Symmetry and Orthogonality in TU-Games

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Allocation rules and TU games

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- In particular, we develop the notions of symmetry and orthogonality in TU-games

Outline

1. Symmetry

2. Orthogonality

Preliminaries

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- $\Delta_i(v, S)$ denotes the contribution of i to S in game v
- m^π denotes the vector of contributions of the players given ordering π

Preliminaries

Classic allocation rules and properties

Prenucleolus	
Nucleolus	
Shapley value	
τ value	
Core-center	
Eq. division	
Eq. surplus div.	

Preliminaries

Classic allocation rules and properties

	EFF
Prenucleolus	✓
Nucleolus	✓
Shapley value	✓
τ value	✓
Core-center	✓
Eq. division	✓
Eq. surplus div.	✓

Preliminaries

Classic allocation rules and properties

	EFF	IR
Prenucleolus	✓	✓*
Nucleolus	✓	✓
Shapley value	✓	✓*
τ value	✓	✓
Core-center	✓	✓
Eq. division	✓	X
Eq. surplus div.	✓	✓*

* This property holds if we restrict attention to zero-monotonic games; which ensures that the imputations set is nonempty and, further, that the prenucleolus and the nucleolus coincide.

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2. We present a new notion of symmetry: **inverse symmetry**
3. We make some concluding considerations regarding symmetry and dummy players

A first game

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Game v

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- $N = \{1, 2, 3\}$

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- Two players i and j are **symmetric** if, for each $S \subseteq N\backslash\{i,j\}$,
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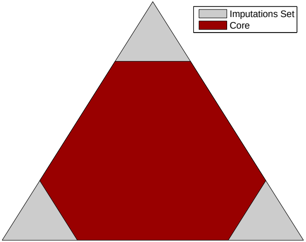

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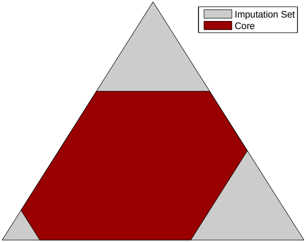
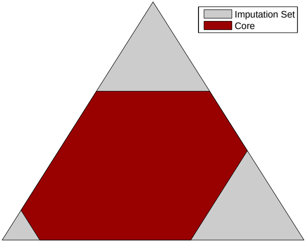
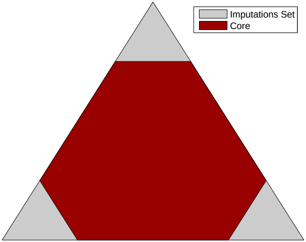


Classic Symmetry

two players are symmetric	all players symmetric
	

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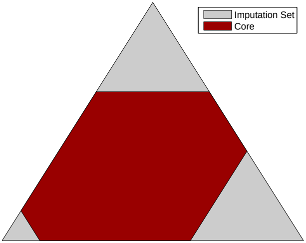	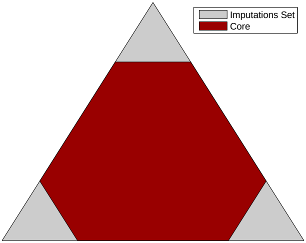

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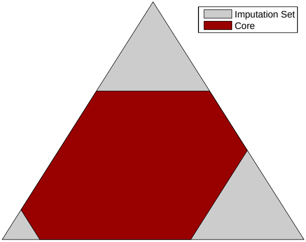	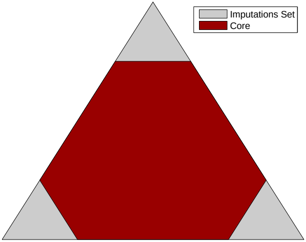

- Symmetries with respect to hyperplanes (mirror symmetries)
- If i and j are symmetric, the **Weber set**, the **core**, and the **core cover** are symmetric with respect to the hyperplane $H^{ij} := \{x \in \mathbb{R}^n : x_i = x_j\}$

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An allocation rule...

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- is **symmetric** if symmetric players get the same

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Lemma

Shapley value, prenucleolus, nucleolus, core-center, tau value, equal division, and equal surplus division coincide for **symmetric games**

A second game

A second game

Game w

- $N = \{1, 2, 3\}$
- $w(1) = 4, w(2) = 2, w(3) = 1$
- $w(12) = 9, w(13) = 8, w(23) = 6$
- $w(N) = 19$

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$\boxed{\textbf{Why???} \quad w = v + (4,2,1)}$

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Translation invariance preserves geometric symmetries:

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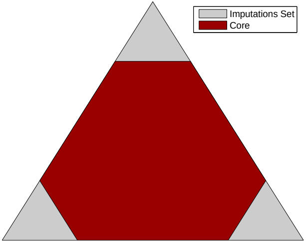

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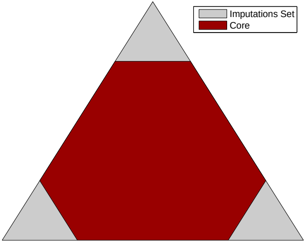

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Translation invariance preserves geometric symmetries: Since most allocation rules satisfy translation invariance, they also coincide for games that are translations of symmetric games

Classic Symmetry

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Are there other classes of games where many solutions coincide?

2-games and PS-games

2-games and PS-games

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Proposition (Nouweland *et. al 1996)*

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v is a **PS-game** if, for each $i \in N$, there is $c_i \in \mathbb{R}$ such that, for each $S \subset N \backslash \{i\}$, $\Delta_i(v, S) + \Delta_i(v, N \backslash (S \cup \{i\})) = c_i$

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Lemma (Kar *et. al 2009)*

Every 2-game is a PS-game

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Lemma (Kar *et. al 2009)*

Every 2-game is a PS-game

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Shapley value and prenucleolus coincide for PS-games

2-games and PS-games

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v is a **2-game** if, for each $S \subseteq N$, $v(S) = \sum_{T \subset S, |T|=2} v(T)$

Proposition (Nouweland *et. al 1996)*

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What drives these results?

A third game

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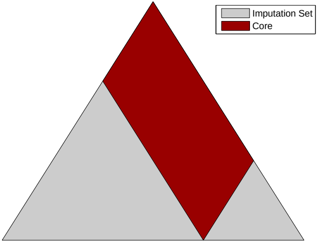

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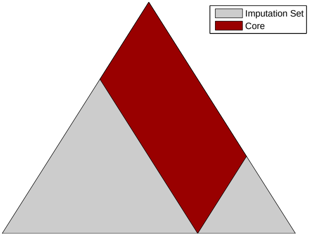

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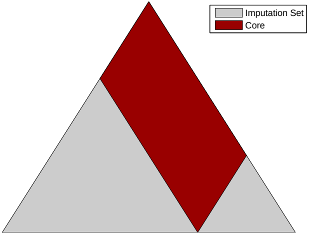

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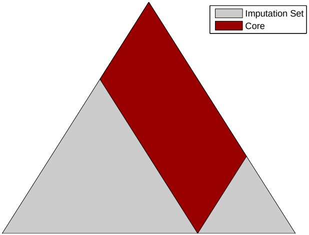

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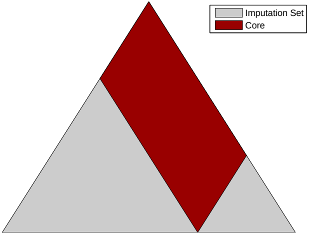

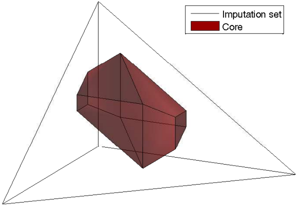

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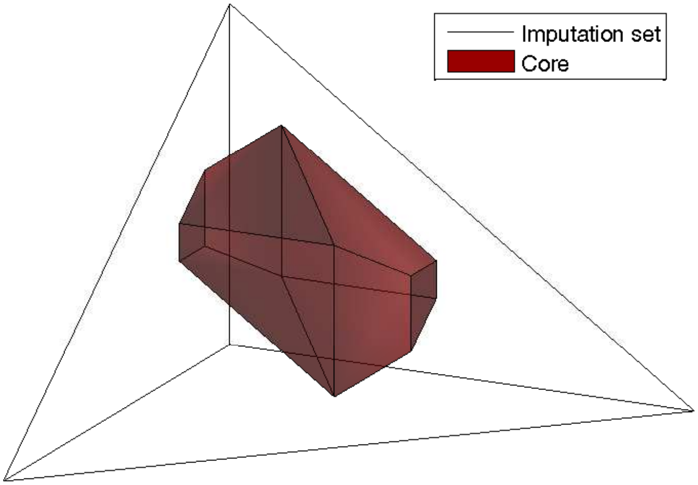

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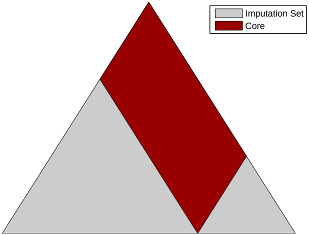

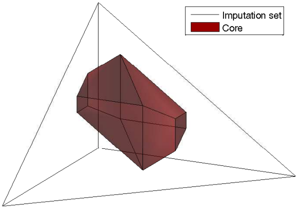

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Prenucleolus	✓	✓	✓
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Shapley Value	✓	✓	✓
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A fourth game

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Game w

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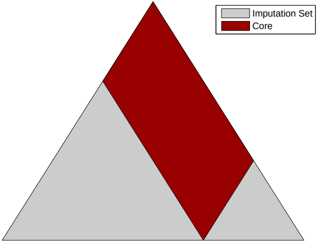

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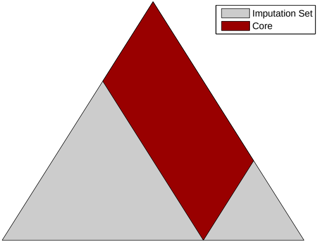

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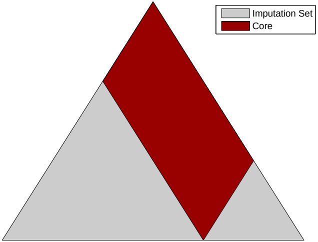

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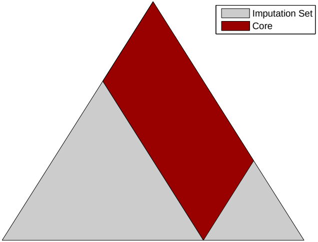

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Proposition (Nouweland *et. al 1996*). Shapley value, prenucleolus, and τ-value coincide for 2-games

Lemma (Kar *et. al 2009*). Every 2-game is a PS-game

Proposition (Kar *et. al 2009*). Shapley value and prenucleolus coincide for PS-games

Lemma

A game is weakly inverse symmetric if and only if it is a PS-game

Corollary. There is a unique allocation rule satisfying **efficiency**, **translation invariance**, and **inverse symmetry** on the class of **PS-games**

Corollary. Shapley value, prenucleolus, nucleolus, core-center, tau value, ~~equal division, and equal surplus division~~ coincide for **PS-games**

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Can we go beyond with these symmetry ideas?

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Example

- $v(1) = 3.5,\ v(2) = 1.5,\ v(3) = 0$
- $v(12) = 6,\ v(13) = 7.5,\ v(23) = 9.5$
- $v(N) = 15$

	Player			
Order π	**1**	**2**	**3**	Efficiency
123	3.5	2.5	9	15
132	3.5	7.5	4	15
213	4.5	1.5	9	15
231	5.5	1.5	8	15
312	7.5	7.5	0	15
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An allocation rule satisfies **squareness** if, given a square game, all the players get the same

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Corollary

Shapley value, prenucleolus, nucleolus, tau value, core-center, ~~*equal division, and equal surplus division*~~ *coincide for* **dummy-symmetric games**

Additivity and Orthogonality

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Additivity: $\varphi(v+w) = \varphi(v) + \varphi(w)$

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Translation Invariance (+DP):
If w is an additive game, $\varphi(v + w) = \varphi(v) + \varphi(w)$

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$\vdots$ Is there anything in between?

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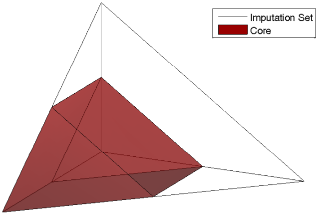

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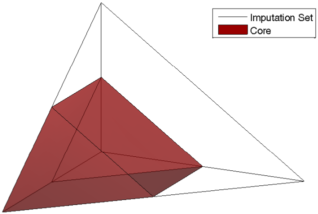

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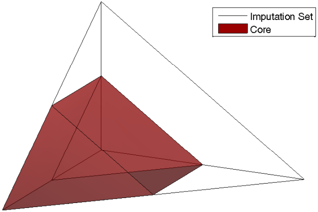

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Orthogonality: $``C(u_{123}) \perp C(u_{14})"$

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Orthogonality $\iff$ Decomposability

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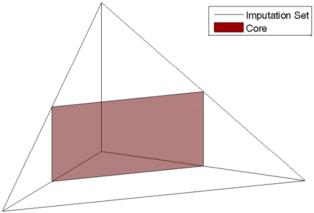
Orthogonality

The games u_{12} and u_{34} are orthogonal



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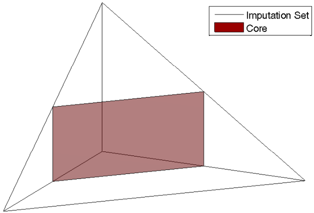

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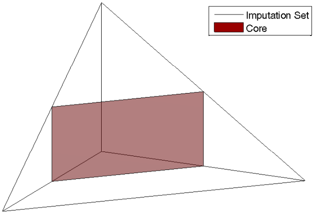

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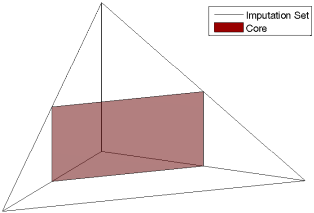

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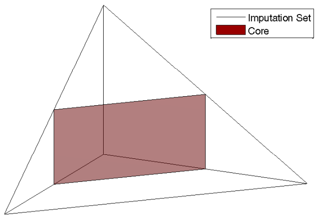
- If a game is sum of orthogonal games, its core is not full dimensional:

Orthogonality

The games u_{12} and u_{34} are orthogonal



- The core of $u_{12} + u_{34}$ can be seen as the Cartesian product of the cores of u_{12} and u_{34}
- For several allocation rules, if v and w are orthogonal, $\varphi(v + w) = \varphi(v) + \varphi(w)$
- If a game is sum of orthogonal games, its core is not full dimensional: **The tent???**

Weak Orthogonality

We can go beyond orthogonality and additivity of orthogonal games

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v and w are...

- **orthogonal** if $|D(v) \cup D(w)| = N$
- **weakly orthogonal** if $|D(v) \cup D(w)| \geq N - 1$

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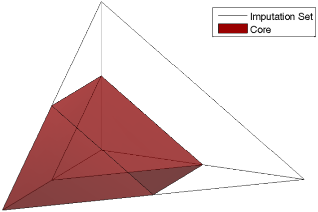

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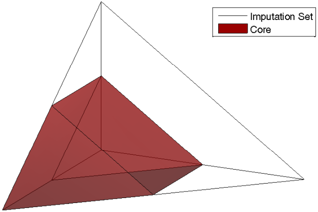

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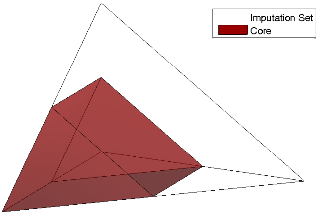

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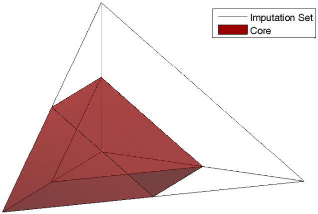

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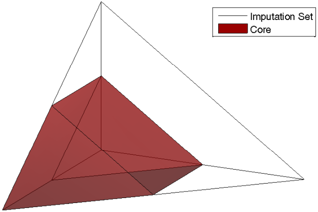

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- Shapley, prenucleolus, nucleolus, core-center, ~~and tau value~~ coincide for $u_{123} + u_{14}$
- $\mathsf{Sh}(u_{123} + u_{14}) = \mathsf{Sh}(u_{123}) + Sh(u_{14}) = (0.8333, 0.3333, 0.3333, 0.5)$
- $\tau(u_{123} + u_{14}) = (0.8, 0.4, 0.4, 0.4)$

Orthogonal Additivity

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Allocation rule
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nucleolus
Shapley value
tau value
core-center
equal division
equal surplus division

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If v and w are weakly orthogonal, $\varphi(v+w)=\varphi(v)+\varphi(w)$

Allocation rule	**Translation Invariance**
prenucleolus	$\checkmark$
nucleolus	$\checkmark$
Shapley value	$\checkmark$
tau value	$\checkmark$
core-center	$\checkmark$
equal division	X
equal surplus division	$\checkmark$

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nucleolus	✓	X
Shapley value	✓	✓
tau value	✓	X
core-center	✓	X
equal division	X	✓
equal surplus division	✓	✓

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nucleolus	$\checkmark$	$\checkmark^*$	X
Shapley value	$\checkmark$	$\checkmark$	$\checkmark$
tau value	$\checkmark$	X	X
core-center	$\checkmark$	$\checkmark$	X
equal division	X	$\checkmark$	$\checkmark$
equal surplus division	$\checkmark$	$\checkmark$	$\checkmark$

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tau value	✓	X	X
core-center	✓	✓	X
equal division	X	✓	✓
equal surplus division	✓	✓	✓

* This property holds if one of the two games has a nonempty core.

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core-center	$\checkmark$	$\checkmark$	X
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Corollary. If a game is sum of weakly orthogonal dummy-symmetric games, then Shapley, prenucleolus, nucleolus, and core-center coincide

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. . . and so does any other solution satisfying **efficiency**, **dummy player**, **symmetry**, and **orthogonal additivity**

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- This analysis allowed to identify classes of games in which different allocation rules coincide
- We have also obtained axiomatic characterizations for these solutions concepts in these classes of games

Symmetry and Orthogonality in TU-Games

Julio González-Díaz[1] Estela Sánchez-Rodríguez[2]

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University of Santiago de Compostela

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