A Natural Selection from the Core of a TU game: The Core-Center

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Outline

The Core-Center: Definition and Properties

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For simplicity, we denote a game (N, v) by v

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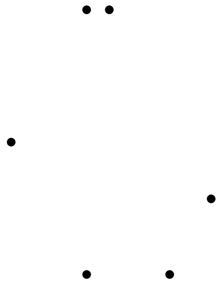

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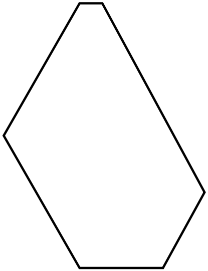

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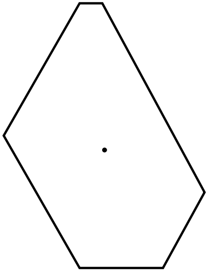

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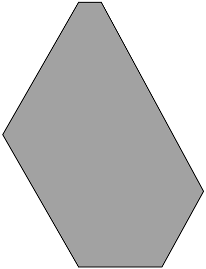

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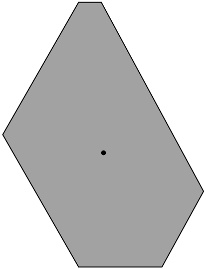

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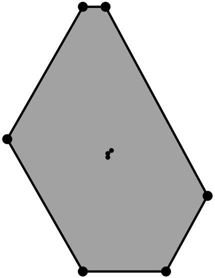

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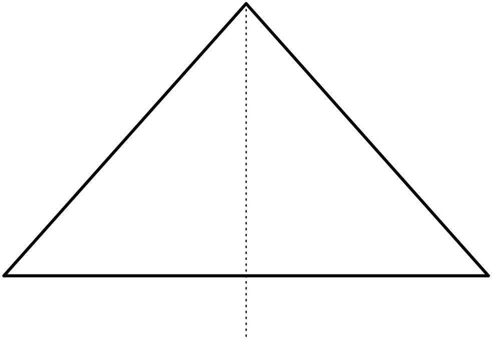

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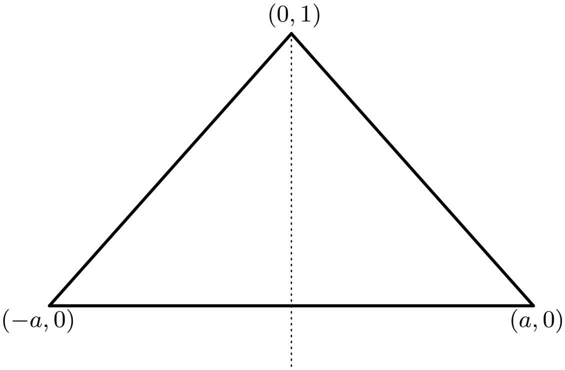

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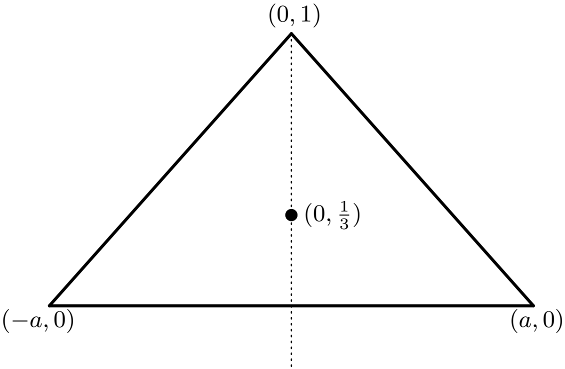

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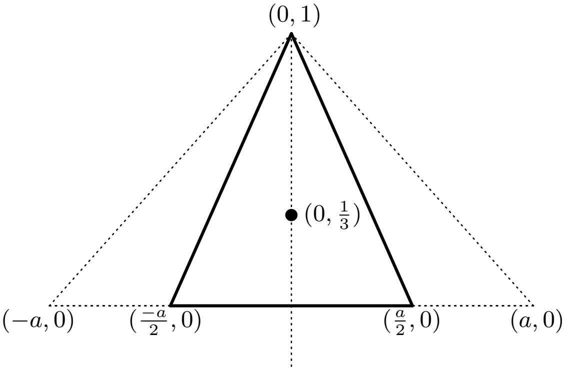

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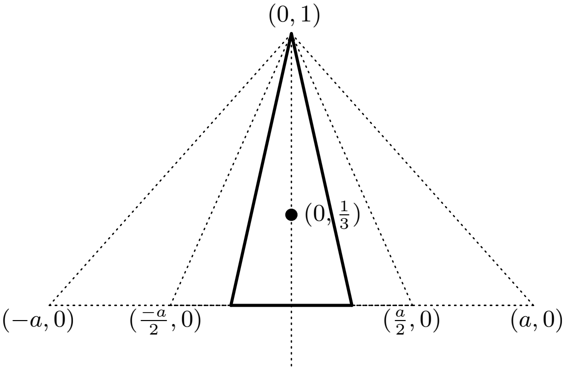

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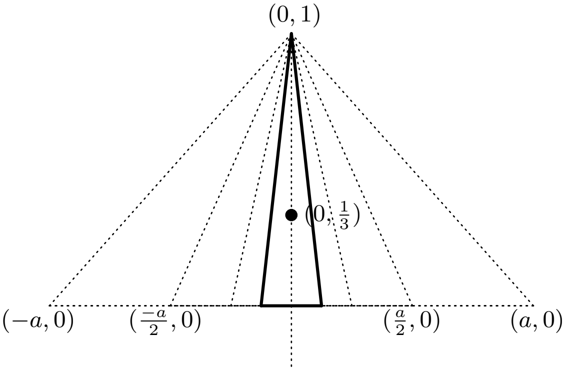

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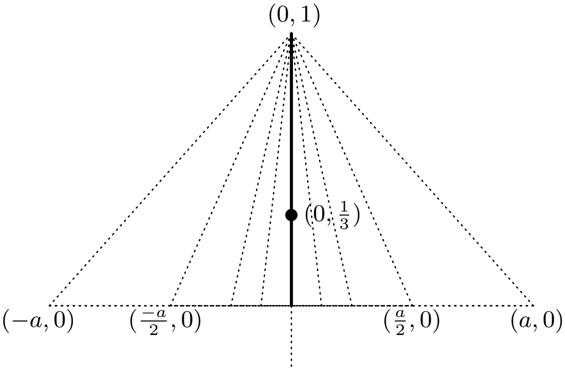

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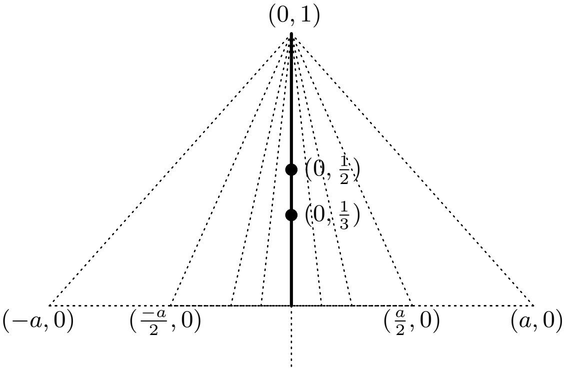

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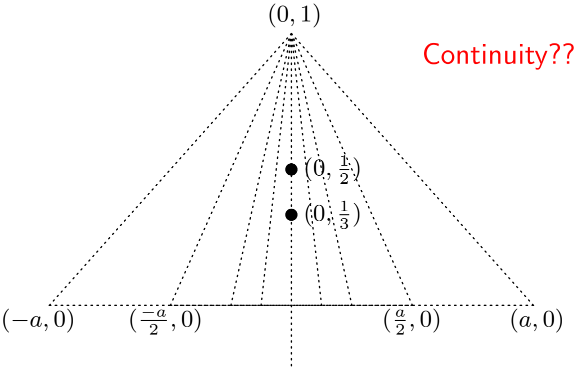

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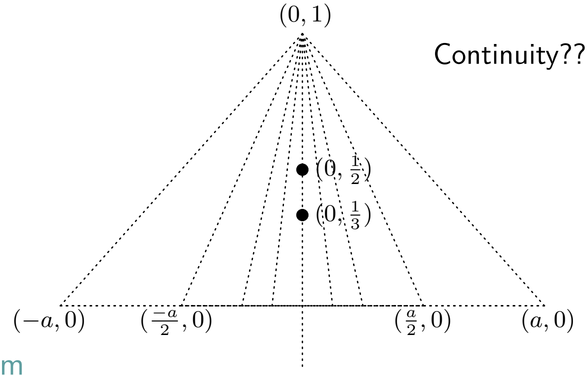

Continuity??

Theorem

The Core-Center is a continuous allocation rule

The Core-Center: Monotonicity

- Take a pair of games v and w

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Strong monotonicity

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Strong monotonicity

Let $i \in N$. If for each $S \subseteq N\backslash\{i\}$,
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- $w(T) > v(T)$ and for each $S \neq T$, $w(S) = v(S)$

The Core-Center: Monotonicity

- Take a pair of games v and w

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Weak coalitional monotonicity

$\sum_{i \in T} \varphi_i(w) \geq \sum_{i \in T} \varphi_i(v)$

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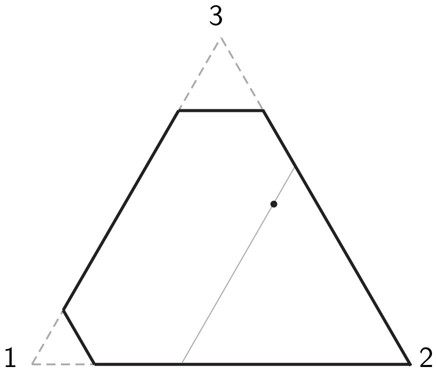
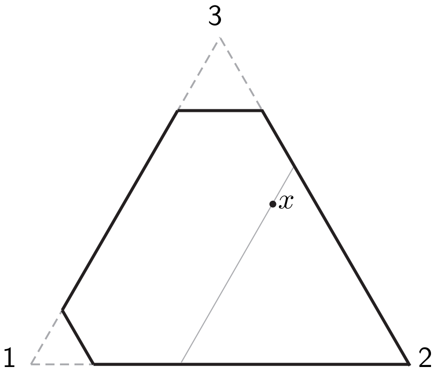
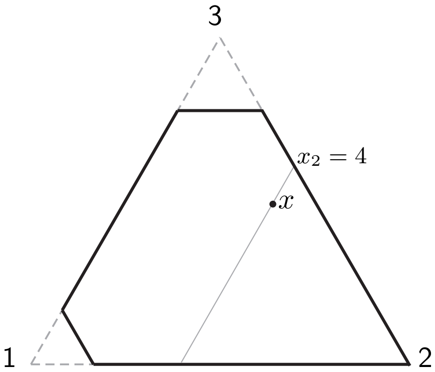
Core-Center $\Longleftrightarrow$ Nucleolus

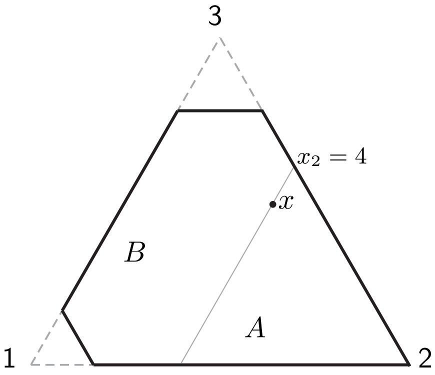
The Core-Center: Specific Properties
A Fairness Property

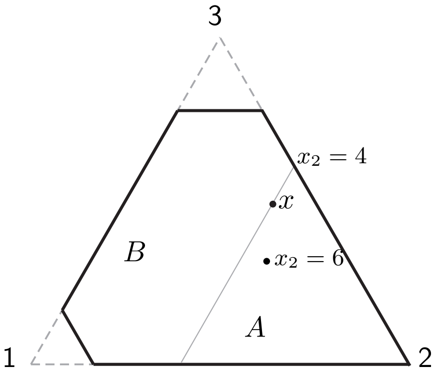



3
x
1
2

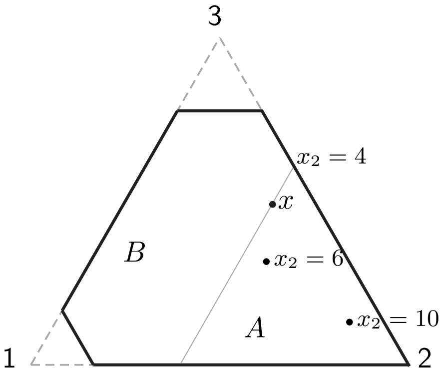


3
$x_2 = 4$
x
1
2

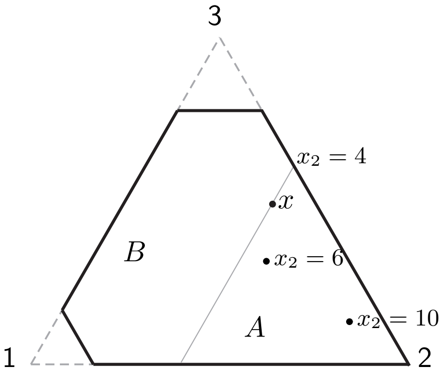

3
$x_2 = 4$
x
B
A
1
2



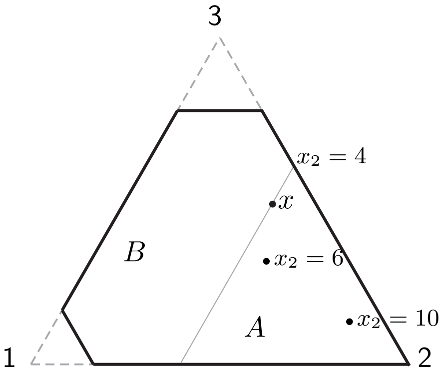
3
$x_2 = 4$
x
B
$x_2 = 6$
A
1
2



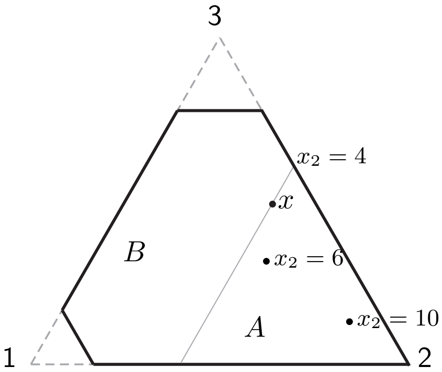
3
$x_2 = 4$
x
B
$x_2 = 6$
$x_2 = 10$
A
1
2



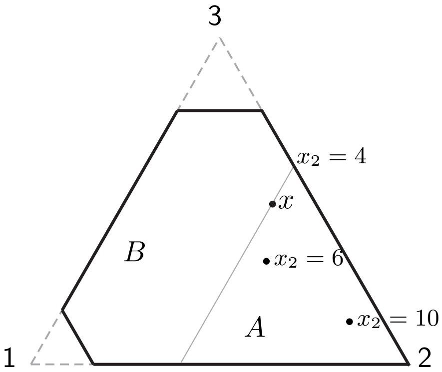
$DS_2(x) =$ ————————



$$DS_2(x) = \frac{\int_B |x_i - y_i| dy}{}$$



$$DS_2(x) = \frac{\int_B |x_i - y_i| dy}{\int_A |x_i - y_i| dy}$$



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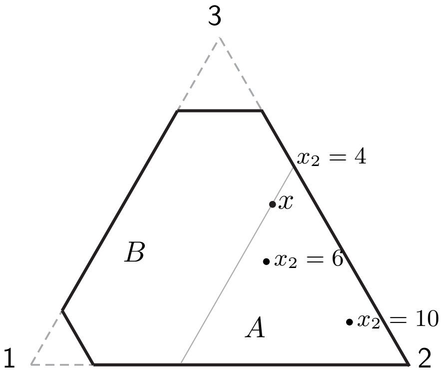
Uniform Distribution



$$DS_2(x) = \frac{\int_B |x_i - y_i| dy}{\int_A |x_i - y_i| dy} \longrightarrow \begin{cases} > & 1 \\ = & 1 \\ < & 1 \end{cases}$$

$x \in \mathbb{R}^n$

$x \in \mathbb{R}^n$ $\mathbb{P}$

$x \in \mathbb{R}^n$ is impartial with respect to $\mathbb{P}$

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Lemma

Let v be a balanced game.

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Lemma

Let v be a balanced game. Take $U(C(v))$

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Lemma

Let v be a balanced game. Take $U(C(v))$
Then, the core-center is the unique efficient allocation which is impartial with respect to $U(C(v))$

Superadditivity:

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φ is a $\mathcal{T}$-solution

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where $\alpha \in [0, 1]$

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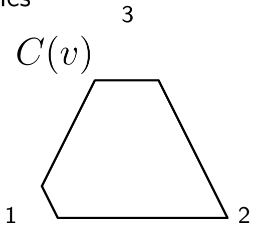
2. *Translation Invariance*

Balanced Games

Balanced Games



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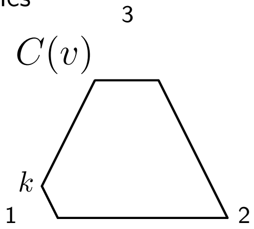
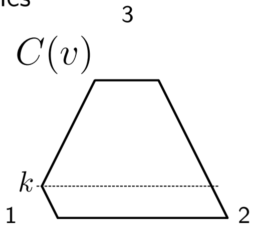
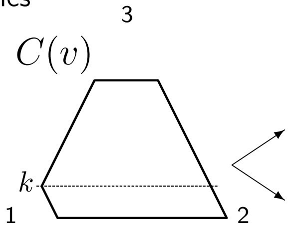


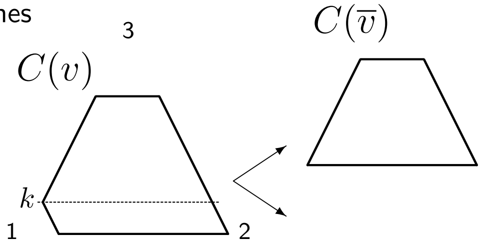
Balanced Games



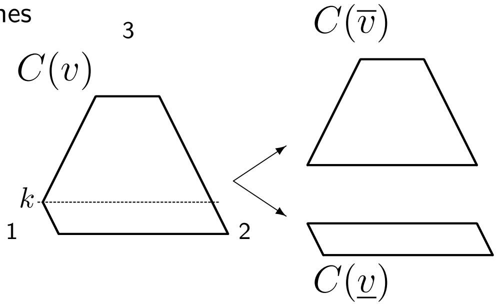
Balanced Games



Balanced Games



Balanced Games



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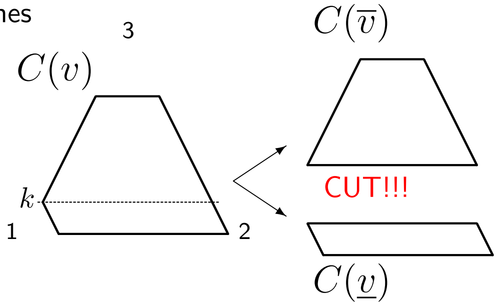

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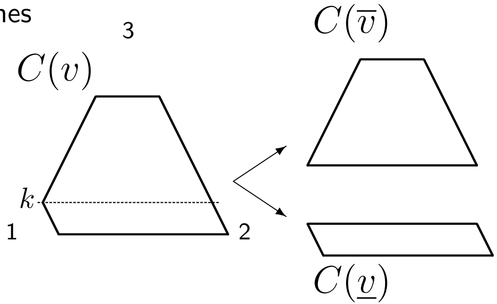
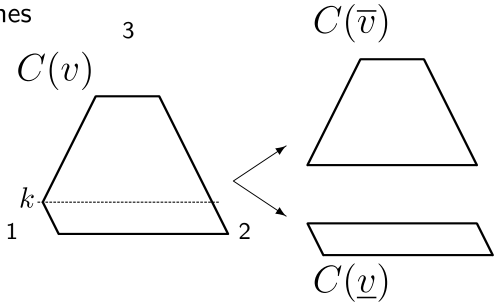


Definition

Balanced Games



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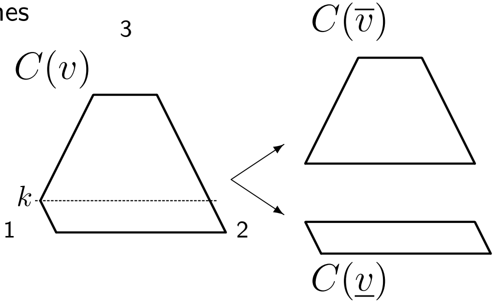
Balanced Games

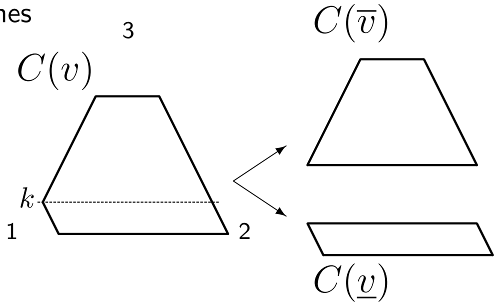


Definition

Let v be a balanced game

Let v' and v'' be two balanced games such that belong to some dissection of v

Balanced Games



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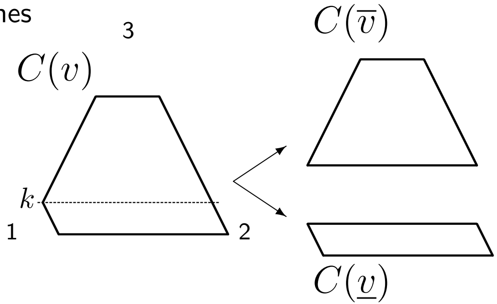
φ satisfies fair additivity with respect to the core if:

Balanced Games



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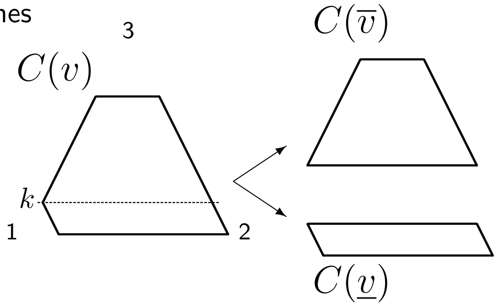
Let v' and v'' be two balanced games such that belong to some dissection of v

φ satisfies fair additivity with respect to the core if:

1. φ is a $\mathcal{RT}$-solution.

Balanced Games



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Let v' and v'' be two balanced games such that belong to some dissection of v

φ satisfies fair additivity with respect to the core if:

1. φ is a $\mathcal{RT}$-solution.
2. $C(v') = C(v'')$ implies that $\alpha_v(v') = \alpha_v(v'')$.

Table of Properties

	Shapley	Nucleolus	Core-Center

Table of Properties

	Shapley	Nucleolus	Core-Center
Efficiency			
Individual Rationality			
Continuity			
Dummy Player			
Symmetry			
Translation and Scale Invariance			

Table of Properties

	Shapley	Nucleolus	Core-Center
Efficiency	✓	✓	✓
Individual Rationality	✓	✓	✓
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Coalitional Monotonicity			
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Additivity			

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Impartiality w.r.t. the core			

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Impartiality w.r.t. the core	X	X	✓

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A Characterization of the Core-center

Extended Weak Symmetry

Definition

$i, j \in N$ are symmetric if

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v symmetric if any two players i and j in N are symmetric

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v is quasi symmetric if any two players i and j in N are q-symmetric

Extended Weak Symmetry

Definition

φ satisfies Weak Symmetry

Definition

φ satisfies Weak Symmetry if for any symmetric game

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$$\varphi_i(N,v) - \varphi_j(N,v) = 0 \quad \forall i,j \in N$$

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Weak Symmetry + Translation Invariance = Extended Weak Symmetry

The Characterization

Theorem

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Let φ be an allocation rule satisfying

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- *Efficiency*

The Characterization

Theorem

Let φ be an allocation rule satisfying

- *Efficiency*
- *Continuity*

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Then, for each $v \in BG$,

The Characterization

Theorem

Let φ be an allocation rule satisfying

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Then, for each $v \in BG$, $\varphi(v)$ coincides with the core-center.

Theorem

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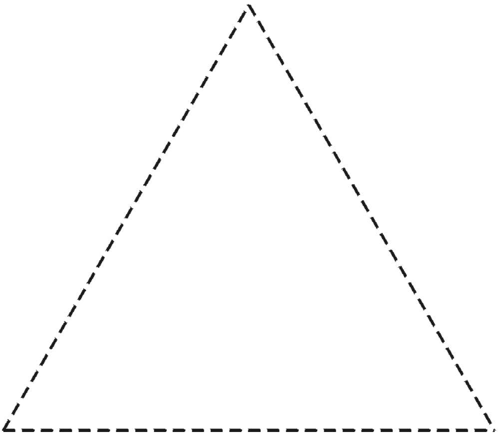
Then, for each $v \in BG$, $\varphi(v)$ coincides with the core-center.

The axioms are independent

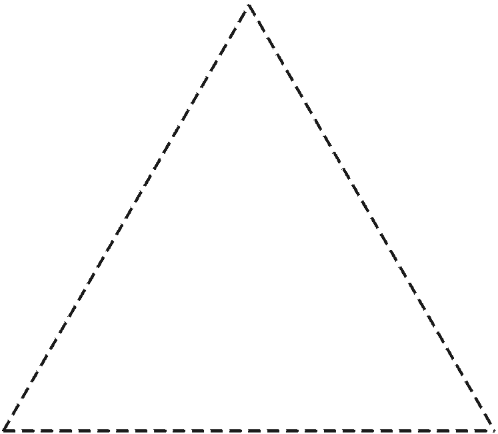
Step 1

Step 1



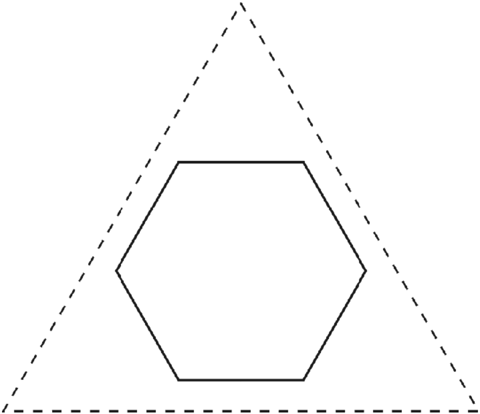
Step 1

Efficiency + EWS



Step 2

Step 2



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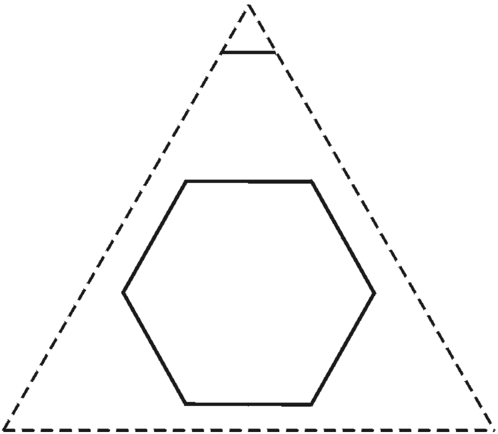

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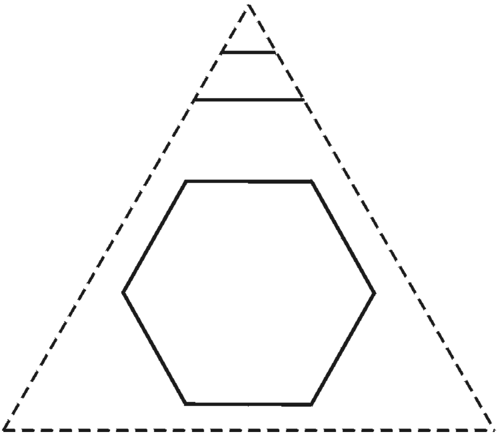

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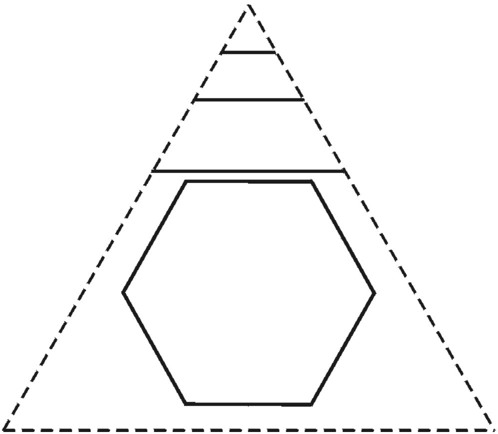

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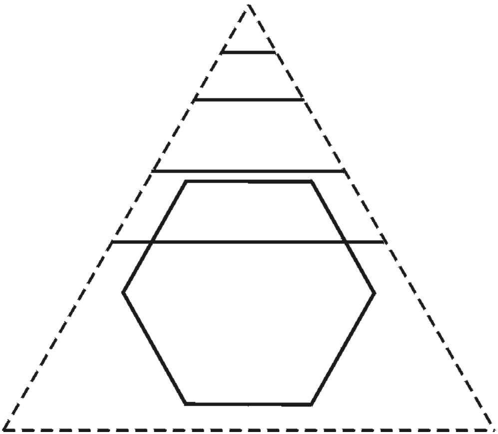

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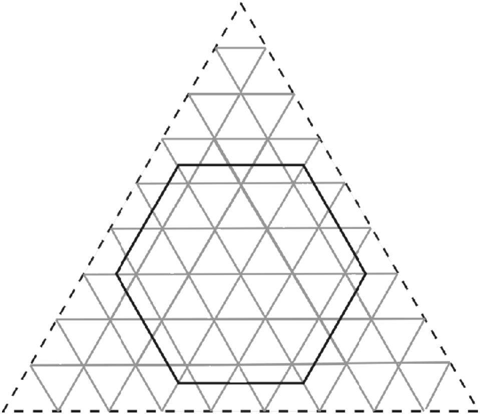

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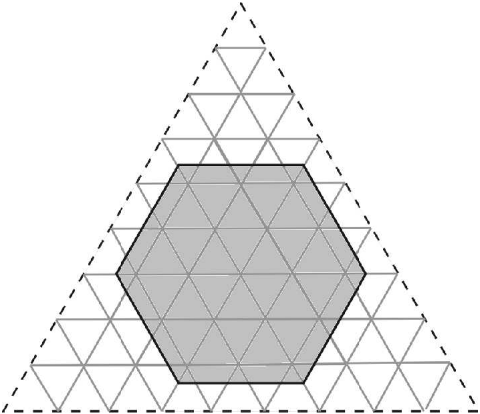

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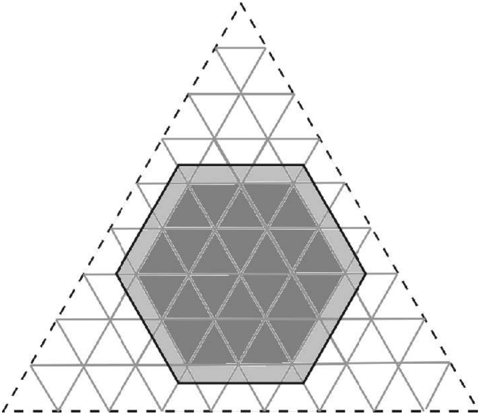
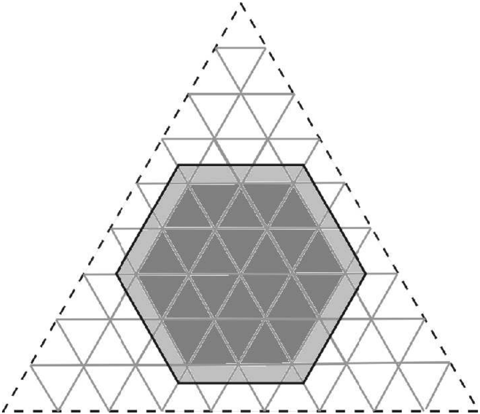


Step 2

Continuity + Fair Additivity



A Natural Selection from the Core of a TU game: The Core-Center

Julio González Díaz

Department of Statistics and Operations Research
Universidade de Santiago de Compostela

June 21th, 2005

Previous concepts

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- Utopia vector, $M(v) \in \mathbb{R}^N$:

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The τ-value

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$$CC(v) := \{x \in \mathbb{R}^N \,:\, \sum_{i \in N} x_i = v(N),\ m(v) \leq x \leq M(v)\}$$

- A game v is compromise admissible if $CC(v) \neq \emptyset$

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The τ^*-value, González-Díaz et al. (2003):

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