Finitely Repeated Games: A Generalized Nash Folk Theorem

Julio González-Díaz

Department of Statistics and Operations Research
Faculty of Mathematics
Universidade de Santiago de Compostela

Outline

Outline

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

- $N = \{1, \ldots, n\}$ is the set of players

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

- $N = \{1, \ldots, n\}$ is the set of players
- $A = \prod_{i=1}^{n} A_i$, where A_i denotes the set of actions for player i

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

- $N = \{1, \dots, n\}$ is the set of players
- $A = \prod_{i=1}^{n} A_i$, where A_i denotes the set of actions for player i
- $\varphi = \prod_{i=1}^{n} \varphi_i$, where $\varphi_i : A \to \mathbb{R}$ is the utility function of player i

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

- $N = \{1, \ldots, n\}$ is the set of players
- $A = \prod_{i=1}^{n} A_i$, where A_i denotes the set of actions for player i
- $\varphi = \prod_{i=1}^{n} \varphi_i$, where $\varphi_i : A \to \mathbb{R}$ is the utility function of player i

Minmax Payoffs:

$$v_i = \min_{a_{-i} \in A-i} \max_{a_i \in A_i} \varphi_i(a_i, a_{-i})$$

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

- $N = \{1, \ldots, n\}$ is the set of players
- $A = \prod_{i=1}^{n} A_i$, where A_i denotes the set of actions for player i
- $\varphi = \prod_{i=1}^{n} \varphi_i$, where $\varphi_i : A \to \mathbb{R}$ is the utility function of player i

Minmax Payoffs:

$$v_i = \min_{a_{-i} \in A-i} \max_{a_i \in A_i} \varphi_i(a_i, a_{-i})$$

Feasible and Individually Rational Payoffs:

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

- $N = \{1, \ldots, n\}$ is the set of players
- $A = \prod_{i=1}^{n} A_i$, where A_i denotes the set of actions for player i
- $\varphi = \prod_{i=1}^{n} \varphi_i$, where $\varphi_i : A \rightarrow \mathbb{R}$ is the utility function of player i

Minmax Payoffs:

$$v_i = \min_{a_{-i} \in A-i} \max_{a_i \in A_i} \varphi_i(a_i, a_{-i})$$

Feasible and Individually Rational Payoffs:

$$F := co\{\varphi(a) \,:\, a \in \varphi(A)\}$$

The Stage Game

A **game** is a triple $G =< N, A, \varphi >$ where:

- $N = \{1, \dots, n\}$ is the set of players
- $A = \prod_{i=1}^{n} A_i$, where A_i denotes the set of actions for player i
- $\varphi = \prod_{i=1}^{n} \varphi_i$, where $\varphi_i : A \to \mathbb{R}$ is the utility function of player i

Minmax Payoffs:

$$v_i = \min_{a_{-i} \in A-i} \max_{a_i \in A_i} \varphi_i(a_i, a_{-i})$$

Feasible and Individually Rational Payoffs:

$$F := co\{\varphi(a) \ : \ a \in \varphi(A)\} \qquad \bar{F} := F \cap \{u \in \mathbb{R}^n : u \geq v\}$$

The Repeated Game

The Repeated Game

- $G(\delta, T)$ denotes the T-fold repetition of the game G with discount parameter δ

The Repeated Game

- $G(\delta, T)$ denotes the T-fold repetition of the game G with discount parameter δ
- Discounted payoffs in the repeated game,

$$\varphi_\delta^T(\sigma) = \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} \varphi_i(a^t)$$

General Considerations

Our framework:

General Considerations

Our framework:

- The sets of actions are compact

General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions

General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions
- Finite Horizon

General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions
- Finite Horizon
- Nash Equilibrium

General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions
- Finite Horizon
- Nash Equilibrium
- Complete Information

General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions
- Finite Horizon
- Nash Equilibrium
- Complete Information
- Perfect Monitoring (Observable mixed actions)

General Considerations

Our framework:

- The sets of actions are compact
- Continuous payoff functions
- Finite Horizon
- Nash Equilibrium
- Complete Information
- Perfect Monitoring (Observable mixed actions)
- Public Randomization (Without loss of generality)

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon		
Finite Horizon		

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon	The "Folk Theorem" (1970s)	
Finite Horizon		

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon	The "Folk Theorem" (1970s)	Fudenberg and Maskin (1986) Abreu et al. (1994) Wen (1994)
Finite Horizon		

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon	The "Folk Theorem" (1970s)	Fudenberg and Maskin (1986) Abreu et al. (1994) Wen (1994)
Finite Horizon	Benoît and Krishna (1987)	

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon	The “Folk Theorem” (1970s)	Fudenberg and Maskin (1986) Abreu et al. (1994) Wen (1994)
Finite Horizon	Benoît and Krishna (1987)	Benoît and Krishna (1985) Smith (1995) Gossner (1995)

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon	The "Folk Theorem" (1970s)	Fudenberg and Maskin (1986) Abreu et al. (1994) Wen (1994)
Finite Horizon	Benoît and Krishna (1987)	Benoît and Krishna (1985) Smith (1995) Gossner (1995)

Necessary and Sufficient conditions

The State of Art

The Folk Theorems

	Nash	Subgame Perfect
Infinite Horizon	The "Folk Theorem" (1970s)	Fudenberg and Maskin (1986) Abreu et al. (1994) Wen (1994)
Finite Horizon	Benoît and Krishna (1987)	Benoît and Krishna (1985) Smith (1995) Gossner (1995)

Necessary and Sufficient conditions

(Benoit & Krishna 1987)

Assumption for the game G

Result

(Benoit & Krishna 1987)

Assumption for the game G

- **Existence of strictly rational Nash payoffs**

Result

(Benoit & Krishna 1987)

Assumption for the game G

- **Existence of strictly rational Nash payoffs**
  For each player i there is a Nash Equilibrium a^i of G such that $\varphi_i(a^i) > v_i$

Result

(Benoit & Krishna 1987)

Assumption for the game G

- **Existence of strictly rational Nash payoffs**

  For each player i there is a Nash Equilibrium a^i of G such that $\varphi_i(a^i) > v_i$

Result

(Benoit & Krishna 1987)

Assumption for the game G

- **Existence of strictly rational Nash payoffs**
  For each player i there is a Nash Equilibrium a^i of G such that $\varphi_i(a^i) > v_i$

Result

- **Every payoff in $\bar{F}$ can be approximated in equilibrium**

(Benoit & Krishna 1987)

Assumption for the game G

- **Existence of strictly rational Nash payoffs**
  For each player i there is a Nash Equilibrium a^i of G such that $\varphi_i(a^i) > v_i$

Result

- **Every payoff in $\bar{F}$ can be approximated in equilibrium**
  For each $u \in \bar{F}$ and each $\varepsilon > 0$, there are T_0 and δ_0 such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash Equilibrium σ of $G(\delta, T)$ satisfying that $\|\varphi_\delta^T(\sigma) - u\| < \varepsilon$

(Benoit & Krishna 1987)

Assumption for the game G

- **Existence of strictly rational Nash payoffs**
  For each player i there is a Nash Equilibrium a^i of G such that $\varphi_i(a^i) > v_i$

Result

- **Every payoff in $\bar{F}$ can be approximated in equilibrium**
  For each $u \in \bar{F}$ and each $\varepsilon > 0$, there are T_0 and δ_0 such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash Equilibrium σ of $G(\delta, T)$ satisfying that $\|\varphi_\delta^T(\sigma) - u\| < \varepsilon$

(Benoit & Krishna 1987)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium

(Benoit & Krishna 1987)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium
- Assume the existence of a Nash Equilibrium a of G such that

$$\text{for each } i \in N, \quad \varphi_i(a) > v_i$$

(Benoit & Krishna 1987)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium
- Assume the existence of a Nash Equilibrium a of G such that

$$\text{for each } i \in N, \quad \varphi_i(a) > v_i$$

- Equilibrium path

$$\underbrace{u, u, \ldots, u, u,}_{T-L \text{ stages}} \underbrace{\varphi(a), \ldots, \varphi(a)}_{L \text{ stages}}$$

(Benoit & Krishna 1987)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium
- Assume the existence of a Nash Equilibrium a of G such that

$$\text{for each } i \in N, \quad \varphi_i(a) > v_i$$

- Equilibrium path

$$\underbrace{u, u, \ldots, u, u,}_{T-L \text{ stages}} \underbrace{\varphi(a), \ldots, \varphi(a)}_{L \text{ stages}}$$

- Deviation of agent i

$$\underbrace{u_i, \ldots, u_i,}_{T-L+1 \text{ stages}} \; M_i \; , \underbrace{v_i, \ldots, v_i}_{L \text{ stages}}$$

Why Nash Equilibrium?

Example (A game for which the **Nash** folk theorem is needed)

	L	M	R
T	2,2	9,1	1,0
M	1,9	0,0	0,0
B	0,1	0,0	0,0

Why Nash Equilibrium?

Example (A game for which the **Nash** folk theorem is needed)

	L	M	R
T	2,2	9,1	1,0
M	1,9	0,0	0,0
B	0,1	0,0	0,0

- subgame perfection + Smith (1995)

Why Nash Equilibrium?

Example (A game for which the **Nash** folk theorem is needed)

	L	M	R
T	2,2	9,1	1,0
M	1,9	0,0	0,0
B	0,1	0,0	0,0

- subgame perfection + Smith (1995) $\longrightarrow (2,2)$

Why Nash Equilibrium?

Example (A game for which the Nash folk theorem is needed)

	L	M	R
T	2,2	9,1	1,0
M	1,9	0,0	0,0
B	0,1	0,0	0,0

- subgame perfection + Smith (1995) $\longrightarrow (2,2)$
- Nash + Benoît and Krishna (1987)

Why Nash Equilibrium?

Example (A game for which the Nash folk theorem is needed)

	L	M	R
T	2,2	9,1	1,0
M	1,9	0,0	0,0
B	0,1	0,0	0,0

- subgame perfection + Smith (1995) $\longrightarrow (2,2)$
- Nash + Benoît and Krishna (1987) $\longrightarrow (5,5)$

Outline

Minmax Bettering Ladders

Smith (1995): Recursively distinct Nash payoffs

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Minmax Payoff (0,0,0)

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Minmax Payoff (0,0,0)
- Nash Equilibrium (T,l,L), Payoff (0,0,3)

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Minmax Payoff (0,0,0)
- Nash Equilibrium (T,l,L), Payoff (0,0,3) (B-K not met)

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Minmax Payoff (0,0,0)
- Nash Equilibrium (T,l,L), Payoff (0,0,3) (B-K not met)
- Player 3 can be threatened

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R
- The profile $\alpha^3 =$(T,l,R) is a Nash Equilibrium of the reduced game with Payoff (0,3,-1)

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R
- The profile α^3 =(T,l,R) is a Nash Equilibrium of the reduced game with Payoff (0,3,-1)
- Now player 2 can be threatened

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

Minmax Bettering Ladders

Example

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R and player 2 to play r

Minmax Bettering Ladders

Example

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R and player 2 to play r
- The profile α^{32} =(T,r,R) is a Nash Equilibrium of the reduced game with Payoff (1,-1,-1)

Minmax Bettering Ladders

Example

	L: l	L: m	L: r	R: l	R: m	R: r
T	0,0,3	0,-1,0	0,-1,0	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,0	0,-1,0	0,-1,0	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,0	0,-1,0	0,-1,0	-1,0,-1	-1,-1,-1	0,-1,-1

- Player 3 is forced to play R and player 2 to play r
- The profile α^{32} =(T,r,R) is a Nash Equilibrium of the reduced game with Payoff (1,-1,-1)
- Now player 1 can be threatened

Minmax Bettering Ladders

Formal Definition

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$				

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$				
game	G				

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$				
game	G				
“Nash equilibrium”	σ^1				

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1			
game	G				
"Nash equilibrium"	σ^1				

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1			
game	G	$G(a_{N_1})$			
"Nash equilibrium"	σ^1				

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1			
game	G	$G(a_{N_1})$			
"Nash equilibrium"	σ^1	σ^2			

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	...		
game	G	$G(a_{N_1})$	...		
"Nash equilibrium"	σ^1	σ^2	...		

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	
game	G	$G(a_{N_1})$	$\dots$		
"Nash equilibrium"	σ^1	σ^2	$\dots$		

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	
"Nash equilibrium"	σ^1	σ^2	$\dots$		

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\ldots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\ldots$	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	$\ldots$	σ^h	$---$

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1	...	N_{h-1}	N_h
game	G	$G(a_{N_1})$	...	$G(a_{N_{h-1}})$	— — —
"Nash equilibrium"	σ^1	σ^2	...	σ^h	— — —

A **minimax-bettering ladder** of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	$---$

A **minimax-bettering ladder** of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$

- $\mathcal{N} := \{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ subsets of N

Minmax Bettering Ladders

Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	$---$

A **minimax-bettering ladder** of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$

- $\mathcal{N} := \{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ subsets of N
- $\mathcal{A} := \{a_{N_1} \in A_{N_1}, \dots, a_{N_{h-1}} \in A_{N_{h-1}}\}$

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1	$\dots$	N_{h-1}	N_h
game	G	$G(a_{N_1})$	$\dots$	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	$\dots$	σ^h	$---$

A **minimax-bettering ladder** of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$

- $\mathcal{N} := \{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ subsets of N
- $\mathcal{A} := \{a_{N_1} \in A_{N_1}, \dots, a_{N_{h-1}} \in A_{N_{h-1}}\}$
- $\Sigma := \{\sigma^1, \dots, \sigma^h\}$

Minmax Bettering Ladders
Formal Definition

reliable players	$\emptyset$	N_1	...	N_{h-1}	N_h
game	G	$G(a_{N_1})$	...	$G(a_{N_{h-1}})$	$---$
"Nash equilibrium"	σ^1	σ^2	...	σ^h	$---$

A **minimax-bettering ladder** of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$

- $\mathcal{N} := \{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ subsets of N
- $\mathcal{A} := \{a_{N_1} \in A_{N_1}, \ldots, a_{N_{h-1}} \in A_{N_{h-1}}\}$
- $\Sigma := \{\sigma^1, \ldots, \sigma^h\}$

N_h is the top rung of the ladder

Minmax Bettering Ladders

Some properties

Minmax Bettering Ladders

Some properties

- A ladder with top rung N_h is maximal if there is no ladder with top rung $N_{h'}$ such that $N_h \subsetneq N_{h'}$

Minmax Bettering Ladders

Some properties

- A ladder with top rung N_h is maximal if there is no ladder with top rung $N_{h'}$ such that $N_h \subsetneq N_{h'}$
- A game G is decomposable as a complete minimax-bettering ladder if it has a minimax-bettering ladder with N as its top rung

Minmax Bettering Ladders

Some properties

- A ladder with top rung N_h is maximal if there is no ladder with top rung $N_{h'}$ such that $N_h \subsetneq N_{h'}$
- A game G is decomposable as a complete minimax-bettering ladder if it has a minimax-bettering ladder with N as its top rung

Lemma

All the maximal ladders of a game G have the same top rung

The New Folk Theorem
(Julio González-Díaz 2003)

Assumption for the game G

Result

The New Folk Theorem
(Julio González-Díaz 2003)

Assumption for the game G

- **Existence of a complete minmax bettering ladder**

Result

The New Folk Theorem
(Julio González-Díaz 2003)

Assumption for the game G

- **Existence of a complete minmax bettering ladder**

Result

- **Every payoff in $\bar{F}$ can be approximated in equilibrium**

The New Folk Theorem
(Julio González-Díaz 2003)

Assumption for the game G

- **Existence of a complete minmax bettering ladder**

Result

- **Every payoff in $\bar{F}$ can be approximated in equilibrium**

Remark

Unlike Benoît and Krishna's result, this theorem provides a necessary and sufficient condition

The New Folk Theorem
(Julio González-Díaz 2003)

Assumption for the game G

- **Existence of a complete minmax bettering ladder**

Result

- **Every payoff in $\bar{F}$ can be approximated in equilibrium**

Remark

Unlike Benoît and Krishna's result, this theorem provides a necessary and sufficient condition

Why the word **generalized**?

The New Folk Theorem
(Julio González-Díaz 2003)

Example (Idea of the proof)

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

The New Folk Theorem
(Julio González-Díaz 2003)

Example (Idea of the proof)

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Nash Equilibrium: α=(T,l,L), payoff (0,0,3). Hence, player 3 is reliable

The New Folk Theorem
(Julio González-Díaz 2003)

Example (Idea of the proof)

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

L

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

R

- Nash Equilibrium: α=(T,l,L), payoff (0,0,3). Hence, player 3 is reliable
- "Nash Equilibrium": α^3 =(T,l,R), payoff (0,3,-1). Hence, player 2 is reliable

The New Folk Theorem
(Julio González-Díaz 2003)

Example (Idea of the proof)

L:

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R:

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0,-1,-1

- Nash Equilibrium: α=(T,l,L), payoff (0,0,3). Hence, player 3 is reliable
- "Nash Equilibrium": α^3 =(T,l,R), payoff (0,3,-1). Hence, player 2 is reliable
- "Nash Equilibrium": α^{32} =(T,r,R), payoff (1,-1,-1). Hence, player 1 is reliable

The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.

The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.
- Equilibrium Path

$$\underbrace{u, u, \ldots, u, u,}_{T - \sum L_i \text{ stages}} \underbrace{\varphi(\alpha^{32}), \ldots, \varphi(\alpha^{32})}_{L_1 \text{ stages}} \ldots \underbrace{\varphi(\alpha), \ldots, \varphi(\alpha)}_{L_3 \text{ stages}}$$

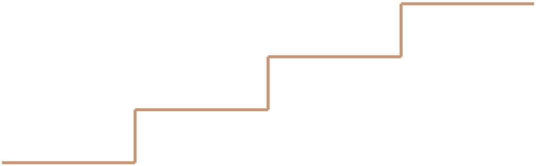
The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.
- Equilibrium Path

$$\underbrace{u, u, \ldots, u, u,}_{T - \sum L_i \text{ stages}} \underbrace{\varphi(\alpha^{32}), \ldots, \varphi(\alpha^{32})}_{L_1 \text{ stages}} \ldots \underbrace{\varphi(\alpha), \ldots, \varphi(\alpha)}_{L_3 \text{ stages}}$$

- The **ladder**



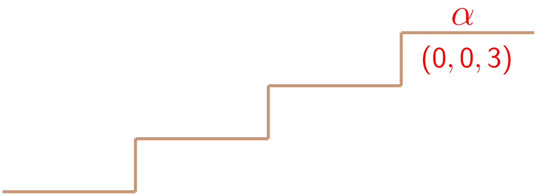
The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.
- Equilibrium Path

$$\underbrace{u, u, \ldots, u, u,}_{T - \sum L_i \text{ stages}} \underbrace{\varphi(\alpha^{32}), \ldots, \varphi(\alpha^{32})}_{L_1 \text{ stages}} \ldots \underbrace{\varphi(\alpha), \ldots, \varphi(\alpha)}_{L_3 \text{ stages}}$$

- The **ladder**



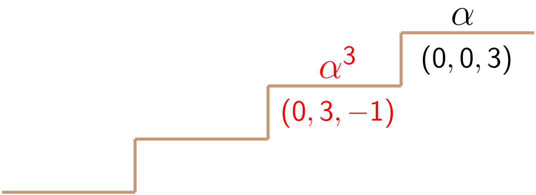
The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.
- Equilibrium Path

$$\underbrace{u, u, \ldots, u, u,}_{T-\sum L_i \text{ stages}} \underbrace{\varphi(\alpha^{32}), \ldots, \varphi(\alpha^{32})}_{L_1 \text{ stages}} \ldots \underbrace{\varphi(\alpha), \ldots, \varphi(\alpha)}_{L_3 \text{ stages}}$$

- The **ladder**



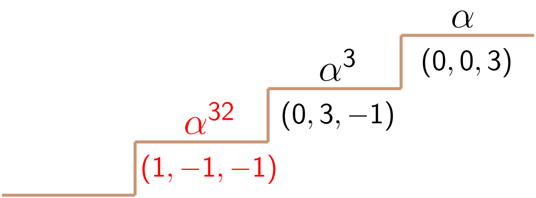
The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.
- Equilibrium Path

$$\underbrace{u, u, \ldots, u, u,}_{T - \sum L_i \text{ stages}} \underbrace{\varphi(\alpha^{32}), \ldots, \varphi(\alpha^{32})}_{L_1 \text{ stages}} \ldots \underbrace{\varphi(\alpha), \ldots, \varphi(\alpha)}_{L_3 \text{ stages}}$$

- The **ladder**



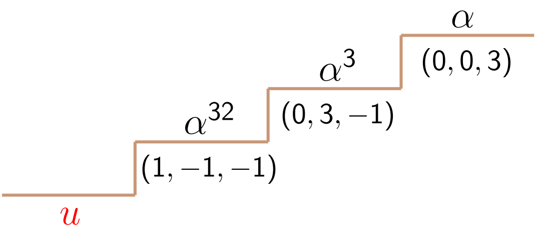
The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.
- Equilibrium Path

$$\underbrace{u, u, \ldots, u, u,}_{T-\sum L_i \text{ stages}} \underbrace{\varphi(\alpha^{32}), \ldots, \varphi(\alpha^{32})}_{L_1 \text{ stages}} \ldots \underbrace{\varphi(\alpha), \ldots, \varphi(\alpha)}_{L_3 \text{ stages}}$$

- The **ladder**



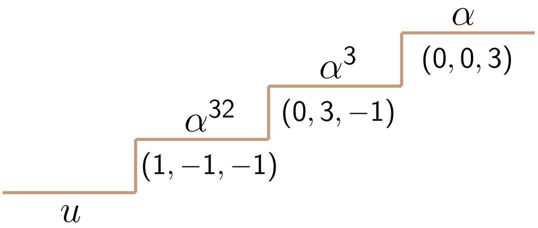
The New Folk Theorem
(Julio González-Díaz 2003)

Idea of the proof

- We want to approximate the payoff $u > v$ in equilibrium.
- Equilibrium Path

$$\underbrace{u, u, \ldots, u, u,}_{T - \sum L_i \text{ stages}} \underbrace{\varphi(\alpha^{32}), \ldots, \varphi(\alpha^{32})}_{L_1 \text{ stages}} \ldots \underbrace{\varphi(\alpha), \ldots, \varphi(\alpha)}_{L_3 \text{ stages}}$$

- The **ladder**



Generalized Nash Folk Theorem

Some more background

Generalized Nash Folk Theorem

Some more background

- Henceforth the set of players N is fixed

Generalized Nash Folk Theorem

Some more background

- Henceforth the set of players N is fixed
- Let $\mathrm{TR}_{N'}$ be the set of games with a maximal ladder with top rung $N' \subseteq N$

Generalized Nash Folk Theorem

Some more background

- Henceforth the set of players N is fixed
- Let $\mathrm{TR}_{N'}$ be the set of games with a maximal ladder with top rung $N' \subseteq N$
- Players in N' are reliable.

Generalized Nash Folk Theorem

Some more background

- Henceforth the set of players N is fixed
- Let $\mathrm{TR}_{N'}$ be the set of games with a maximal ladder with top rung $N' \subseteq N$
- Players in N' are reliable. Players in $N \backslash N'$ are not

Generalized Nash Folk Theorem

Some more background

- Henceforth the set of players N is fixed
- Let $\mathrm{TR}_{N'}$ be the set of games with a maximal ladder with top rung $N' \subseteq N$
- Players in N' are reliable. Players in $N \backslash N'$ are not
- By definition, if $a \in A$ is such that all the players in $N \backslash N'$ are best responding, then all of them receive their minmax payoff.

Generalized Nash Folk Theorem

Some more background

- Henceforth the set of players N is fixed
- Let $\mathrm{TR}_{N'}$ be the set of games with a maximal ladder with top rung $N' \subseteq N$
- Players in N' are reliable. Players in $N \backslash N'$ are not
- By definition, if $a \in A$ is such that all the players in $N \backslash N'$ are best responding, then all of them receive their minmax payoff. (otherwise N' is not the top rung of a maximal ladder)

Generalized Nash Folk Theorem

Some more background

- Henceforth the set of players N is fixed
- Let $\mathrm{TR}_{N'}$ be the set of games with a maximal ladder with top rung $N' \subseteq N$
- Players in N' are reliable. Players in $N \backslash N'$ are not
- By definition, if $a \in A$ is such that all the players in $N \backslash N'$ are best responding, then all of them receive their minmax payoff. (otherwise N' is not the top rung of a maximal ladder)
- In every Nash equilibrium of $G(\delta, T)$, players in $N \backslash N'$ must be best responding at every stage

Generalized Nash Folk Theorem

Some more background

Generalized Nash Folk Theorem

Some more background

- Let G be a game with top rung N'

Generalized Nash Folk Theorem

Some more background

- Let G be a game with top rung N'
- Let $\hat{a} \in A_{N'}$

Generalized Nash Folk Theorem

Some more background

- Let G be a game with top rung N'
- Let $\hat{a} \in A_{N'}$

$$(\hat{a}, \sigma) \in A$$

Generalized Nash Folk Theorem

Some more background

- Let G be a game with top rung N'
- Let $\hat{a} \in A_{N'}$

$$\{(\hat{a}, \sigma) \in A \ : \ \sigma \text{ Nash eq. of } G(\hat{a})\}$$

Generalized Nash Folk Theorem

Some more background

- Let G be a game with top rung N'
- Let $\hat{a} \in A_{N'}$
- Let $\Lambda(\hat{a}) := \{(\hat{a}, \sigma) \in A \ : \ \sigma$ Nash eq. of $G(\hat{a})\}$

Generalized Nash Folk Theorem

Some more background

- Let G be a game with top rung N'
- Let $\hat{a} \in A_{N'}$
- Let $\Lambda(\hat{a}) := \{(\hat{a}, \sigma) \in A \; : \; \sigma$ Nash eq. of $G(\hat{a})\}$
- $\Lambda = \bigcup_{\hat{a} \in A_{N'}} \Lambda(\hat{a})$

Generalized Nash Folk Theorem

Some more background

- Let G be a game with top rung N'
- Let $\hat{a} \in A_{N'}$
- Let $\Lambda(\hat{a}) := \{(\hat{a}, \sigma) \in A \;:\; \sigma$ Nash eq. of $G(\hat{a})\}$
- $\Lambda = \bigcup_{\hat{a} \in A_{N'}} \Lambda(\hat{a})$
- $\bar{F}_{N'} := \bar{F} \cap co\{\varphi(\lambda) \;:\; \lambda \in \Lambda\}$

Generalized Nash Folk Theorem

Theorem (Main result)

Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in$ TR$_{N'}$.

Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in TR_{N'}$. Let $u \in F$.

Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in$ TR$_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if

Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in$ TR$_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in$ TR$_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Remark

Given a game G we have characterized the whole set of payoffs attainable as a Nash equilibrium in some repeated game associated with G

Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in$ TR$_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof

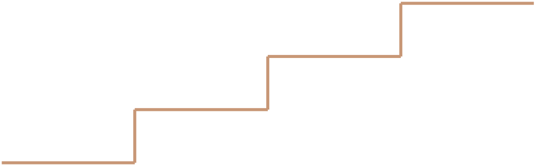
Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in$ TR$_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof

"$\Leftarrow$" The **ladder**



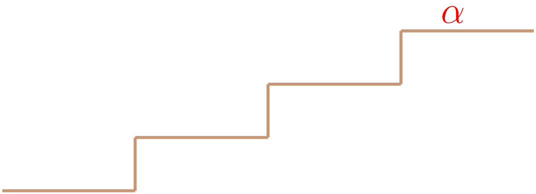
Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in TR_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof

"$\Leftarrow$" The **ladder**



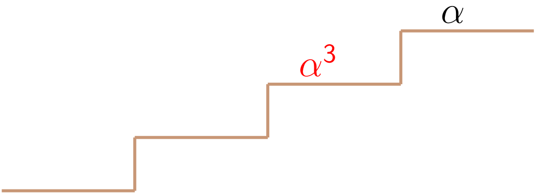
Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in$ TR$_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) *if and only if $u \in \bar{F}_{N'}$.*

Idea of the proof

"$\Leftarrow$" The **ladder**



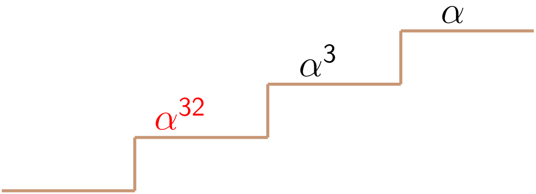
Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in TR_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof

"$\Leftarrow$" The **ladder**



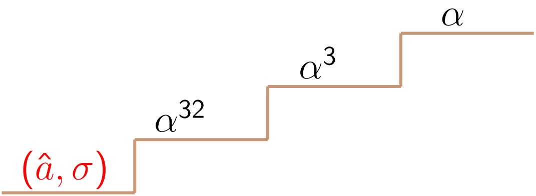
Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in TR_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof

"$\Leftarrow$" The **ladder**



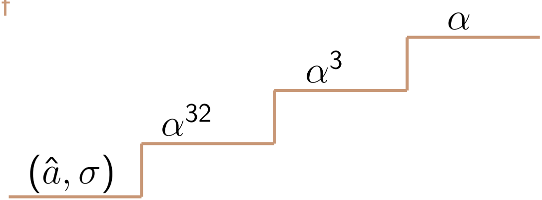
Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in TR_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof

"$\Leftarrow$" The **ladder**



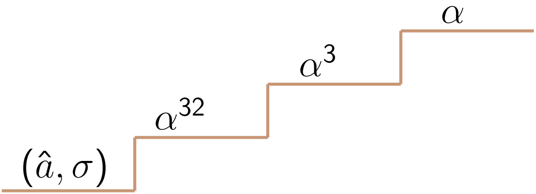
"$\Rightarrow$" Let $u \notin \bar{F}_{N'}$.

Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in TR_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof



"$\Leftarrow$" The **ladder**

"$\Rightarrow$" Let $u \notin \bar{F}_{N'}$. For each strategy of the repeated game, take the last stage in which an action not in Λ is played.

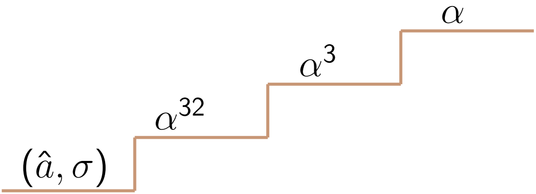
Generalized Nash Folk Theorem

Theorem (Main result)

Let $G \in TR_{N'}$. Let $u \in F$. Then, we can approximate u in Nash equilibrium of $G(\delta, T)$ (for some δ and T) if and only if $u \in \bar{F}_{N'}$.

Idea of the proof

“$\Leftarrow$” The **ladder**



“$\Rightarrow$” Let $u \notin \bar{F}_{N'}$. For each strategy of the repeated game, take the last stage in which an action not in Λ is played. A player in $N \backslash N'$ can deviate without being punished

Outline

Unobservable Mixed Actions

Example

L

	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

M

	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

R

	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

Unobservable Mixed Actions

Example

L

	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

M

	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

R

	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

- The minmax payoff is (0,0,0)

Unobservable Mixed Actions

Example

L:

	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

M:

	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

R:

	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

- The minmax payoff is (0,0,0)
- (T,l,L) is a Nash Equilibrium with payoff (0,0,2). Hence, player 3 is reliable

Unobservable Mixed Actions

Example

L:

	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

M:

	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

R:

	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

- The minmax payoff is (0,0,0)
- (T,l,L) is a Nash Equilibrium with payoff (0,0,2). Hence, player 3 is reliable
- If player 3 randomizes (0,0.5,0.5) the subgame has an equilibrium with payoff (0.5,0.5,-4.5).

Unobservable Mixed Actions

Example

L:

	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

M:

	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

R:

	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

- The minmax payoff is (0,0,0)
- (T,l,L) is a Nash Equilibrium with payoff (0,0,2). Hence, player 3 is reliable
- If player 3 randomizes (0,0.5,0.5) the subgame has an equilibrium with payoff (0.5,0.5,-4.5). The game has a complete ladder

Unobservable Mixed Actions

Example

L

	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

M

	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

R

	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

- The minmax payoff is (0,0,0)
- (T,l,L) is a Nash Equilibrium with payoff (0,0,2). Hence, player 3 is reliable
- If player 3 randomizes (0,0.5,0.5) the subgame has an equilibrium with payoff (0.5,0.5,-4.5). The game has a complete ladder
- Player 3 is not indifferent between M and R

Unobservable mixed actions

- The results concerning necessity results still carry over

Unobservable mixed actions

- The results concerning necessity results still carry over
- We have not found a proof for the sufficiency ones

Conclusions

Conclusions

Conclusions

Conclusions

- We have extended the result in Benoît and Krishna (1987)

Conclusions

Conclusions

- We have extended the result in Benoît and Krishna (1987)
- We have generalized the result in Benoît and Krishna (1987)

Conclusions

Conclusions

- We have extended the result in Benoît and Krishna (1987)
- We have generalized the result in Benoît and Krishna (1987)
- Our main result establishes a necessary and sufficient condition for the finite horizon Nash folk theorem

Conclusions

Conclusions

- We have extended the result in Benoît and Krishna (1987)
- We have generalized the result in Benoît and Krishna (1987)
- Our main result establishes a necessary and sufficient condition for the finite horizon Nash folk theorem
- Can the same result be obtained if we drop the assumption of observable? mixed actions

References

Abreu, D., P. K. Dutta, and L. Smith (1994): "The Folk Theorem for Repeated Games: A NEU Condition," *Econometrica*, 62, 939–948.

Benoît, J.-P. and V. Krishna (1985): "Finitely Repeated Games," *Econometrica*, 53, 905–922.

——— (1987): "Nash Equilibria of Finitely Repeated Games," *International Journal of Game Theory*, 16, 197–204.

Fudenberg, D. and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54, 533–554.

Gossner, O. (1995): "The Folk Theorem for Finitely Repeated Games with Mixed Strategies," *International Journal of Game Theory*, 24, 95–107.

Smith, L. (1995): "Necessary and Sufficient Conditions for the Perfect Finite Horizon Folk Theorem," *Econometrica*, 63, 425–430.

Wen, Q. (1994): "The "Folk Theorem" for Repeated Games with Complete Information," *Econometrica*, 62, 949–954.

...

THANKS