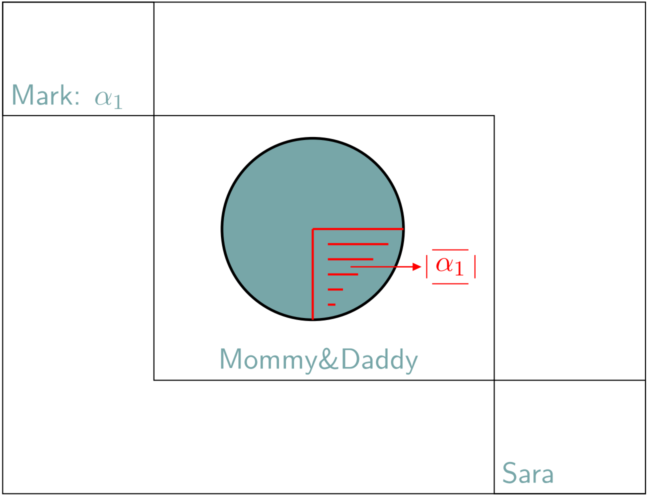

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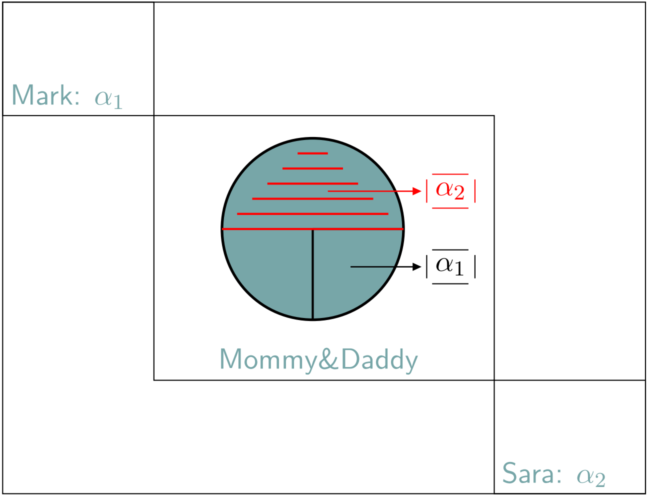

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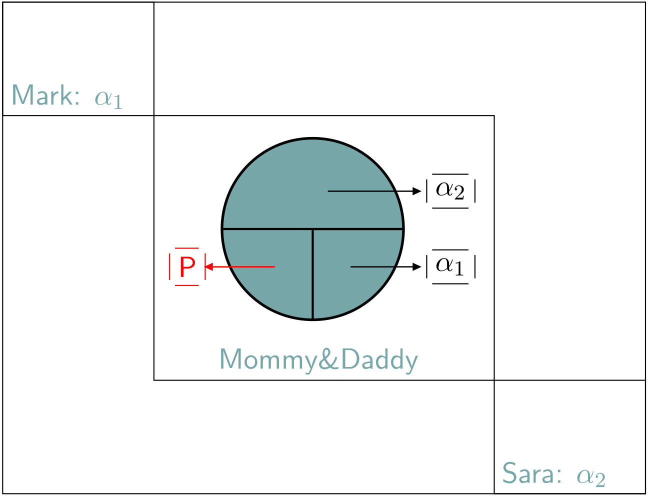

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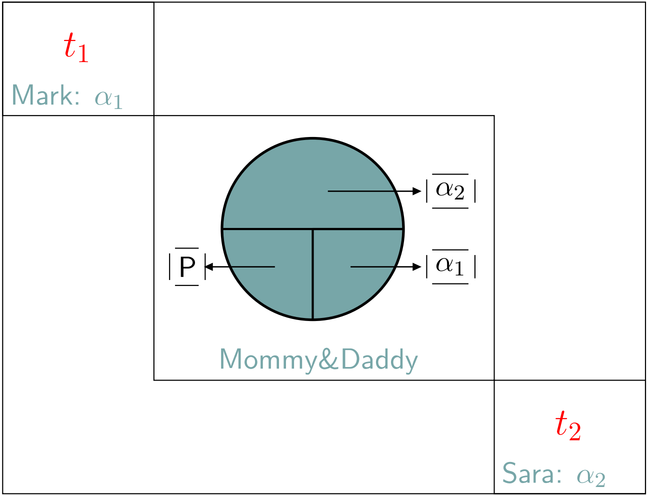

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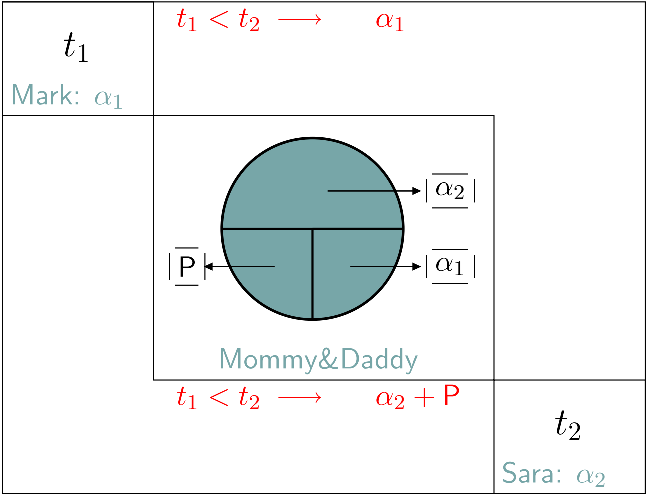

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Mark: α_1

$t_1 < t_2 \longrightarrow \alpha_1$

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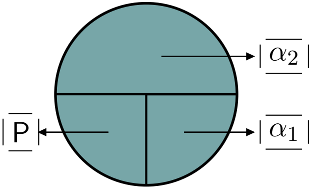

Mommy&Daddy

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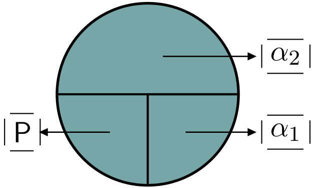

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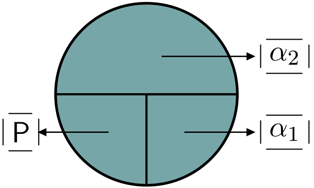

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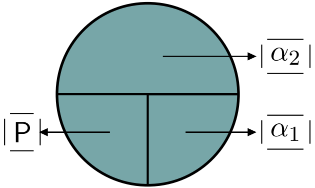

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Ties??

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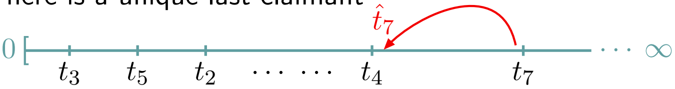

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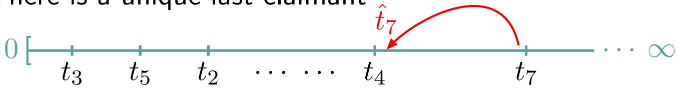

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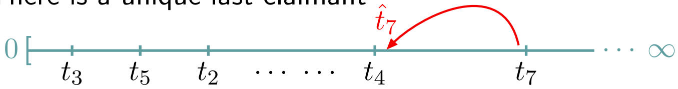

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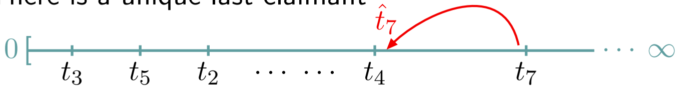

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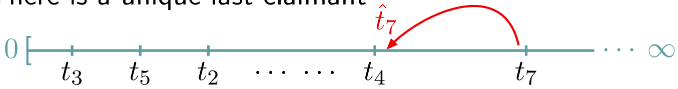

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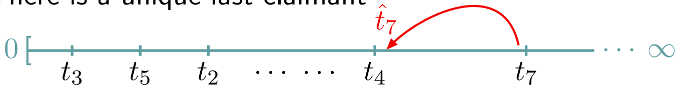

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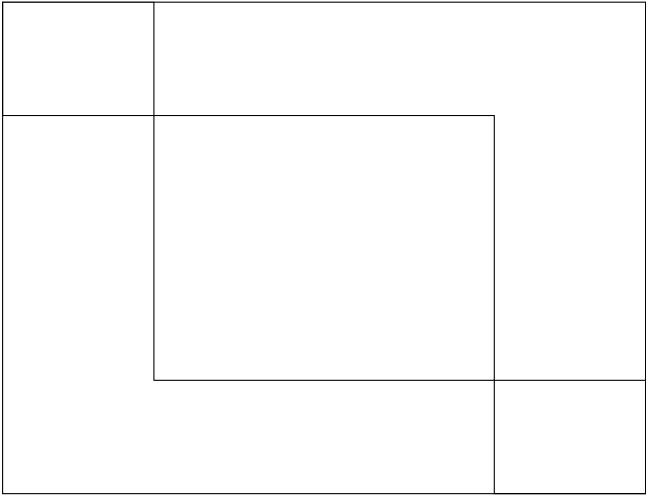

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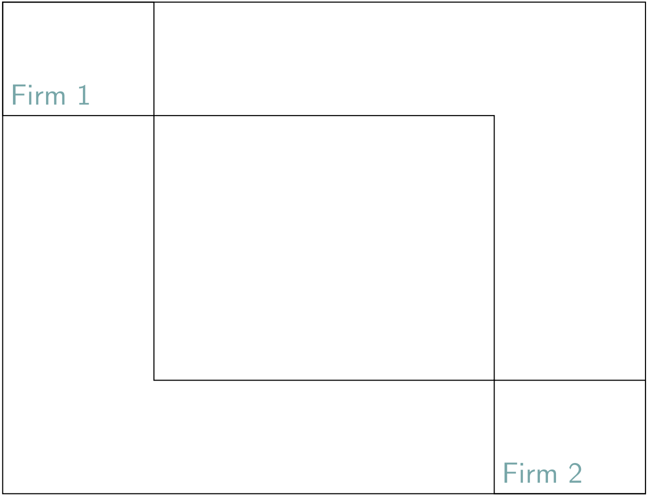

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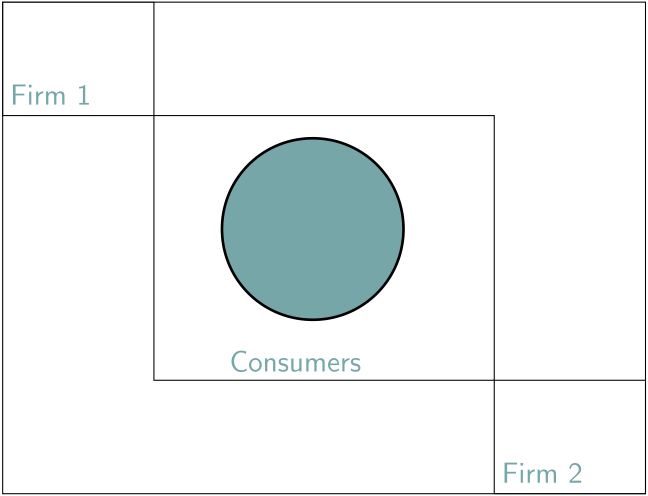

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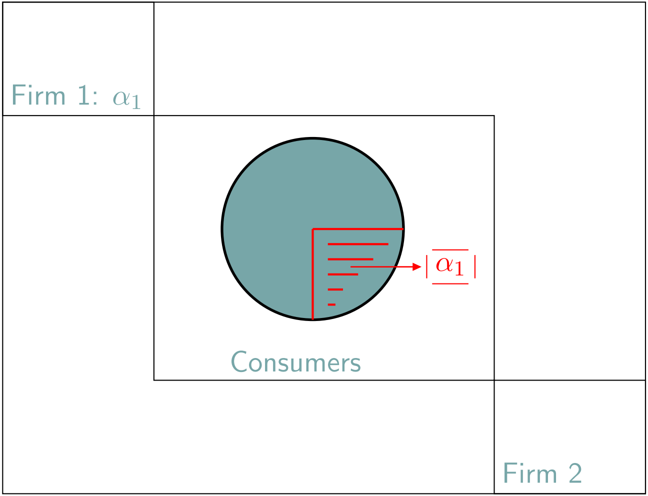

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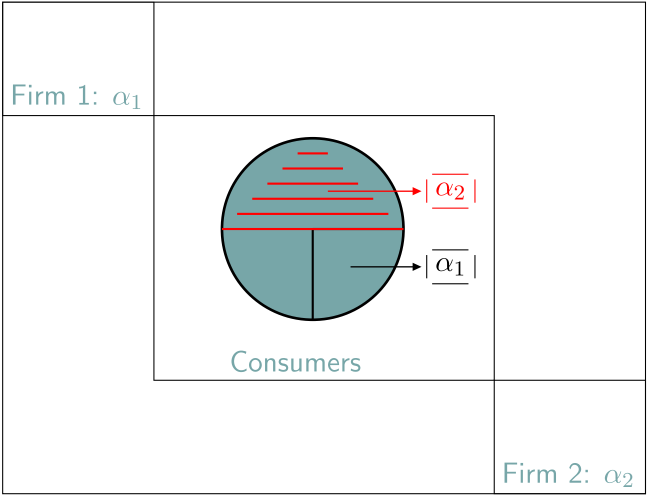

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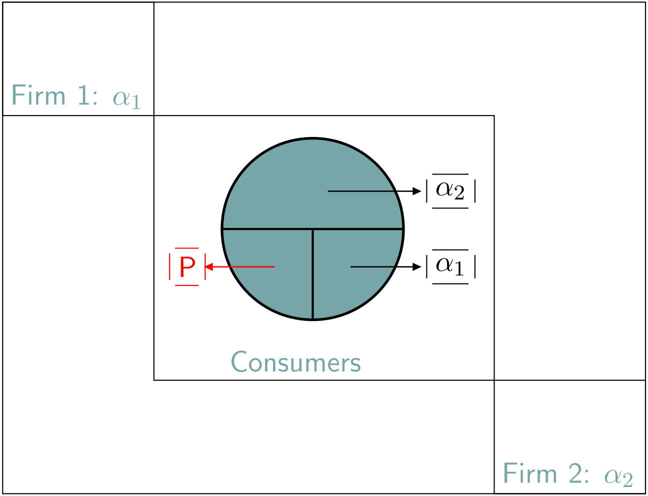

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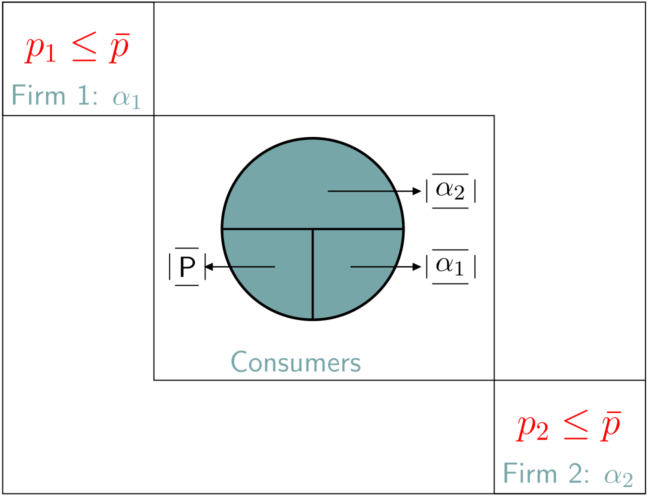

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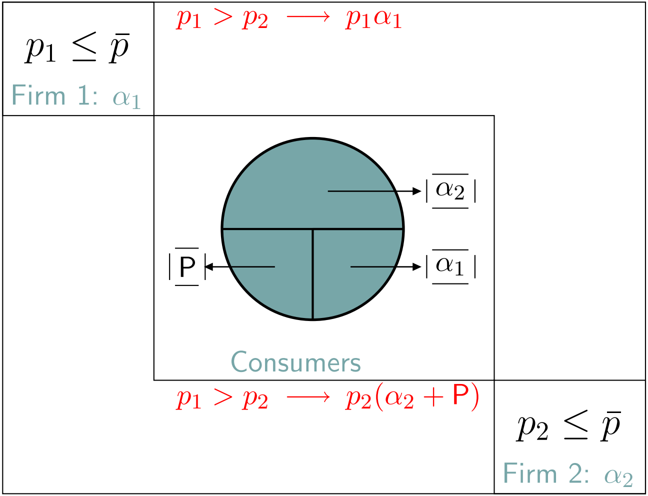

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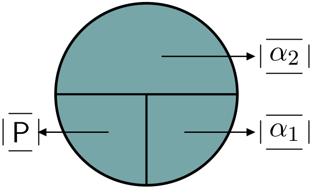

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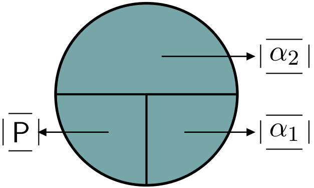

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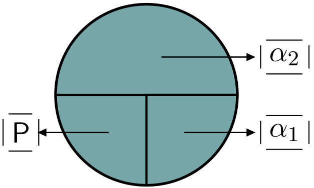

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The pricing game

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- Only the firm with less loyal consumers gains by "competing"
- Strategic consumers pay no more than loyal consumers

The Characterization Result and the Pricing Game

The pricing game

- N firms. Each one with α_i loyal consumers
- Strategic consumers: P
- Higher admissible price: $\bar{p}$

The equilibrium of the pricing game

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- Only the firm with less loyal consumers gains by "competing"
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Outline

1. Motivation
2. Winner-takes-all Contests
3. Various Models of Contests
4. Results

Winner-takes-all Contests

The Game

Winner-takes-all Contests

The Game

- The players want to get a **prize**

Winner-takes-all Contests

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Primitives

Winner-takes-all Contests

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Winner-takes-all Contests

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Winner-takes-all Contests

Tie Payoff Functions

Winner-takes-all Contests

Tie Payoff Functions

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Winner-takes-all Contests

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Winner-takes-all Contests

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Winner-takes-all Contests

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Winner-takes-all Contests
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Examples

Winner-takes-all Contests

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- $T_i(e, S) = \begin{cases} \frac{p_i(e)}{|S|} & i \in S \\ 0 & \text{otherwise} \end{cases}$

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- $T_i(e, S) = \begin{cases} p_i(e) & \{i\} = S \\ 0 & \text{otherwise} \end{cases}$

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Productivity functions

Winner-takes-all Contests

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Assumptions

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For each $i \in N$, $b_i(\cdot)$ is strictly decreasing

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$$\bar{e}_i := \sup_{e \in [0,M]} \{b_i(0) \leq b_i(e) + p_i(e)\}$$

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For each $i \in N$, $\bar{e}_i < M$

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Impact (trade-off) functions: for each $i \in N$, $I_i(e) = \frac{b_i(0) - b_i(e)}{p_i(e)}$

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Impact (trade-off) functions: for each $i \in N$, $I_i(e) = \frac{b_i(0) - b_i(e)}{p_i(e)}$

- Assumption: No-crossing

For each pair $i, j \in N$, if there is e^* such that $I_i(e^*) < I_j(e^*)$, then $I_i(e) < I_j(e)$ for all e

A First Result

A First Result

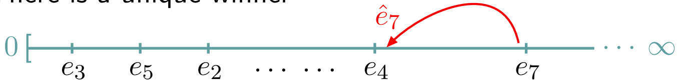
Proposition

If the contest C^f_{pure} satisfies All-pay and M-bounding, then it does not have any Nash equilibrium.

A First Result

There is no Nash equilibrium in pure strategies

- There is a unique winner



$$\pi_7(\sigma) = b_i(e_7) + p_i(e_7) \; < \; \pi_7(\sigma_{-7}, \hat{e}_7) = b_i(\hat{e}_7) + p_i(\hat{e}_7)$$

- There are several winners

$$\pi_7(\sigma) = b_i(e_7) + 0 \; < \; \pi_7(\sigma_{-7}, \hat{e}_7) = b_i(\hat{e}_7) + p_i(\hat{e}_7)$$

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We need mixed strategies

A First Result

Proposition

If the contest C^f_{pure} satisfies All-pay and M-bounding, then it does not have any Nash equilibrium.

We need mixed strategies

No ties with positive probability in equilibrium

Outline

Generalized Models

	All-pay	Winner-pays	M-bounding	No-crossing

Generalized Models

1. First Price Auction

	All-pay	Winner-pays	M-bounding	No-crossing

Generalized Models

	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	**X**	✓	✓	✓

Generalized Models

1. First Price Auction

2. All-Pay Auction

(Politically Contestable Rents)

All-pay	Winner-pays	M-bounding	No-crossing
X	✓	✓	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				

Generalized Models

	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	**X**	✓	✓	✓
2. All-Pay Auction	✓	**X**	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers				

Generalized Models

	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	**X**	✓	✓	✓
2. All-Pay Auction	✓	**X**	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	**X**	✓	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition				

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
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Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales				

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth				

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	**X**	✓	✓	✓
2. All-Pay Auction	✓	**X**	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	**X**	✓	✓
4. Bertrand Competition	**X**	✓	✓	✓
5. Varian's Model of Sales	✓	**X**	✓	✓
6. Federalism and Economic Growth	**X**	✓	**X**	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers				

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers	✓	X	✓	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers	✓	X	✓	✓
8. Litigation Systems				

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers	✓	X	✓	✓
8. Litigation Systems	✓	X	✓	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	**X**	✓	✓	✓
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(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	**X**	✓	✓
4. Bertrand Competition	**X**	✓	✓	✓
5. Varian's Model of Sales	✓	**X**	✓	✓
6. Federalism and Economic Growth	**X**	✓	**X**	✓
7. Market Makers	✓	**X**	✓	✓
8. Litigation Systems	✓	**X**	✓	✓
9. Timing Games				

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	**X**	✓	✓	✓
2. All-Pay Auction	✓	**X**	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	**X**	✓	✓
4. Bertrand Competition	**X**	✓	✓	✓
5. Varian's Model of Sales	✓	**X**	✓	✓
6. Federalism and Economic Growth	**X**	✓	**X**	✓
7. Market Makers	✓	**X**	✓	✓
8. Litigation Systems	✓	**X**	✓	✓
9. Timing Games	✓	**X**	✓	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	**X**	✓	✓	✓
2. All-Pay Auction	✓	**X**	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	**X**	✓	✓
4. Bertrand Competition	**X**	✓	✓	✓
5. Varian's Model of Sales	✓	**X**	✓	✓
6. Federalism and Economic Growth	**X**	✓	**X**	✓
7. Market Makers	✓	**X**	✓	✓
8. Litigation Systems	✓	**X**	✓	✓
9. Timing Games	✓	**X**	✓	✓

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction (Politically Contestable Rents)	✓	X	✓	✓
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers	✓	X	✓	✓
8. Litigation Systems	✓	X	✓	✓
9. Timing Games	✓	X	✓	✓

Discretizing??

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers	✓	X	✓	✓
8. Litigation Systems	✓	X	✓	✓
9. Timing Games	✓	X	✓	✓

Discretizing?? **No Crossing??**

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers	✓	X	✓	✓
8. Litigation Systems	✓	X	✓	✓
9. Timing Games	✓	X	✓	✓

Discretizing?? No Crossing??

Other models

Generalized Models	All-pay	Winner-pays	M-bounding	No-crossing
1. First Price Auction	X	✓	✓	✓
2. All-Pay Auction	✓	X	✓	✓
(Politically Contestable Rents)				
3. Politically Contestable Transfers	✓	X	✓	✓
4. Bertrand Competition	X	✓	✓	✓
5. Varian's Model of Sales	✓	X	✓	✓
6. Federalism and Economic Growth	X	✓	X	✓
7. Market Makers	✓	X	✓	✓
8. Litigation Systems	✓	X	✓	✓
9. Timing Games	✓	X	✓	✓

Discretizing?? **No Crossing??**

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- Second Price Auction

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Discretizing?? **No Crossing??**

Other models

- Second Price Auction
- Second Price All-Pay Auction
- War of Attrition

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9. Timing Games	✓	X	✓	✓

Discretizing?? No Crossing??

Other models

- Second Price Auction → First Price Auction
- Second Price All-Pay Auction → (First Price) All-pay Auction
- War of Attrition → Timing Games

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Discretizing?? **No Crossing??**

Other models

- Second Price Auction
- Second Price All-Pay Auction
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Classification

Classification

All-pay (b_i functions strictly decreasing)

- All-pay auction (Politically contestable rents)
- Politically contestable Transfers
- Varian's model of sales
- Market makers
- Litigation systems
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All-pay (b_i functions strictly decreasing)

- All-pay auction (Politically contestable rents)
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- Varian's model of sales
- Market makers
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- (Silent) Timing games

Winner-pays (p_i functions strictly decreasing)

- First price auction
- Bertrand competition
- Federalism and economic growth (No M-bounding)

Discussion

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Positive Features of the model

Limitations of the model

Discussion

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- Powerful to model asymmetries

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Outline

1. Motivation
2. Winner-takes-all Contests
3. Various Models of Contests
4. Results

Characterization under **All-pay** and M-**bounding**

Theorem (Characterization under **All-pay** and M-**bounding**)

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Theorem (Characterization under **All-pay** and M-**bounding**)

- *If either* $n = 2$ *or* $\bar{e}_1 > \bar{e}_2 > \bar{e}_3$, *then* EP^f *has a unique Nash equibrium*

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- *If either $n = 2$ or $\bar{e}_1 > \bar{e}_2 > \bar{e}_3$, then EP^f has a unique Nash equibrium*

$$G_1^*(e) = \begin{cases} 0 & e < 0 \\ I_2(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_2^*(e) = \begin{cases} 0 & e < 0 \\ I_1^*(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_i^*(e) = \begin{cases} 0 & e < 0 \\ 1 & e \geq 0 \end{cases}$$

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Implications of the result

Characterization under **Winner-pays**

Theorem (Characterization under **Winner-pays** and M-**bounding**)

Assume that, for each $i \in N$, $b_i(\cdot)$ *equals constant* $b_i \in \mathbb{R}$.

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Assume that, for each $i \in N$, $b_i(\cdot)$ equals constant $b_i \in \mathbb{R}$.

- *Let $\bar{e}_1 > \bar{e}_2$. Then, EP^f has no Nash equilibrium in pure strategies but it has a continuum of mixed Nash equilibria. The equilibrium payoffs are such that $\eta_1 \in (b_1, b_1 + p_1(\bar{e}_2)]$ and, for each $i \neq 1$, $\eta_i = b_i$.*

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Implications of the result:

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Implications of the result: Auctions

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Implications of the result: Auctions and Bertrand competition

Characterization under **Winner-pays**

Corollary

Characterization under **Winner-pays**

Corollary

Take a general Bertrand competition model (BM) with n firms

Characterization under **Winner-pays**

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If the cost function is the same for all firms and exhibits strictly decreasing average costs,

Characterization under **Winner-pays**

Corollary

Take a general Bertrand competition model (BM) with n firms
If the cost function is the same for all firms and exhibits strictly decreasing average costs,
then there is no Nash equilibrium (neither pure, nor mixed)

Characterizations

Characterizations **without M-bounding**?

Characterizations

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Ties

Conclusions

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- Generalization of the results included in the models satisfying All-pay assumption

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A Unifying Model of Winner-takes-all Contests

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and
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