Rankings and Tournaments: A new approach

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(joint with Miguel Brozos-Vázquez, Marco Antonio Campo-Cabana, and José Carlos Díaz-Ramos)

Outline

1. Motivation
2. The model
3. Ranking Methods
4. Our contribution

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- Ranking candidates in labor markets

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- Characterized axiomatically by Palacios-Huerta and Volij (2004)

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- Only makes sense for Round-Robin tournaments (because of IIA).

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- Anonymity
- Quasi-flatness preservation
- Negative responsiveness to losses

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- Violates the axiom:

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- Asymmetric treatment of victories with respect to losses (because of negative responsiveness to losses)
- Violates the axiom:

  The ranking proposed for A is the inverse of the one proposed for A^t

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- Bradley and Terry (1952) took the (standard) logistic distribution: $F(r_i - r_j) = \frac{e^{r_i}}{e^{r_i} + e^{r_j}}$ $\quad (F(0) = 1/2,\ \lim_{d \to +\infty} F(d) = 1,\ \lim_{d \to -\infty} F(d) = 0)$

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- Assume that there is a distribution function F such that the expected score of a player with strength r_i in a match against a player with strength r_j is given by $F(r_i - r_j)$.
- The function F is called rating function
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- In chess tournaments also a logistic distribution has proved to fit the observed data quite well

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- What about non-asymptotic behavior?

Outline

Idea

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Fair-Bets: The iterative method (linear system) to use all the information of the tournament

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(lack of robustness on F, computation costs)

Recursive Performance

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r_i^1 = "performance of i"

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- The proposed rating is robust in F
- According to the proposed rating, the performance of each player coincides with his own rating

Round Robin

When restricting attention to round robin tournaments we get:

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$$\text{Scores} = \text{Maximum Likelihood} = \text{Recursive Performance} \neq \text{Fair Bets}$$

Some numeric examples

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$$
\begin{array}{c}
A_1 \\
\begin{pmatrix}
0 & 1 & 0.9 & 0.9 \\
1 & 0 & 0.9 & 0 \\
0.1 & 0.1 & 0 & 0 \\
0.1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
\qquad
\begin{array}{rrrr}
\overset{(av.scores)}{s} & \text{RP} & \text{ML} & \text{FB} \\
0.7 & 1.010 & 1.099 & 0.703 \\
0.633 & 1.037 & 1.099 & 0.703 \\
0.1 & -1.017 & -1.099 & 0.078 \\
0.1 & -1.031 & -1.099 & 0.078
\end{array}
$$

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$$A_1 = \begin{pmatrix} 0 & 1 & 0.9 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}$$

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0.633	1.037	1.099	0.703
0.1	−1.017	−1.099	0.078
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$$A_2 = \begin{pmatrix} 0 & 1 & 90 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}$$

$\overset{(av.scores)}{s}$	RP	ML	FB
0.892	1.127	1.099	0.703
0.633	0.971	1.099	0.703
0.1	−1.049	−1.099	0.078
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$$A_3 = \begin{pmatrix} 0 & 0.9 & 90 & 0.9 \\ 1.1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}$$

$\overset{(av.scores)}{s}$	RP	ML	FB
0.891	1.087	1.055	0.637
0.667	1.082	1.227	0.764
0.1	−1.085	−1.140	0.071
0.1	−1.084	−1.142	0.070

Directions for future research

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- Axiomatic analysis of the recursive performance ranking method

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- Develop comparative studies of the different ranking methods in applied settings

Rankings and Tournaments: A new approach

Julio González-Díaz

Kellogg School of Management (CMS-EMS)
Northwestern University
and
Research Group in Economic Analysis
Universidad de Vigo
....

(joint with Miguel Brozos-Vázquez, Marco Antonio Campo-Cabana, and José Carlos Díaz-Ramos)

Some tournaments

Return

Some tournaments

$$
\begin{array}{cccccc}
A_2 & \overset{(av.scores)}{s} & ++ & ++ & ++ \\
\begin{pmatrix} 0 & 1 & 90 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} &
\begin{array}{r} 0.892 \\ 0.633 \\ 0.1 \\ 0.1 \end{array} &
\begin{array}{c} ---\\ ---\\ ---\\ --- \end{array} &
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\begin{array}{c} ---\\ ---\\ ---\\ --- \end{array}
\end{array}
$$

Return

Some tournaments

$$\begin{array}{cccccc} A_1 & \overset{(av.scores)}{s} & ++ & ++ & ++ \\ \left(\begin{matrix} 0 & 1 & 0.9 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{matrix}\right) & \begin{matrix} 0.7 \\ 0.633 \\ 0.1 \\ 0.1 \end{matrix} & \begin{matrix} - -- \\ - -- \\ - -- \\ - -- \end{matrix} & \begin{matrix} - -- \\ - -- \\ - -- \\ - -- \end{matrix} & \begin{matrix} - -- \\ - -- \\ - -- \\ - -- \end{matrix} \end{array}$$

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A_3 & \overset{(av.scores)}{s} & ++ & ++ & ++ \\
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\begin{array}{c} - -- \\ - -- \\ - -- \\ - -- \end{array} &
\begin{array}{c} - -- \\ - -- \\ - -- \\ - -- \end{array} &
\begin{array}{c} - -- \\ - -- \\ - -- \\ - -- \end{array}
\end{array}
$$

Return

Some tournaments

$$\begin{array}{cccc}
 & \overset{(av.scores)}{s} & & \\
\overset{A_1}{\begin{pmatrix} 0 & 1 & 0.9 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}} & \begin{array}{r} 0.7 \\ 0.633 \\ 0.1 \\ 0.1 \end{array} & \begin{array}{ccc} ++ & ++ & ++ \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \end{array}
\end{array}$$

$$\begin{array}{cccc}
 & \overset{(av.scores)}{s} & & \\
\overset{A_2}{\begin{pmatrix} 0 & 1 & 90 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}} & \begin{array}{r} 0.892 \\ 0.633 \\ 0.1 \\ 0.1 \end{array} & \begin{array}{ccc} ++ & ++ & ++ \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \end{array}
\end{array}$$

$$\begin{array}{cccc}
 & \overset{(av.scores)}{s} & & \\
\overset{A_3}{\begin{pmatrix} 0 & 0.9 & 90 & 0.9 \\ 1.1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}} & \begin{array}{r} 0.891 \\ 0.667 \\ 0.1 \\ 0.1 \end{array} & \begin{array}{ccc} ++ & ++ & ++ \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \\ -\,-- & -\,-- & -\,-- \end{array}
\end{array}$$

Some tournaments

$$\begin{array}{c} A_1 \\ \begin{pmatrix} 0 & 1 & 0.9 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{rrrr} \overset{(av.scores)}{s} & ++ & ++ & ++ \\ 0.7 & 1.010 & 1.099 & 0.703 \\ 0.633 & 1.037 & 1.099 & 0.703 \\ 0.1 & -1.017 & -1.099 & 0.078 \\ 0.1 & -1.031 & -1.099 & 0.078 \end{array}$$

$$\begin{array}{c} A_2 \\ \begin{pmatrix} 0 & 1 & 90 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{rrrr} \overset{(av.scores)}{s} & ++ & ++ & ++ \\ 0.892 & 1.127 & 1.099 & 0.703 \\ 0.633 & 0.971 & 1.099 & 0.703 \\ 0.1 & -1.049 & -1.099 & 0.078 \\ 0.1 & -1.048 & -1.099 & 0.078 \end{array}$$

$$\begin{array}{c} A_3 \\ \begin{pmatrix} 0 & 0.9 & 90 & 0.9 \\ 1.1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{rrrr} \overset{(av.scores)}{s} & ++ & ++ & ++ \\ 0.891 & 1.087 & 1.055 & 0.637 \\ 0.667 & 1.082 & 1.227 & 0.764 \\ 0.1 & -1.085 & -1.140 & 0.071 \\ 0.1 & -1.084 & -1.142 & 0.070 \end{array}$$

Return

Some tournaments

$$\begin{array}{ccccc} A_1 & \overset{(av.scores)}{s} & ++ & ++ & ++ \\ \begin{pmatrix} 0 & 1 & 0.9 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{r} 0.7 \\ 0.633 \\ 0.1 \\ 0.1 \end{array} & \begin{array}{r} 1.010 \\ 1.037 \\ -1.017 \\ -1.031 \end{array} & \begin{array}{r} 1.099 \\ 1.099 \\ -1.099 \\ -1.099 \end{array} & \begin{array}{r} 0.703 \\ 0.703 \\ 0.078 \\ 0.078 \end{array} \end{array}$$

$$\begin{array}{ccccc} A_2 & \overset{(av.scores)}{s} & ++ & ++ & ++ \\ \begin{pmatrix} 0 & 1 & 90 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{r} 0.892 \\ 0.633 \\ 0.1 \\ 0.1 \end{array} & \begin{array}{r} 1.127 \\ 0.971 \\ -1.049 \\ -1.048 \end{array} & \begin{array}{r} 1.099 \\ 1.099 \\ -1.099 \\ -1.099 \end{array} & \begin{array}{r} 0.703 \\ 0.703 \\ 0.078 \\ 0.078 \end{array} \end{array}$$

$$\begin{array}{ccccc} A_3 & \overset{(av.scores)}{s} & ++ & ++ & ++ \\ \begin{pmatrix} 0 & 0.9 & 90 & 0.9 \\ 1.1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{r} 0.891 \\ 0.667 \\ 0.1 \\ 0.1 \end{array} & \begin{array}{r} 1.087 \\ 1.082 \\ -1.085 \\ -1.084 \end{array} & \begin{array}{r} 1.055 \\ 1.227 \\ -1.140 \\ -1.142 \end{array} & \begin{array}{r} 0.637 \\ 0.764 \\ 0.071 \\ 0.070 \end{array} \end{array}$$

Some tournaments

$$A_1 = \begin{pmatrix} 0 & 1 & 0.9 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}$$

$\overset{(av.scores)}{s}$	RP	ML	FB
0.7	1.010	1.099	0.703
0.633	1.037	1.099	0.703
0.1	– 1.017	– 1.099	0.078
0.1	– 1.031	– 1.099	0.078

$$A_2 = \begin{pmatrix} 0 & 1 & 90 & 0.9 \\ 1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}$$

$\overset{(av.scores)}{s}$	RP	ML	FB
0.892	1.127	1.099	0.703
0.633	0.971	1.099	0.703
0.1	– 1.049	– 1.099	0.078
0.1	– 1.048	– 1.099	0.078

$$A_3 = \begin{pmatrix} 0 & 0.9 & 90 & 0.9 \\ 1.1 & 0 & 0.9 & 0 \\ 10 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}$$

$\overset{(av.scores)}{s}$	RP	ML	FB
0.891	1.087	1.055	0.637
0.667	1.082	1.227	0.764
0.1	– 1.085	– 1.140	0.071
0.1	– 1.084	– 1.142	0.070

Return