Essays on Competition and Cooperation in Game Theoretical Models

Julio González Díaz

Department of Statistics and Operations Research
Universidade de Santiago de Compostela

Thesis Dissertation
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General Overview

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Part I: Noncooperative Game Theory

Part II: Cooperative Game Theory
(On the Geometry of TU games)

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Part I: Noncooperative Game Theory

- A Silent Battle over a Cake (Chapter 1)

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- **Repeated Games** (Chapters 2 and 3)

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- **The Core-Center** (Chapters 5, 6, and 7)

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The Core-Center

Part I

Noncooperative Game Theory

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And a Nash Equilibrium?

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Unilateral deviations are not profitable

A Silent Battle over a Cake

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- Proof of the existence and uniqueness result of the Nash equilibrium in the general case (n-players)

A Noncooperative Approach to Bankruptcy Problems

Bankruptcy Problems and Bankruptcy Rules

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- Coincides with the proposal of the proportional rule

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Repeated Games

1. A Silent Battle over a Cake
   - Brief Overview
2. A Noncooperative Approach to Bankruptcy Problems
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3. Repeated Games
   - Definitions and Classic Results
   - A Generalized Nash Folk Theorem
   - Unilateral Commitments

Repeated Games

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"The model of a repeated game is designed to examine the logic of longterm interaction. It captures the idea that a player will take into account the effect of his current behavior on the other players' future behavior, and aims to explain phenomena like cooperation, revenge, and threats".
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$$\pi_\delta^T(\sigma) := \frac{1}{T} \sum_{t=1}^{T} \pi(a^t)$$

The Repeated Game

Repeated Games (with complete information)

- Let $G = (N, A, \pi)$
- G_δ^T denotes the T-fold repetition of the game G with discount parameter δ
- $G_\delta^T := (N, S, \pi_\delta^T)$ where
  - $N := \{1, \ldots, n\}$
  - $S := \prod_{i \in N} S_i$, $S_i := A_i^H$
  - Discounted payoffs in the repeated game,

$$\pi_\delta^T(\sigma) := \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} \pi(a^t)$$

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Repeated Games (with complete information)

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Folk Theorems

General Considerations

Our framework:

General Considerations

Our framework:

- The sets of actions are compact

General Considerations

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- The sets of actions are compact
- Continuous payoff functions

General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions
- Finite Horizon

General Considerations

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- Nash Equilibrium

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General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions
- Finite Horizon
- Nash Equilibrium
- Complete Information
- Perfect Monitoring (Observable mixed actions)

The State of Art
The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon		
Finite Horizon		

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon	The "Folk Theorem" (1970s)	Fudenberg and Maskin (1986) Abreu et al. (1994) Wen (1994)
Finite Horizon	Benoît and Krishna (1987)	Benoît and Krishna (1985) Smith (1995) Gossner (1995)

The State of Art

The Folk Theorems

	Nash	**Subgame Perfect**
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Necessary and Sufficient conditions

The State of Art

The Folk Theorems

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Necessary and Sufficient conditions

(Benoît and Krishna, 1987)

Assumption for the game G

Result

(Benoît and Krishna, 1987)

Assumption for the game G

- **Existence of strictly rational Nash payoffs**

Result

(Benoît and Krishna, 1987)

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  For each player i there is a Nash Equilibrium a^i of G such that $\pi_i(a^i) > v_i$

Result

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Assumption for the game G

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Result

- **Every payoff in F can be approximated in equilibrium**

(Benoît and Krishna, 1987)

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- **Existence of strictly rational Nash payoffs**
  For each player i there is a Nash Equilibrium a^i of G such that $\pi_i(a^i) > v_i$

Result

- **Every payoff in F can be approximated in equilibrium**
  For each $u \in F$ and each $\varepsilon > 0$, there are T_0 and δ_0 such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash Equilibrium σ of $G(\delta, T)$ satisfying that $\|\pi_\delta^T(\sigma) - u\| < \varepsilon$

(Benoît and Krishna, 1987)

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Our Contribution

Minmax Bettering Ladders

Our Contribution

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

Our Contribution
Minmax Bettering Ladders

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M	-1,0,0	0,-1,0	0,-1,0
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R:

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- Minmax Payoff (0,0,0)

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M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Minmax Payoff (0,0,0)
- Nash Equilibrium (T,l,L), Payoff (0,0,3)

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Minmax Bettering Ladders

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	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Minmax Payoff (0,0,0)
- Nash Equilibrium (T,l,L), Payoff (0,0,3) (B-K not met)

Our Contribution
Minmax Bettering Ladders

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T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Minmax Payoff (0,0,0)
- Nash Equilibrium (T,l,L), Payoff (0,0,3) (B-K not met)
- Player 3 can be threatened

Our Contribution

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

l	m	r
0,3,-1	0,-1,-1	1,-1,-1
-1,0,-1	-1,-1,-1	0,-1,-1
-1,0,-1	-1,-1,-1	0,-1,-1

Our Contribution
Minmax Bettering Ladders

Example

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

L

l	m	r
0,3,-1	0,-1,-1	1,-1,-1
-1,0,-1	-1,-1,-1	0,-1,-1
-1,0,-1	-1,-1,-1	0,-1,-1

R

- Player 3 is forced to play R

Our Contribution

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R
- The profile α^3 =(T,l,R) is a Nash Equilibrium of the reduced game with Payoff (0,3,-1)

Our Contribution

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R
- The profile $\alpha^3 =$(T,l,R) is a Nash Equilibrium of the reduced game with Payoff (0,3,-1)
- Now player 2 can be threatened

Our Contribution

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

Our Contribution

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R and player 2 to play r

Our Contribution

Minmax Bettering Ladders

Example

	L: l	L: m	L: r	R: l	R: m	R: r
T	0,0,3	0,-1,0	0,-1,0	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,0	0,-1,0	0,-1,0	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,0	0,-1,0	0,-1,0	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R and player 2 to play r
- The profile α^{32} =(T,r,R) is a Nash Equilibrium of the reduced game with Payoff (1,-1,-1)

Our Contribution

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R and player 2 to play r
- The profile α^{32} =(T,r,R) is a Nash Equilibrium of the reduced game with Payoff (1,-1,-1)
- Now player 1 can be threatened

Minmax Bettering Ladders

Formal Definition

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$				

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$				
game	G				

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$				
game	G				
"Nash equilibrium"	σ^1				

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1			
game	G				
"Nash equilibrium"	σ^1				

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1			
game	G	$G(a_{N_1})$			
"Nash equilibrium"	σ^1				

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1			
game	G	$G(a_{N_1})$			
"Nash equilibrium"	σ^1	σ^2			

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\ldots$		
game	G	$G(a_{N_1})$	$\ldots$		
"Nash equilibrium"	σ^1	σ^2	$\ldots$		

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	
game	G	$G(a_{N_1})$	$\dots$		
"Nash equilibrium"	σ^1	σ^2	$\dots$		

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	
"Nash equilibrium"	σ^1	σ^2	$\dots$		

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reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	
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A **minimax-bettering ladder** of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	$---$

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- $\mathcal{N} := \{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ subsets of N

Minmax Bettering Ladders

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reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	$---$
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Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	$---$

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Minmax Bettering Ladders

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reliable players	$\emptyset$	N_1	$\ldots$	N_{h-1}	N_h
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N_h is the top rung of the ladder

Minmax Bettering Ladders

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reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	$---$

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- $\Sigma := \{\sigma^1, \dots, \sigma^h\}$

N_h is the top rung of the ladder

$N_h = N \rightarrow$ **complete minimax-bettering ladder**

The New Folk Theorem
(González-Díaz, 2003)

Assumption for the game G

Result

The New Folk Theorem
(González-Díaz, 2003)

Assumption for the game G

- **Existence of a complete minmax bettering ladder**

Result

The New Folk Theorem
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- **Every payoff in F can be approximated in equilibrium**

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Remark

Unlike Benoît and Krishna's result, this theorem provides a necessary and sufficient condition

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Why the word **generalized**?

Unilateral Commitments

Motivation

Unilateral Commitments

Motivation

- Commitment

Unilateral Commitments

Motivation

- Commitment
- Repeated games

Unilateral Commitments

Motivation

- Commitment
- Repeated games
- Unilateral commitments in repeated games

Unilateral Commitments

Motivation

- Commitment
- Repeated games
- Unilateral commitments in repeated games
- Delegation games

Unilateral Commitments

Definitions

- **The stage game:** $G := (N, A, \pi) \begin{cases} N := \{1, \dots, n\} \\ A := \prod_{i \in N} A_i \\ \pi := (\pi_1, \dots, \pi_n) \end{cases}$
- **The repeated game:** $G_\delta^T := (N, S, \pi_\delta^T) \begin{cases} N := \{1, \dots, n\} \\ S := \prod_{i \in N} S_i \\ (S_i := A_i^H) \\ \pi_\delta^T \end{cases}$

Unilateral Commitments

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- **The UC-extension:** $U(G) := (N, A^U, \pi^U)$

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Unilateral Commitments

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  1. $\emptyset \subsetneq A_i^c \subseteq A_i$,

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  1. $\emptyset \subsetneq A_i^c \subseteq A_i$,
  2. $\alpha_i : \prod_{j \in N} 2^{A_j} \longrightarrow A_i$

Unilateral Commitments

Definitions

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  $A^U := \prod_{i\in N} A_i^U$, where A_i^U is the set of all couples (A_i^c, α_i) such that
  1. $\emptyset \subsetneq A_i^c \subseteq A_i$,
  2. $\alpha_i : \prod_{j\in N} 2^{A_j} \longrightarrow A_i$ and, for each $A^c \in \prod_{j\in N} 2^{A_j}$, $\alpha_i(A^c) \in A_i^c$

Unilateral Commitments

Definitions

- **The stage game:** $G := (N, A, \pi) \begin{cases} N := \{1, \ldots, n\} \\ A := \prod_{i \in N} A_i \\ \pi := (\pi_1, \ldots, \pi_n) \end{cases}$
- **The repeated game:** $G_\delta^T := (N, S, \pi_\delta^T) \begin{cases} N := \{1, \ldots, n\} \\ S := \prod_{i \in N} S_i \\ (S_i := A_i^H) \\ \pi_\delta^T \end{cases}$
- **The UC-extension:** $U(G) := (N, A^U, \pi^U)$

  $A^U := \prod_{i \in N} A_i^U$, where A_i^U is the set of all couples (A_i^c, α_i) such that

  1. $\emptyset \subsetneq A_i^c \subseteq A_i$,
  2. $\alpha_i : \prod_{j \in N} 2^{A_j} \longrightarrow A_i$ and, for each $A^c \in \prod_{j \in N} 2^{A_j}$, $\alpha_i(A^c) \in A_i^c$

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Commitments are Unilateral

Complete Information

Virtually Subgame Perfect Equilibrium

Motivation

Virtually Subgame Perfect Equilibrium

Motivation

Subgame Perfect Equilibrium

Virtually Subgame Perfect Equilibrium

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Subgame Perfect Equilibrium

- Eliminates Nash equilibria based on incredible threats

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Let σ be a strategy profile.

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Virtually Subgame Perfect Equilibrium

Discussion

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Subgame Perfect Vs Virtually Subgame Perfect

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Why do we need VSPE?

Virtually Subgame Perfect Equilibrium
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- In our model, we face very large trees

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We cannot use the classic results for the existence of SPE

The Folk Theorems

Finite Horizon

The Folk Theorems
Finite Horizon

Nash Folk Theorem (without UC)

Existence of a complete minmax bettering ladder

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No assumption is needed for the Nash folk theorem with UC

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G must have a pair of Nash equilibra in which some player gets different payoffs

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The counterpart of Theorem 1 for VSPE does not hold

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Let $\bar{a} \in A$ be a Nash equilibrium of G. Then, the game $U(G)$ has a VSPE with payoff $\pi(\bar{a})$

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	Without UC	**1 stage of UC**	**2 stages of UC**
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The State of Art

Necessary and Sufficient Conditions for the Folk Theorems

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Noncooperative Game Theory

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Noncooperative Game Theory

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Noncooperative Game Theory
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Noncooperative Game Theory

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Noncooperative Game Theory
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- Study Unilateral Commitments in models with incomplete information

Noncooperative Game Theory

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Part II

Cooperative Game Theory

What is a Cooperative Game?

And an Allocation Rule?

What is a Cooperative Game?
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Cooperative game (with transferable utility)

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An allocation $x \in \mathbb{R}^n$ is efficient if $\sum_{i=1}^n x_i = v(N)$

A Geometric Characterization of the τ-value

The τ-value

Let v denote a cooperative game (N is fixed)

The τ-value

Previous concepts

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- Utopia vector, $M(v) \in \mathbb{R}^n$:

The τ-value

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- A game v is compromise admissible if $CC(v) \neq \emptyset$

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The Core-Center

4. A Geometric Characterization of the τ-value
   - Brief Overview

5. The Core-Center
   - The Core-Center: Definition and Properties
   - A Characterization of the Core-Center
   - The Core-Center and the Shapley Value

Some More Background

Fix a game v and an allocation $x \in \mathbb{R}^n$

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A game v is balanced if $C(v) \neq \emptyset$

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$$v = \begin{cases} v(1) = 0 \quad v(2) = 0 \quad v(3) = 0 \\ v(12) = 1 \quad v(13) = 4 \quad v(23) = 7 \\ v(123) = 15 \end{cases}$$

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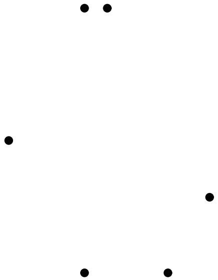

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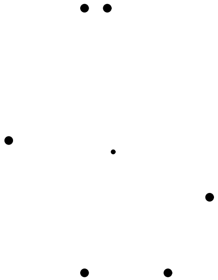

Weber Set & Shapley Value

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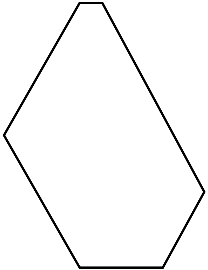

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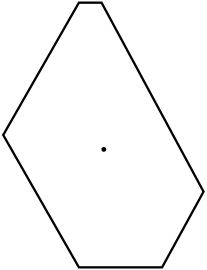
Core-Cover & τ-value



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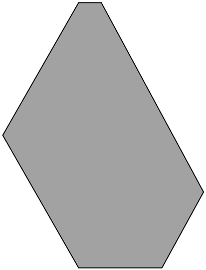

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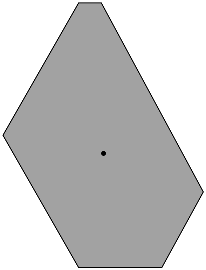
Core & Core-Center



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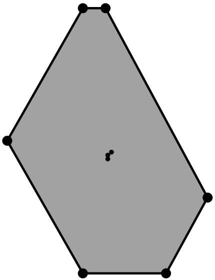

The Core-Center: Basic Properties

Basic Properties

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- Efficiency

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$\varphi : \quad \Omega \subseteq \mathbb{R}^{2^n}$

The Core-Center: Continuity

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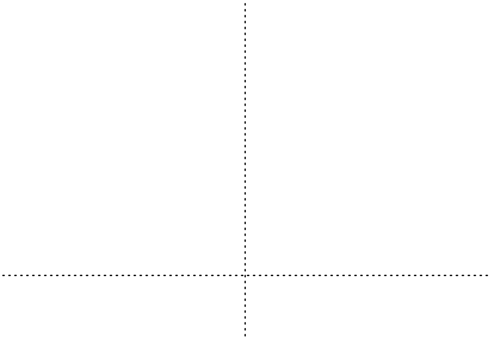

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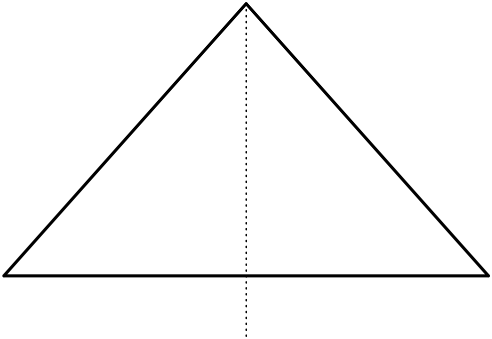

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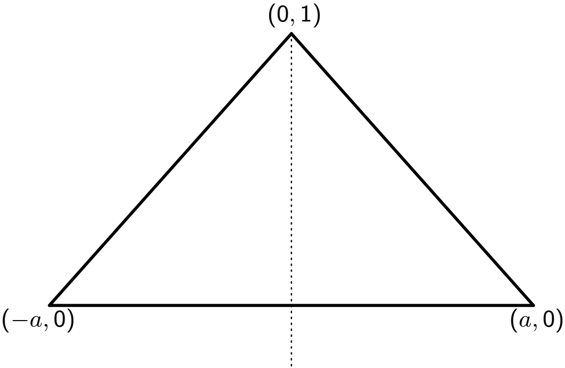

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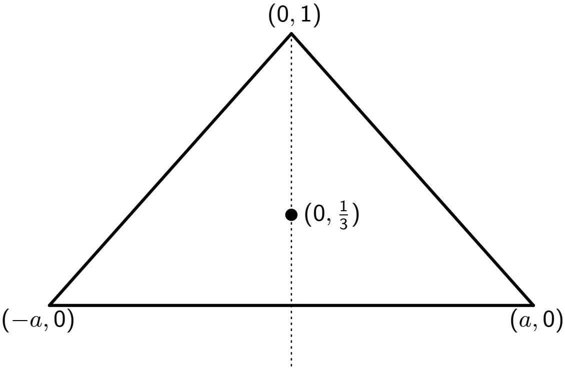

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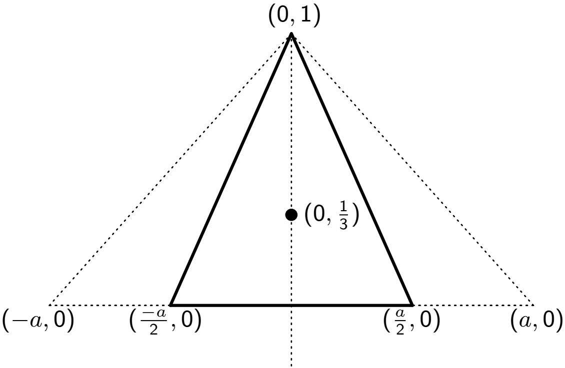

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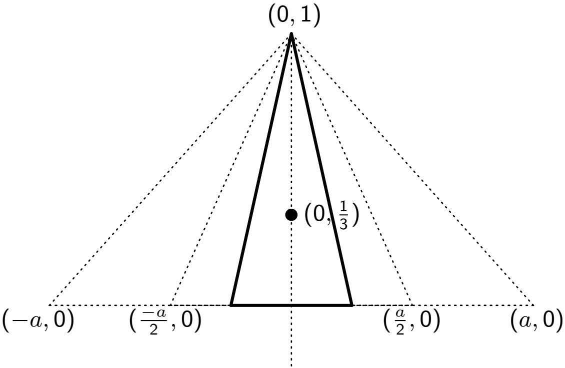

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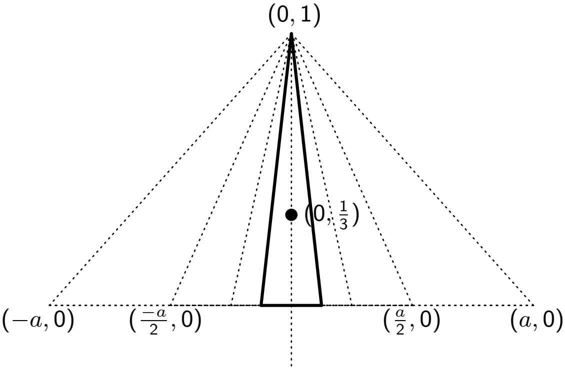

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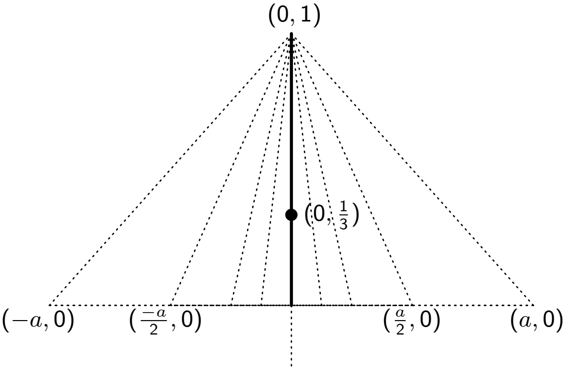

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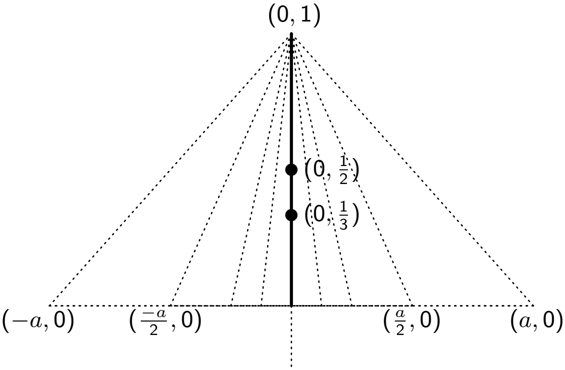

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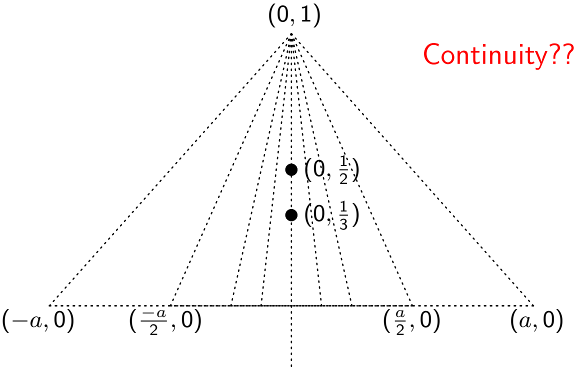

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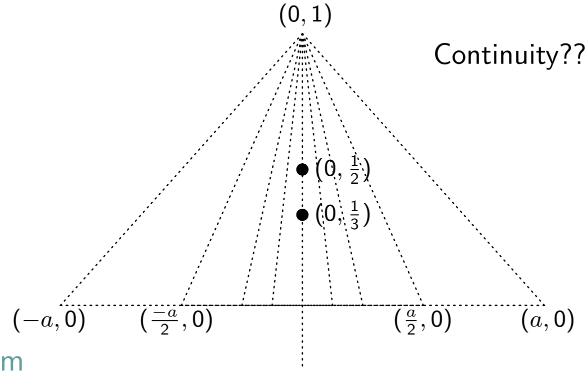

Continuity??

Theorem

The Core-Center is a continuous allocation rule

The Core-Center: Monotonicity

- Take a pair of games v and w

The Core-Center: Monotonicity

- Take a pair of games v and w

  Strong monotonicity

The Core-Center: Monotonicity

- Take a pair of games v and w

Strong monotonicity

Let $i \in N$. If for each $S \subseteq N \backslash \{i\}$,
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The Core-Center: Monotonicity

- Take a pair of games v and w

Strong monotonicity — NOT SATISFIED

Let $i \in N$. If for each $S \subseteq N \backslash \{i\}$,
$w(S \cup \{i\}) - w(S) \geq v(S \cup \{i\}) - v(S)$, then $\varphi_i(N,w) \geq \varphi_i(N,v)$

- $w(T) > v(T)$ and for each $S \neq T$, $w(S) = v(S)$

Coalitional monotonicity

For each $i \in T$,

The Core-Center: Monotonicity

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Weak coalitional monotonicity

$\sum_{i \in T} \varphi_i(w) \geq \sum_{i \in T} \varphi_i(v)$

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Core-Center $\Longleftrightarrow$ Nucleolus

The Core-Center: An Additivity Property

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Superadditivity:

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Superadditivity: If $S \cap T = \emptyset$, then

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$$\bar{v}(S) = \begin{cases} k & T = S \\ v(S) & \text{otherwise} \end{cases}$$

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"CUT"

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$$\varphi(v) = \alpha\varphi(\overline{v}) + (1 - \alpha)\varphi(\underline{v})$$

where $\alpha \in [0, 1]$

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2. *Translation Invariance*

The Core-Center:

Balanced Games

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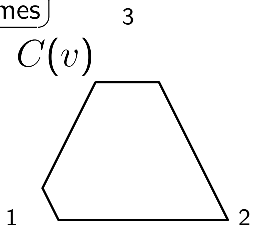
Balanced Games



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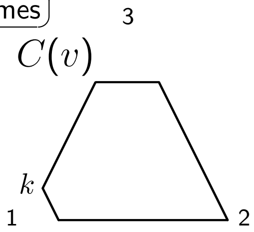

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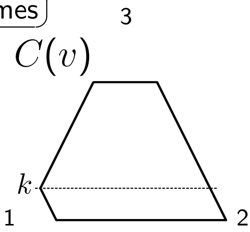

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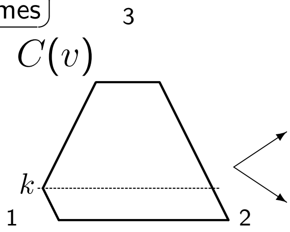

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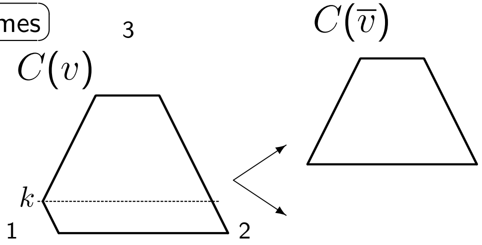

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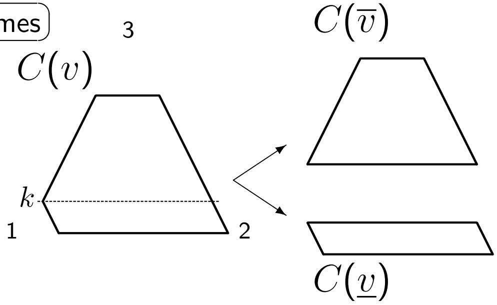

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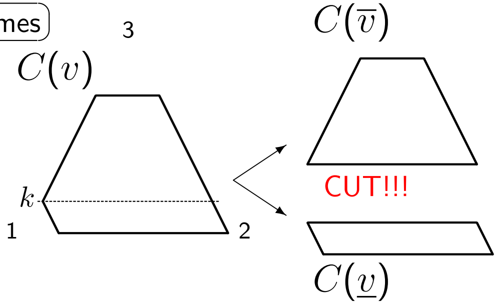

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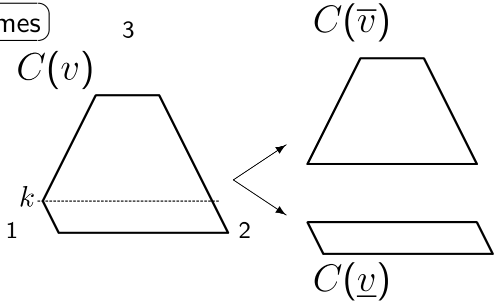

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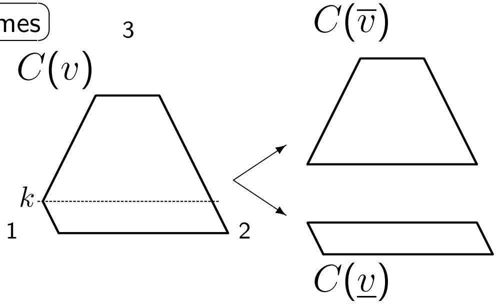

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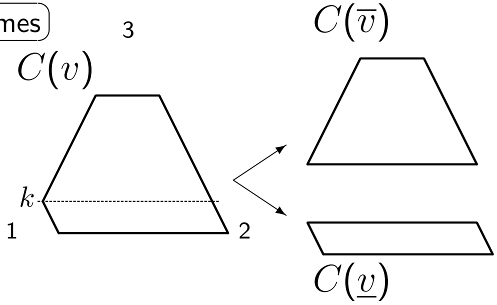

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Let v be a balanced game

Let v' and v'' be two balanced games such that belong to some dissection of v

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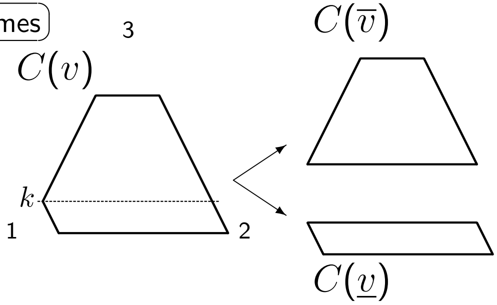

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φ satisfies fair additivity with respect to the core if:

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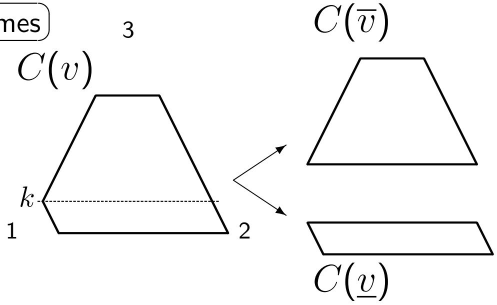

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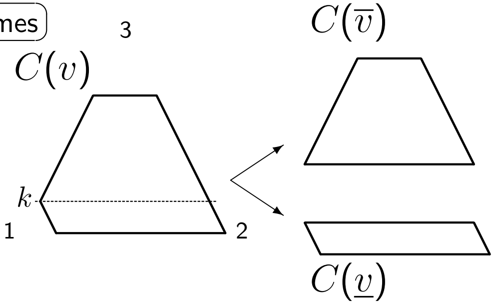

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Let v' and v'' be two balanced games such that belong to some dissection of v

φ satisfies fair additivity with respect to the core if:

1. φ is a $\mathcal{RT}$-solution
2. $C(v') = C(v'')$ implies that $\alpha_v(v') = \alpha_v(v'')$

Table of Properties

	Shapley	Nucleolus	Core-Center

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	Shapley	Nucleolus	Core-Center
Efficiency			
Individual Rationality			
Continuity			
Dummy Player			
Symmetry			
Translation and Scale Invariance			

Table of Properties

	Shapley	Nucleolus	Core-Center
Efficiency	✓	✓	✓
Individual Rationality	✓	✓	✓
Continuity	✓	✓	✓
Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓

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	Shapley	Nucleolus	Core-Center
Efficiency	✓	✓	✓
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Continuity	✓	✓	✓
Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability			

Table of Properties

	Shapley	Nucleolus	Core-Center
Efficiency	✓	✓	✓
Individual Rationality	✓	✓	✓
Continuity	✓	✓	✓
Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓

Table of Properties

	Shapley	Nucleolus	Core-Center
Efficiency	✓	✓	✓
Individual Rationality	✓	✓	✓
Continuity	✓	✓	✓
Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓
Strong Monotonicity			
Coalitional Monotonicity			
Aggregate Monotonicity			
Weak Coalitional Monotonicity			

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Continuity	✓	✓	✓
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Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓
Strong Monotonicity	✓	X	X
Coalitional Monotonicity	✓	X	X
Aggregate Monotonicity	✓	X	X
Weak Coalitional Monotonicity	✓	✓	✓

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Continuity	✓	✓	✓
Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓
Strong Monotonicity	✓	X	X
Coalitional Monotonicity	✓	X	X
Aggregate Monotonicity	✓	X	X
Weak Coalitional Monotonicity	✓	✓	✓
Additivity			

Table of Properties

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Continuity	✓	✓	✓
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Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓
Strong Monotonicity	✓	X	X
Coalitional Monotonicity	✓	X	X
Aggregate Monotonicity	✓	X	X
Weak Coalitional Monotonicity	✓	✓	✓
Additivity	✓	X	X

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Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓
Strong Monotonicity	✓	X	X
Coalitional Monotonicity	✓	X	X
Aggregate Monotonicity	✓	X	X
Weak Coalitional Monotonicity	✓	✓	✓
Additivity	✓	X	X
Consistency (Davis/Maschler)			
Consistency (Hart/Mas-Collel)			

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Individual Rationality	✓	✓	✓
Continuity	✓	✓	✓
Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓
Strong Monotonicity	✓	X	X
Coalitional Monotonicity	✓	X	X
Aggregate Monotonicity	✓	X	X
Weak Coalitional Monotonicity	✓	✓	✓
Additivity	✓	X	X
Consistency (Davis/Maschler)	X	✓	X
Consistency (Hart/Mas-Collel)			

Table of Properties

	Shapley	Nucleolus	Core-Center
Efficiency	✓	✓	✓
Individual Rationality	✓	✓	✓
Continuity	✓	✓	✓
Dummy Player	✓	✓	✓
Symmetry	✓	✓	✓
Translation and Scale Invariance	✓	✓	✓
Stability	X	✓	✓
Strong Monotonicity	✓	X	X
Coalitional Monotonicity	✓	X	X
Aggregate Monotonicity	✓	X	X
Weak Coalitional Monotonicity	✓	✓	✓
Additivity	✓	X	X
Consistency (Davis/Maschler)	X	✓	X
Consistency (Hart/Mas-Collel)	✓	X	X

Table of Properties

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Fair Additivity w.r.t. the core			

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Additivity	✓	X	X
Consistency (Davis/Maschler)	X	✓	X
Consistency (Hart/Mas-Collel)	✓	X	X
Fair Additivity w.r.t. the core	X	X	✓

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Additivity	✓	X	X
Consistency (Davis/Maschler)	X	✓	X
Consistency (Hart/Mas-Collel)	✓	X	X
Fair Additivity w.r.t. the core	X	X	✓

The Characterization

The Characterization

$\boxed{\textbf{Fair Additivity} \iff \textbf{Volume}}$

The Characterization

Fair Additivity $\Longleftrightarrow$ Volume

Theorem

The Characterization

Fair Additivity $\Longleftrightarrow$ Volume

Theorem

Let φ be an allocation rule satisfying

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Fair Additivity $\Longleftrightarrow$ Volume

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The Characterization

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The Characterization

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The Characterization

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The Characterization

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Let φ be an allocation rule satisfying

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- *Weak Symmetry*
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- *Fair Additivity with respect to the core*

The Characterization

Fair Additivity $\Longleftrightarrow$ Volume

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Then, for each $v \in BG$,

The Characterization

Fair Additivity $\Longleftrightarrow$ Volume

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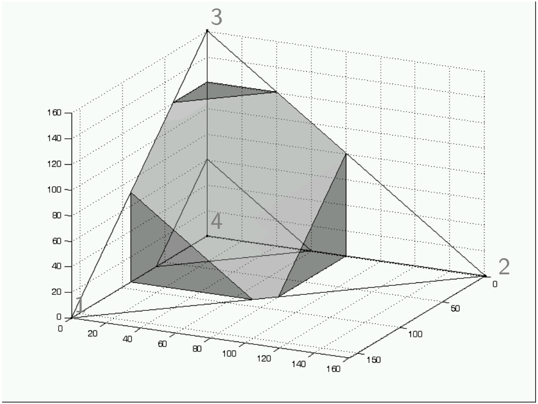
The axioms are independent

The Core-Center and the Shapley Value

Convex Games

The Core-Center and the Shapley Value

Convex Games



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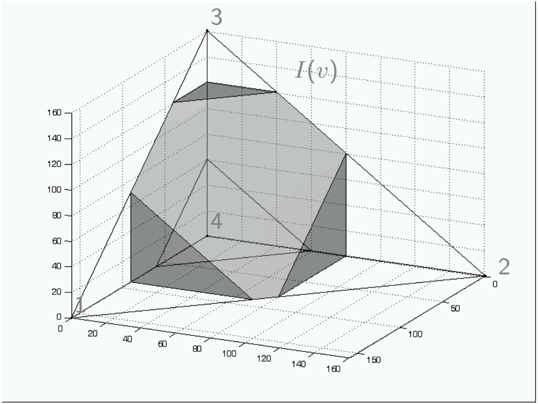

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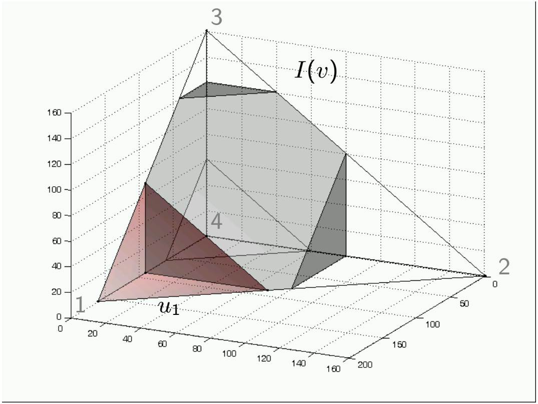

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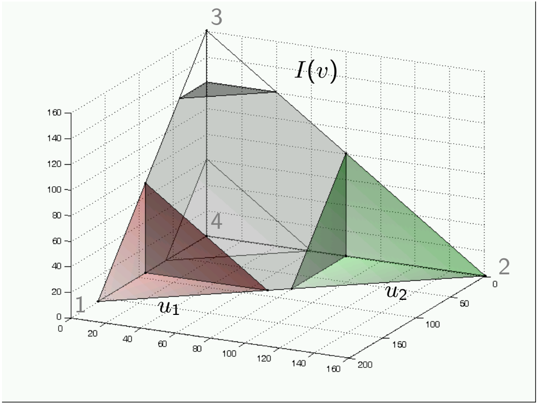

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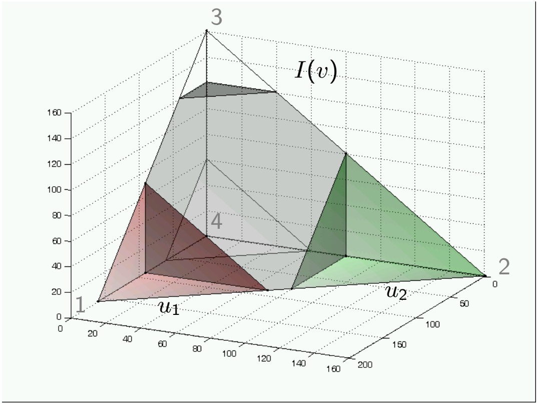

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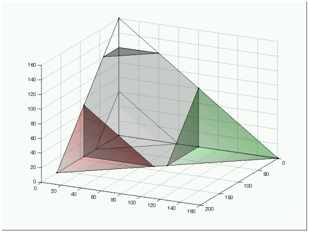

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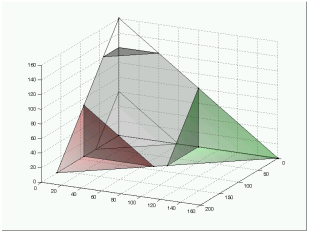
- We define games u_1, u_2, u_3, u_4



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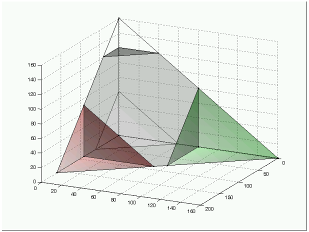

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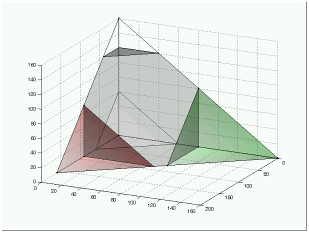

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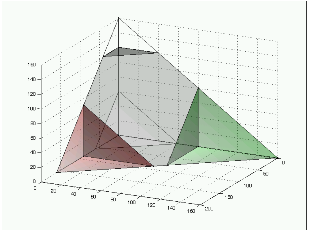

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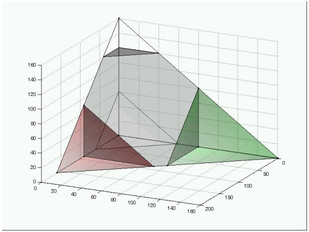

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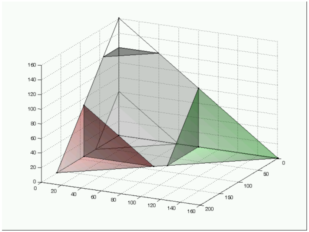

$\mu(N, v)$

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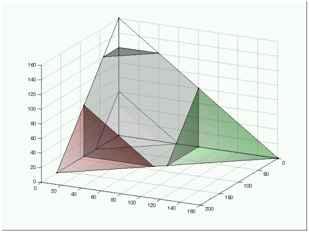

$$\mu(N,v) \overset{\text{Fair Add.}}{=}$$

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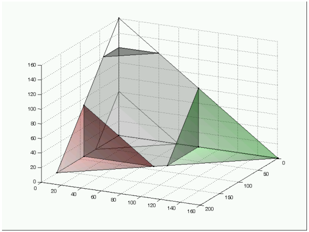

$$\mu(N,v) \overset{\text{Fair Add.}}{=} \mu(N,u_0) - \sum_{i\in N} \mu(N,u_i)$$

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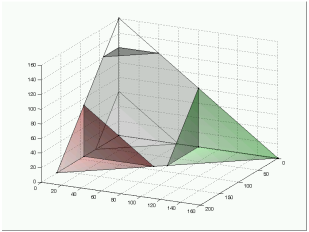

$$\mu(N,v) \overset{\text{Fair Add.}}{=} \frac{w_0}{w}\,\mu(N,u_0) - \sum_{i\in N} \frac{w_i}{w}\,\mu(N,u_i)$$

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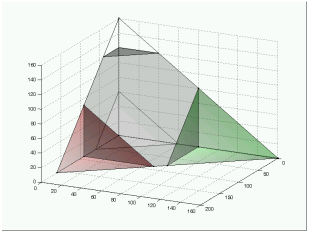

$$\mu(N,v) \overset{\text{Fair Add.}}{=} \frac{w_0}{w}\,\mu(N,u_0) - \sum_{i\in N}\frac{w_i}{w}\,\mu(N,u_i)$$

$$= \frac{w_0}{w}\,\text{Sh}(N,u_0) - \sum_{i\in N}\frac{w_i}{w}\,\text{Sh}(N,u_i)$$

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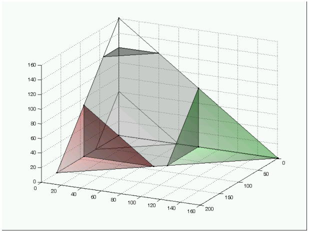

$$\begin{aligned} \mu(N,v) &\overset{\text{Fair Add.}}{=} \frac{w_0}{w}\,\mu(N,u_0) - \sum_{i\in N} \frac{w_i}{w}\,\mu(N,u_i) \\ &= \frac{w_0}{w}\,\mathrm{Sh}(N,u_0) - \sum_{i\in N} \frac{w_i}{w}\,\mathrm{Sh}(N,u_i) \\ &\overset{\text{Shap Add.}}{=} \end{aligned}$$

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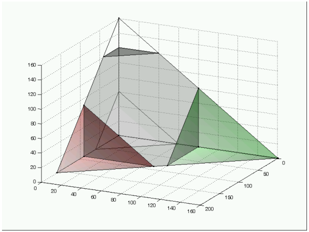

$$\begin{aligned}\mu(N,v) &\overset{\text{Fair Add.}}{=} \frac{w_0}{w}\mu(N,u_0) - \sum_{i\in N}\frac{w_i}{w}\mu(N,u_i)\\ &= \frac{w_0}{w}\mathrm{Sh}(N,u_0) - \sum_{i\in N}\frac{w_i}{w}\mathrm{Sh}(N,u_i)\\ &\overset{\text{Shap Add.}}{=} \mathrm{Sh}(N,v^*)\end{aligned}$$

The Core-Center and the Shapley Value

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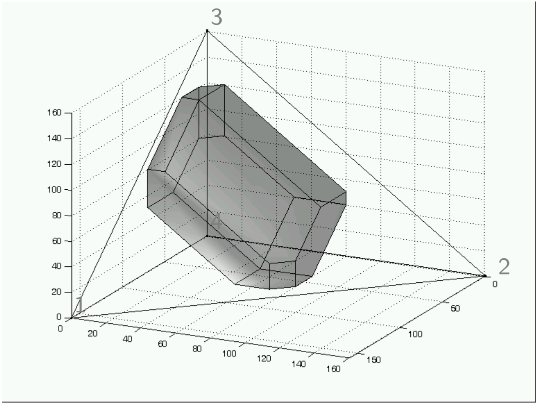

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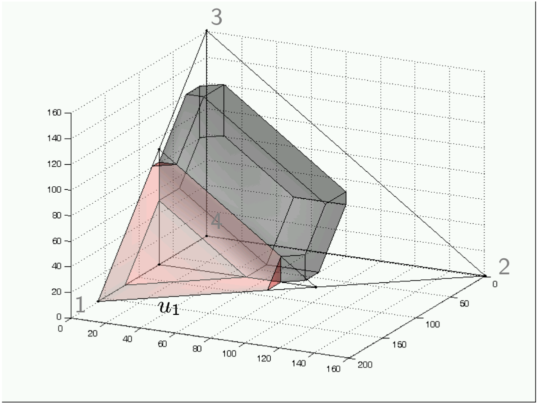

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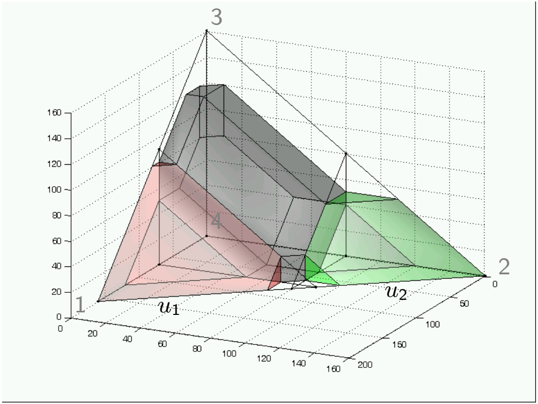

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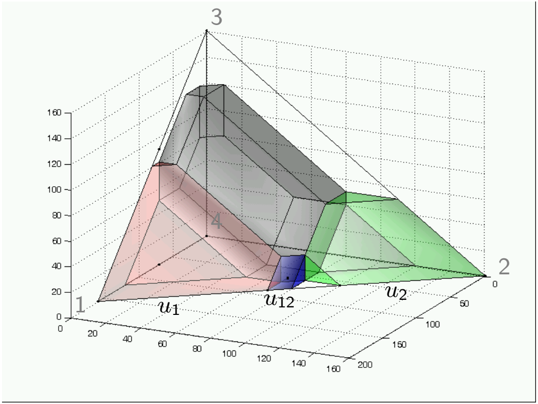

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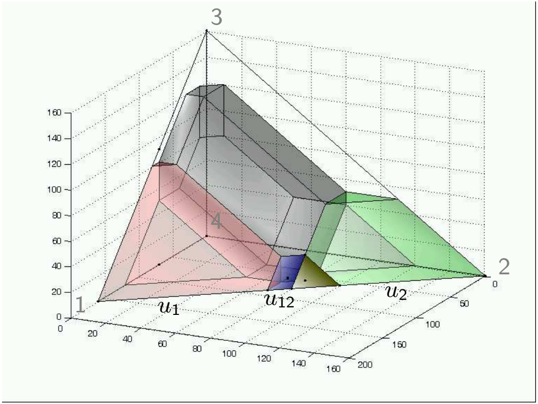

Cooperative Game Theory

Conclusions

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Cooperative Game Theory
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- We have shown a geometric characterization of the τ value

Cooperative Game Theory

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Cooperative Game Theory
Future Research

Future Research

Cooperative Game Theory
Future Research

Future Research

- Try to find different characterizations of the core-center

Cooperative Game Theory

Future Research

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- Try to find different characterizations of the core-center
- Deepen in the relation between the core-center and the Shapley value

Cooperative Game Theory
Future Research

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- Try to find different characterizations of the core-center
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- Look for noncooperative foundations for the core-center (implementation)

Cooperative Game Theory
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Cooperative Game Theory
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- Look for extensions to different classes of games

Cooperative Game Theory

References

References

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Universidade de Santiago de Compostela

Thesis Dissertation
June 29th, 2005

Part I: Noncooperative Game Theory

Part II: Cooperative Game Theory