Sharing a Cake

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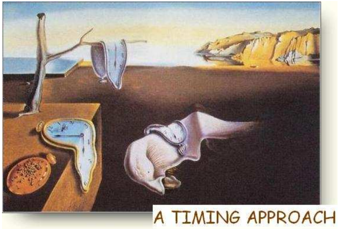
Tilburg University



Timing Games

Timing Games

- Chicken game

Timing Games

- Chicken game
- Patent race

Timing Games

"Noisy" timing games

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Two families of timing games

1. War of attrition games
2. Preemption games

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- J. Reinganum, 1981
  (Review of Economic Studies)

Two families of timing games

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Timing Games

"Noisy" timing games

- Chicken game
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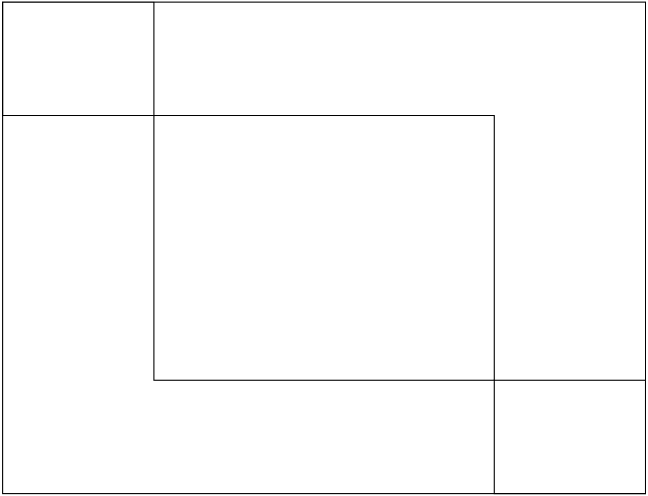
"Silent" timing games

- J. Reinganum, 1981 (Review of Economic Studies)
- H. Hamers, 1993 (Mathematical Methods of OR)

Two families of timing games

1. War of attrition games
2. Preemption games

First Example: Sharing a Cake



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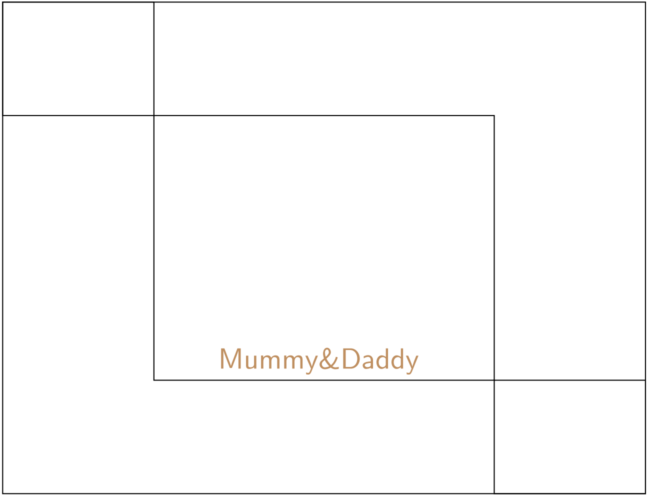

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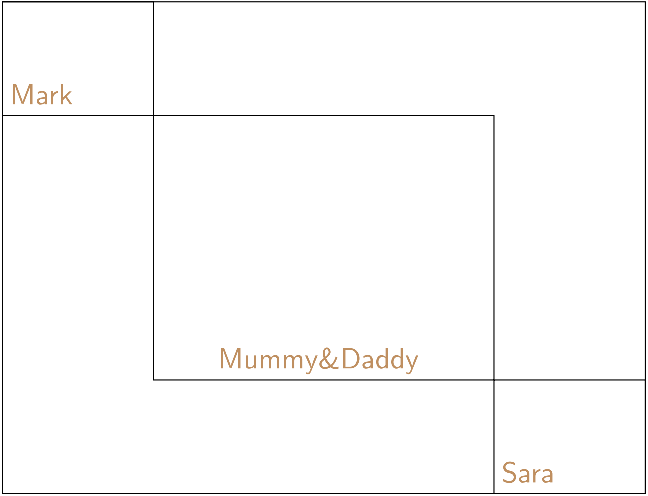

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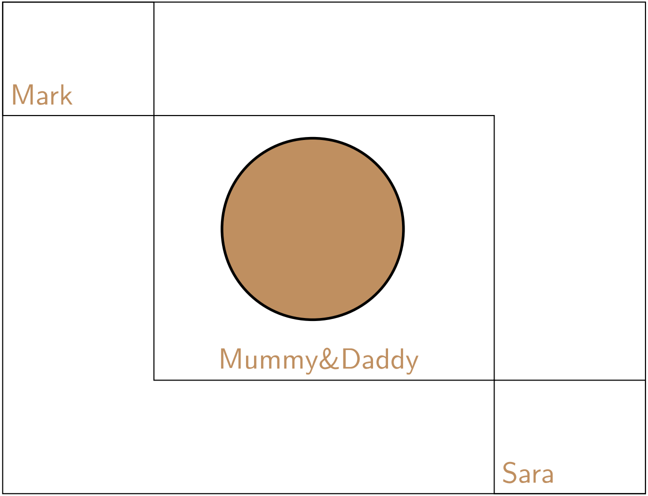

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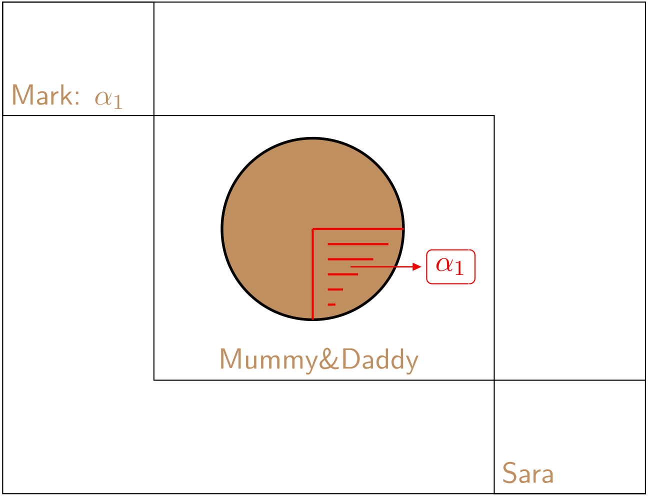

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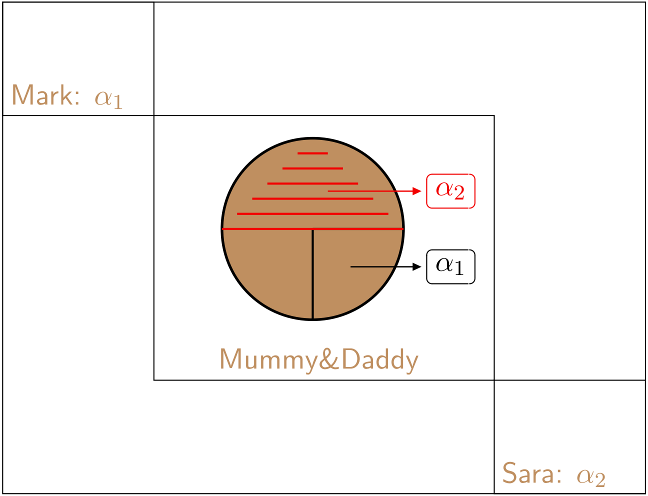

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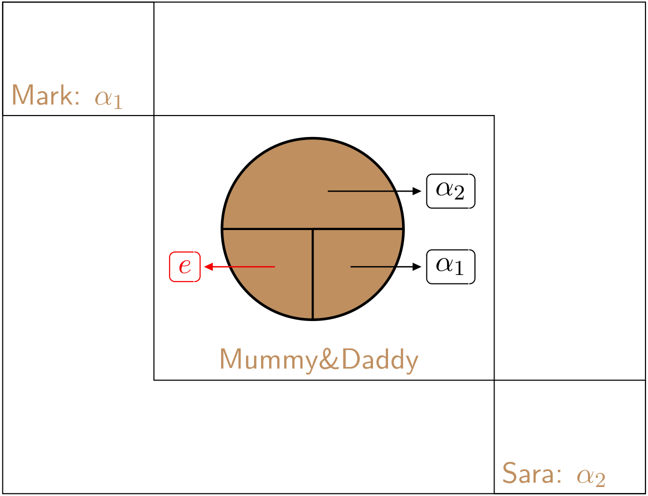

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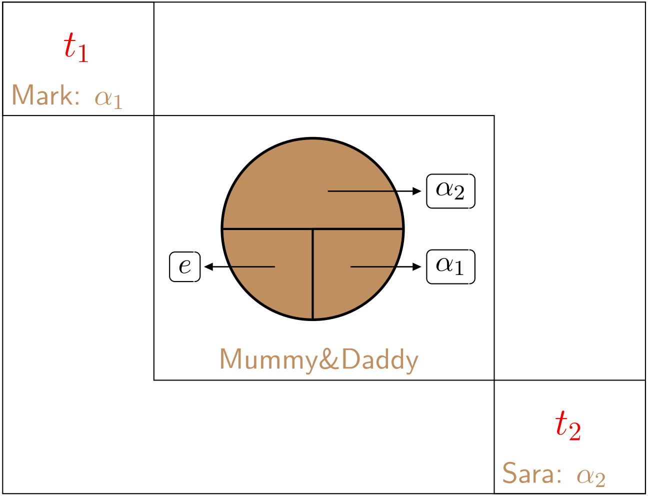

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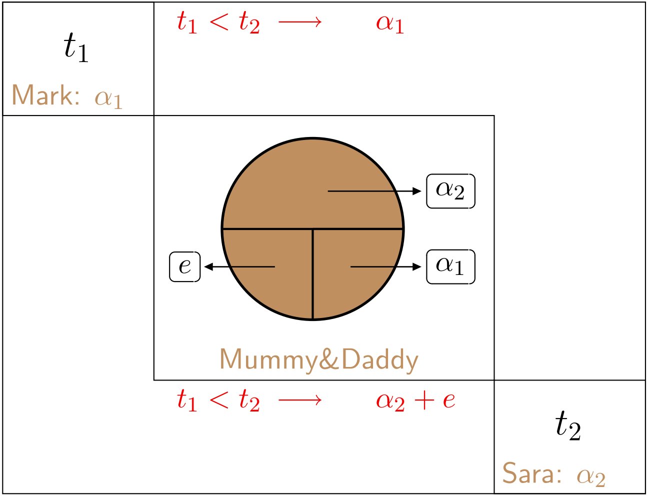


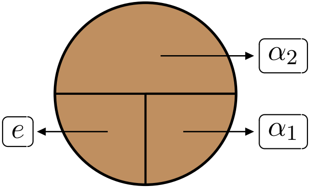
First Example: Sharing a Cake

t_1

Mark: α_1

$t_1 < t_2 \longrightarrow \alpha_1$

$t_1 > t_2 \longrightarrow \alpha_1 + e$



Mummy&Daddy

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t_2

Sara: α_2

First Example: Sharing a Cake

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$t_1 < t_2 \longrightarrow \alpha_1$

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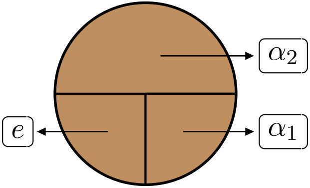

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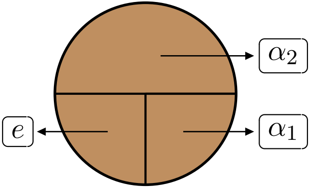

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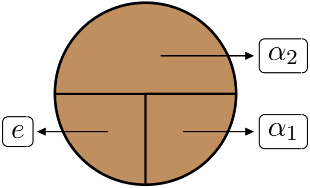

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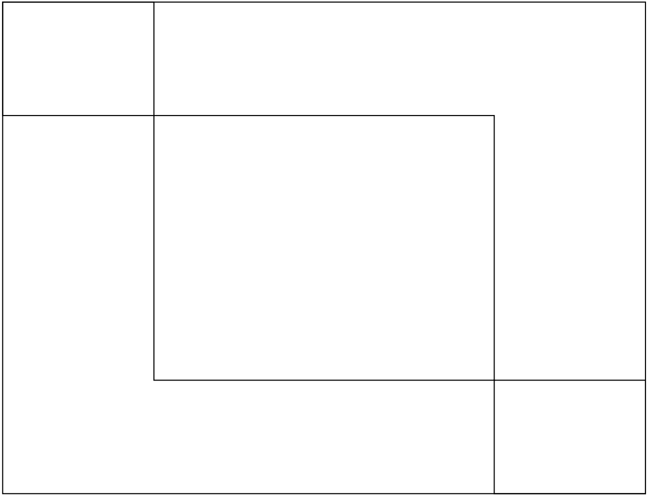

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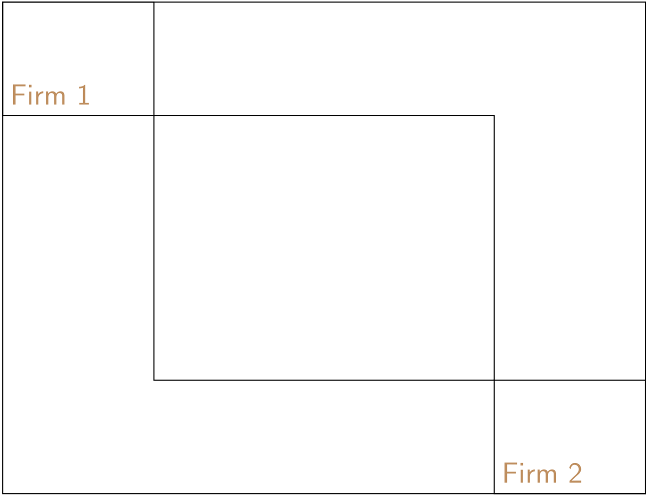

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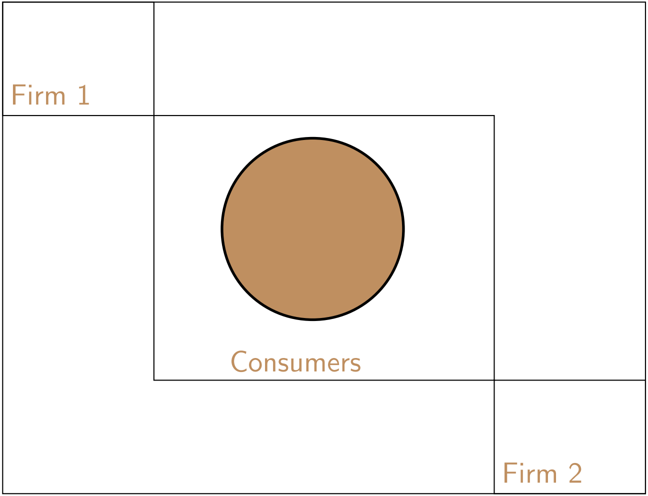

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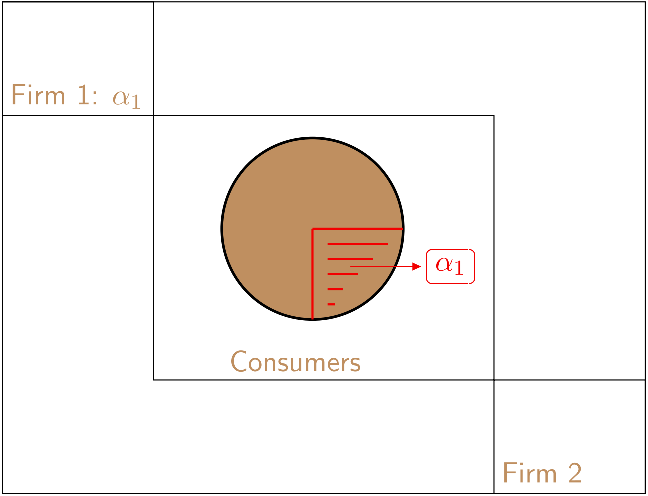

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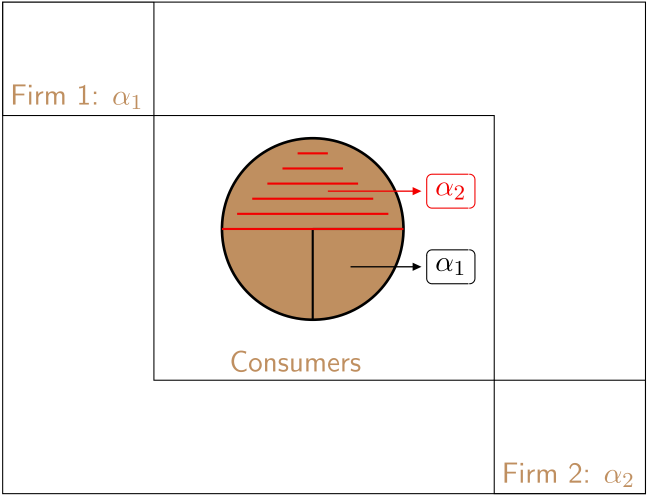

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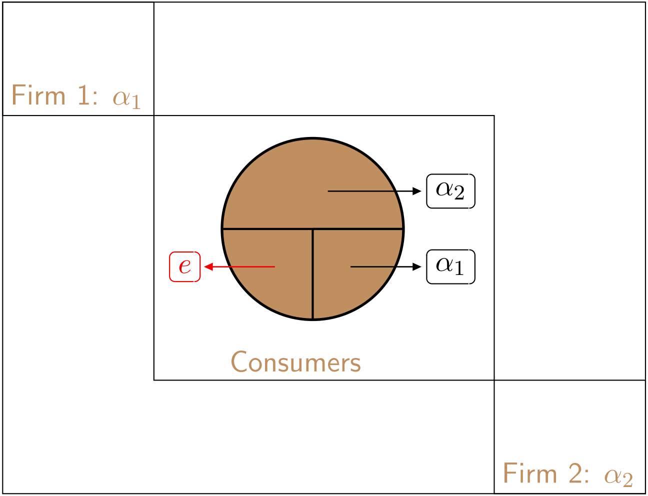

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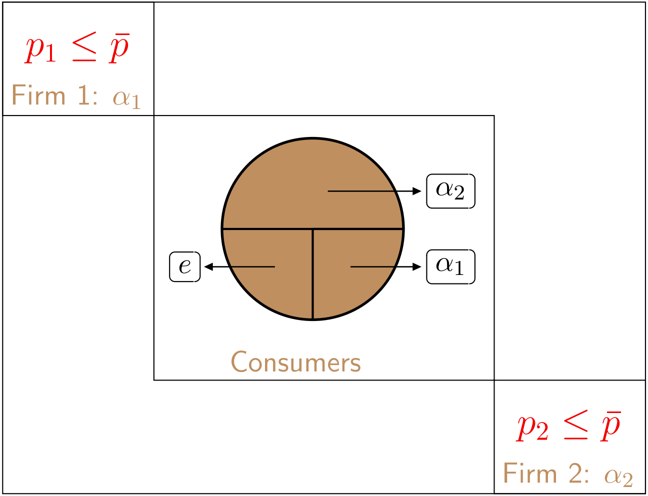

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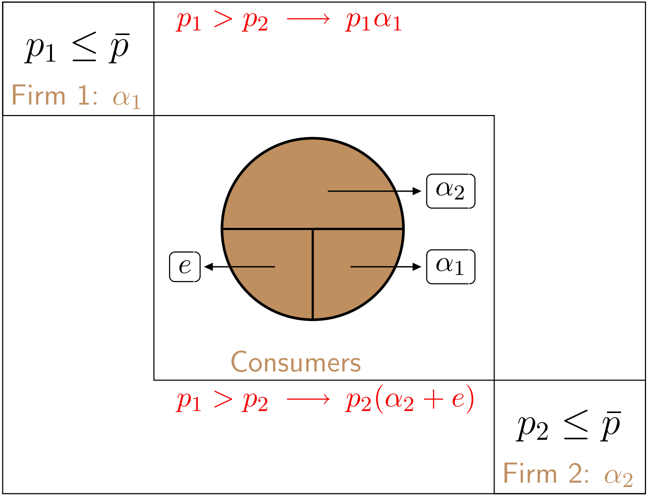


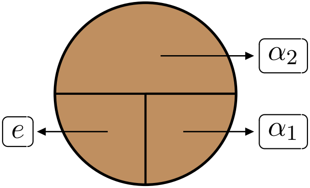
Second Example: Sharing a Market

$p_1 \leq \bar{p}$

Firm 1: α_1

$p_1 > p_2 \longrightarrow p_1\alpha_1$

$p_1 < p_2 \longrightarrow p_1(\alpha_1 + e)$



Consumers

$p_1 > p_2 \longrightarrow p_2(\alpha_2 + e)$

$p_1 < p_2 \longrightarrow p_2\alpha_2$

$p_2 \leq \bar{p}$

Firm 2: α_2

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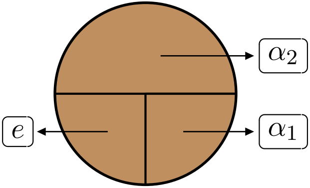

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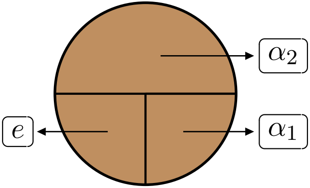

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The Models

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Timing Game
(Sharing a Cake)

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Primitives $\alpha\ ,\ \delta$

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Primitives $\alpha \, , \, \delta$

The Game $\Gamma^{\text{pure}} =< N, \{A_1, A_2\}, \{\pi_1, \pi_2\} >$

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$$\pi_i(a_1, a_2) = \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$

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The Models

	Timing Game (Sharing a Cake)	Pricing Game (Sharing a Market)
Primitives	$\alpha\ ,\ \delta$	
The Game	$\Gamma^{\text{pure}} =< N, \{A_1, A_2\}, \{\pi_1, \pi_2\} >$	

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Outline

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The Cake Sharing Game

The Model

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- $N = \{1, \ldots, n\}$ is the set of players
- Let $\alpha \in \mathbb{R}^N_+$ be the initial rights vector:
  $1 - (\alpha_1 + \cdots + \alpha_n) = e > 0 \qquad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$

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Cake sharing game with **pure strategies**

$$\Gamma_{\alpha,\delta}^{\text{pure}} =< N, \{A_i\}_{i \in N}, \{\pi_i\}_{i \in N} >$$

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- π_i is the payoff function of player $i \in N$, defined by:

$$\pi_i(t_1, \ldots, t_n) = \begin{cases} \delta^{t_i}\alpha_i & \text{if } t_i \leq \max_{j \neq i} t_j \\ \delta^{t_i}(\alpha_i + e) & \text{if } t_i > \max_{j \neq i} t_j \end{cases}$$

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Ties??

Discussion of the model

Discussion of the model

Discussion of the model

Objectives

Discussion of the model

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Objectives

Assumptions of the model

Discussion of the model

Objectives

Assumptions of the model

- Continuous time

Discussion of the model

Objectives

Assumptions of the model

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Discussion of the model

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- $\alpha_1 < \alpha_2 < \ldots < \alpha_n$

Discussion of the model

Objectives

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Differences with "Noisy" timing games

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- Substantial change in payoff funtions

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$L_1(t_1)$	

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A negative result

A negative result

There is no Nash equilibrium in pure strategies

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There is no Nash equilibrium in pure strategies

- There is a unique last claimant

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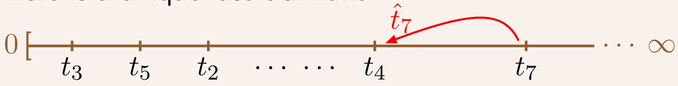

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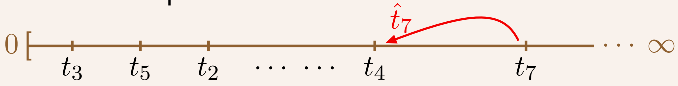

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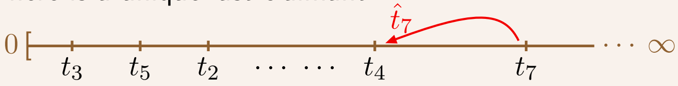

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A mixed strategy is a distribution function G, defined on $[0, \infty)$

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- **Hamers (1993)** proves the existence and uniqueness of the Nash equilibrium of any two player cake sharing game
- **Koops (2001)** finds several properties that Nash equilibria of three player cake sharing game must satisfy

Outline

The result (two player case)

Theorem 1 (Hamers (1993))

The result (two player case)

Theorem 1 (Hamers (1993))

Let $\Gamma_{\alpha,\delta}$ *be a 2-player cake sharing game and* $\bar{t} := \log_\delta \frac{\alpha_2}{\alpha_2+e}$. *Define* $G^* = (G_1^*, G_2^*) \in \mathcal{G} \times \mathcal{G}$ *by*

$$G_1^*(t) = \begin{cases} \dfrac{\alpha_2 - \alpha_2\delta^t}{\delta^t e} & \text{if } 0 \leq t \leq \bar{t} \\ 1 & \text{if } t > \bar{t} \end{cases}$$

$$G_2^*(t) = \begin{cases} \dfrac{\alpha_2(\alpha_1 + e) - \alpha_1(\alpha_2 + e)\delta^t}{\delta^t(\alpha_2 + e)e} & \text{if } 0 \leq t \leq \bar{t} \\ 1 & \text{if } t > \bar{t} \end{cases}$$

Then G^* *is the unique Nash equilibrium of* $\Gamma_{\alpha,\delta}$. *The payoffs are*

$$\bar{\pi}_1 = \frac{\alpha_2(\alpha_1 + e)}{\alpha_2 + e} \qquad \bar{\pi}_2 = \alpha_2$$

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An Example

Example 1

Player 1: $\alpha_1 = 0.1$

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Discount factor: $\delta = 0.9$

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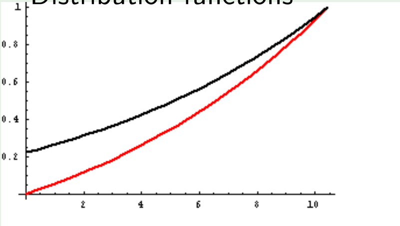

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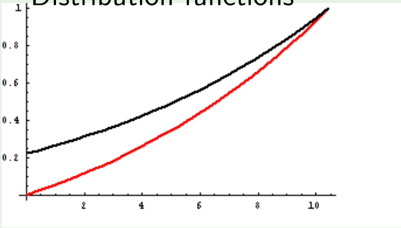
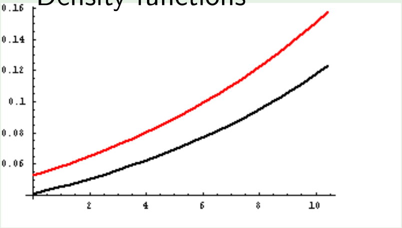


Density functions



An Example

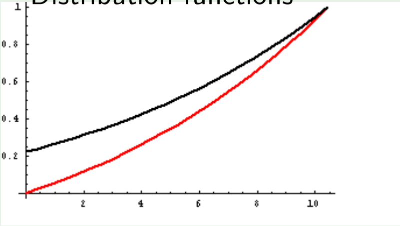
Example 1

Player 1: $\alpha_1 = 0.1$ — Equilibrium Payoff: 0.2333

Player 2: $\alpha_2 = 0.3$ — Equilibrium Payoff: 0.3

Discount factor: $\delta = 0.9$

Distribution functions



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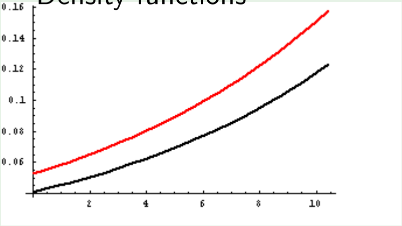

The result (n-player case)

Theorem 2

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Let $\Gamma_{\alpha,\delta}$ be an n-player cake sharing game with $n \geq 3$. Then $\Gamma_{\alpha,\delta}$ has a unique Nash equilibrium in which players $3, \ldots, n$ put probability 1 at 0 and players 1 and 2 play the game with total cake size $\alpha_1 + \alpha_2 + e$.

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- Allowing for equalities in the initial rights

The result and the pricing game

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The pricing game

- N firms. Each one with α_i loyal consumers
- Strategic consumers: e
- Higher admissible price: $\bar{p}$

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- Only the firm with less loyal consumers gains by "fighting"
- Strategic consumers pay less than loyal consumers

Our Contribution

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- Alternative proof of the existence and uniqueness result of the Nash equilibrium in the two player case

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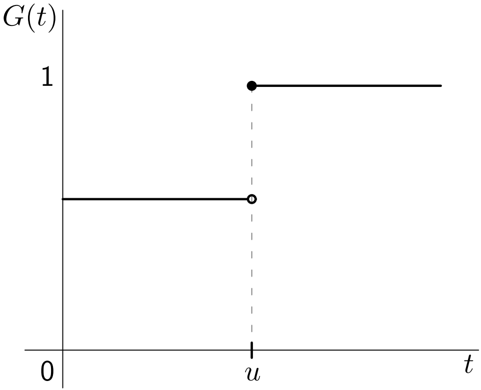
- Alternative proof of the existence and uniqueness result of the Nash equilibrium in the two player case
- Proof of the existence and uniqueness result of the Nash equilibrium in the general case (n-players)

Outline

Lemma 1 (No jumps)



Lemma 1 (No jumps)

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Let $\Gamma_{\alpha,\delta}$ *be an* n*-player cake sharing game and let* $G = (G_i)_{i \in N} \in \mathcal{G}^N$ *be a Nash equilibrium of* $\Gamma_{\alpha,\delta}$. *Then,* $J(G_i) \cap (0, \infty) = \emptyset$ *for every* $i \in N$.

▸ Proof

Lemma 1 (No jumps)

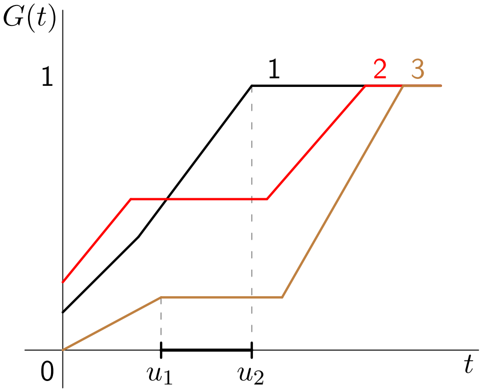
Lemma 1

Let $\Gamma_{\alpha,\delta}$ be an n-player cake sharing game and let $G = (G_i)_{i \in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{\alpha,\delta}$. Then, $J(G_i) \cap (0, \infty) = \emptyset$ for every $i \in N$.

Proof

No jumps in $(0, \infty)$

Lemma 2 (No one grows alone)



-No Jumps

Lemma 2 (No one grows alone)

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Let $\Gamma_{\alpha,\delta}$ be an n-player cake sharing game and let the profile $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{\alpha,\delta}$. Let $i \in N$ and $t \in S(G_i)$. There exists $j \in N\backslash\{i\}$ such that $t \in S(G_j)$.

Proof

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No distribution function grows alone

Lemma 2 (No one grows alone)

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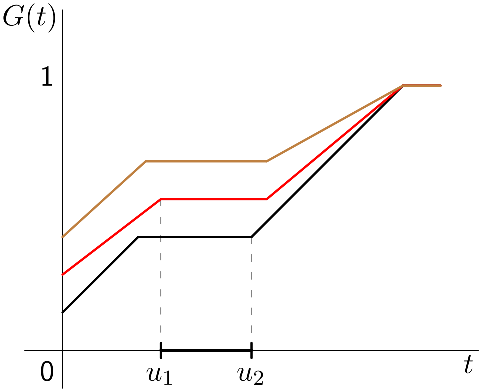
No distribution function grows alone

Lemma 2 + 2-player: The supports coincide

Lemma 3 (No stop&go)



-No Jumps
-No one grows alone
[2]Common Support

Lemma 3 (No stop&go)

Lemma 3

Let $G = (G_i)_{i \in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{\alpha,\delta}$. Suppose $t \in [0, \infty)$ is such that $t \notin S(G_j)$ for every $j \in N$. Then $(t, \infty) \cap S(G_j) = \emptyset$ for every $j \in N$.

Proof

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▸ Proof

No stop&go

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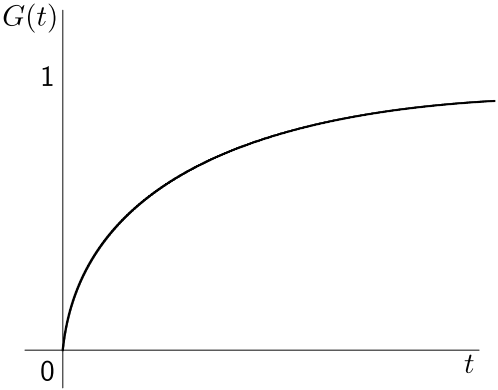
▸ Proof

No stop&go

Lemma 3 + 2-player: Strictly increasing distribution functions (till they get value 1)

Lemma 4 (Bounded Support)



-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing

Lemma 4 (Bounded Support)

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Let $G = (G_i)_{i \in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{\alpha,\delta}$. Then, $S(G_i)$ is a compact set for every $i \in N$. ▸ Proof

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Bounded Support

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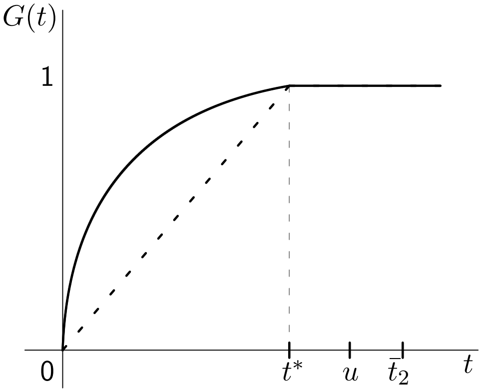
Bounded Support

Corollary 1

$S(G_1) \subset [0, \bar{t}_2]$.

Lemma 5 (n=2) (Supports are $[0, \bar{t}_2]$)



-No Jumps
-No one grows alone
^{2}Common Support
-No stop&go
^{2}Strictly Increasing
-Bounded support

Lemma 5 (n=2) (Supports are $[0, \bar{t}_2]$)

Lemma 5 (n=2)

Let $\Gamma_{\alpha,\delta}$ *be a 2-player cake sharing game and let* $G = (G_1, G_2) \in \mathcal{G} \times \mathcal{G}$ *be a Nash equilibrium of* $\Gamma_{\alpha,\delta}$*. Let* $\bar{t}_2 := \log_\delta \frac{\alpha_2}{\alpha_2 + e}$*. Then* $S(G_1) = S(G_2) = [0, \bar{t}_2]$.

▸ Proof

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▸ Proof

(n=2) The supports are $[0, \bar{t}_2]$

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▸ Proof

(n=2) The supports are $[0, \bar{t}_2]$

Corollary 2 (n=2)

Player 1 puts probability 0 at 0

Proof of Theorem 1

Proof of Theorem 1.

□

-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support
-[2]Supports are $[0, \bar{t}_2]$
-[2]Player 1 puts prob 0 at 0

Proof of Theorem 1

Proof of Theorem 1.

$S(G_1) = S(G_2) = [0, \bar{t}_2]$

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Proof of Theorem 1

Proof of Theorem 1.

$S(G_1) = S(G_2) = [0, \bar{t}_2]$

There exist constants c and d such that

$$c = \pi_1(t, G_2) = \delta^t(\alpha_1 + eG_2(t)) \quad t \in [0, \bar{t}_2]$$
$$d = \pi_2(G_1, t) = \delta^t(\alpha_2 + eG_1(t)) \quad t \in [0, \bar{t}_2]$$

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Since $G_1(0) = 0$, $d = \pi_2^G(0) = \alpha_2$.

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Then $G_1(t) = \frac{\alpha_2 - \alpha_2\delta^t}{e\delta^t}$

□

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Similarly, since $G_2(\bar{t}_2) = 1$

$c = \pi_1^G(\bar{t}_2) = \delta^{\bar{t}_2}(\alpha_1 + e) = \frac{\alpha_2(\alpha_1+e)}{\alpha_2+e}$, ...

□

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These strategies are Nash by definition. □

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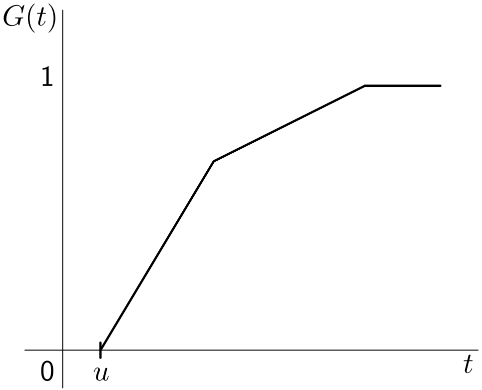
Lemma 6 (0 is in the support of every strategy)



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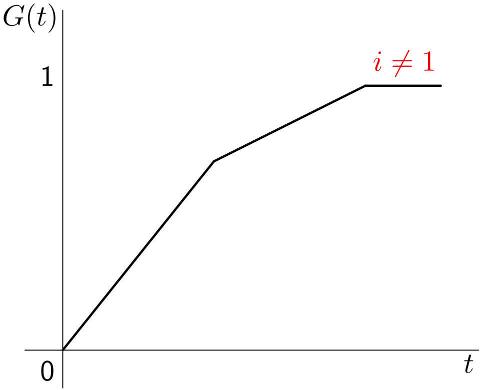
-Bounded support

Lemma 7 (Every player but player 1 jumps at 0)



- -No Jumps
- -No one grows alone
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Lemma 6 and Lemma 7

(0 is in the support of every strategy & Every player but player 1 jumps at 0)

Lemma 6 (and 7)

Let $\Gamma_{\alpha,\delta}$ *be an* n*-player cake sharing game with* $n \geq 3$ *and let* $G = (G_i)_{i \in N} \in \mathcal{G}^N$ *be a Nash equilibrium of* $\Gamma_{\alpha,\delta}$*. Then* $0 \in S(G_j)$ *for every* $j \in N$*. Moreover* $G_j(0) > 0$ *for every* $j \in N \backslash 1$*.*

▸ Proof

Lemma 6 and Lemma 7

(0 is in the support of every strategy & Every player but player 1 jumps at 0)

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▸ Proof

0 is in the support of every strategy

Lemma 6 and Lemma 7

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Proof

0 is in the support of every strategy

Every player but player 1 jumps at 0

Lemma 8 (Nash Payoffs)

Lemma 8

Let $G = (G_i)_{i \in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{\alpha,\delta}$ and let $\bar{\pi} = (\eta_i)_{i \in N}$ be the corresponding vector of equilibrium payoffs. Then

$$\begin{aligned} \bar{\pi}_1 &= \frac{\alpha_2(\alpha_1 + e)}{\alpha_2 + e} \text{ and} \\ \bar{\pi}_i &= \alpha_i \text{ for every } i \in N \backslash \{1\} \end{aligned}$$

▸ Proof

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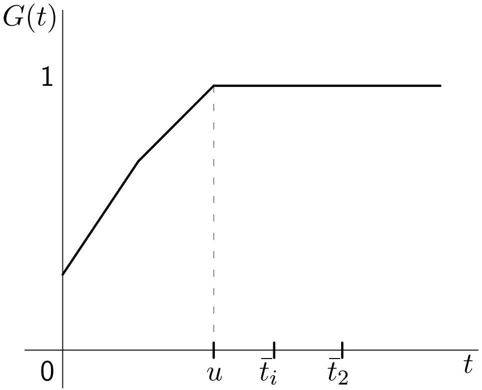
$$\bar{\pi}_1 = \frac{\alpha_2(\alpha_1 + e)}{\alpha_2 + e} \text{ and}$$
$$\bar{\pi}_i = \alpha_i \text{ for every } i \in N \backslash \{1\}$$

▸ Proof

Nash payoffs are $\bar{\pi}_1 = \frac{\alpha_2(\alpha_1+e)}{\alpha_2+e}$ **and** $\bar{\pi}_i = \alpha_i \ (i \neq 1)$

Lemma 9 (Players $3, \dots, n$ play $t = 0$)



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-$i \neq 1$ jumps at 0
-$\bar{\pi}_i = \frac{\alpha_2(\alpha_1+e)}{(\alpha_2+e)}$
-$\bar{\pi}_i = \alpha_i, \; i \neq 1$

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Let $G = (G_i)_{i \in N}$ *be a Nash equilibrium of the* n*-player cake sharing game* $\Gamma_{\alpha,\delta}$ *with* $n \geq 3$. *Then for every* $i \in N \backslash \{1, 2\}$, G_i *corresponds to pure strategy* $t = 0$.

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▸ Proof

Players $3, \ldots, n$ play $t = 0$

Proof of Theorem 2

Proof of Theorem 2.

□

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-No stop&go
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-$i \neq 1$ jumps at 0
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-$3, \ldots, n$ play $t = 0$

Proof of Theorem 2

Proof of Theorem 2.

Agents 1 and 2 play the game with cake size $\alpha_1 + \alpha_2 + e$

□

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Strategy $t = 0$ is optimal for players $3, \ldots, n$ □

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Outline

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Results

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Conclusions

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- Different degrees of loyalty

References

FUDENBERG, D., R. GILBERT, J. STIGLITZ, AND J. TIROLE (1983): "Preemption, Leapfrogging and Competition in Patent Races," *European Economic Review*, 22, 3–31.

HAMERS, H. (1993): "A Silent Duel over a Cake," *Methods and Models of Operations Research*, 37, 119–127.

HENDRICKS, K., A. WEISS, AND C. WILSON (1988): "The War of Attrition in Continuous Time with Complete Information," *International Economic Review*, 29, 663–680.

KOOPS, O. (2001): "A Silent Battle over a Cake," Master's thesis, Tilburg University.

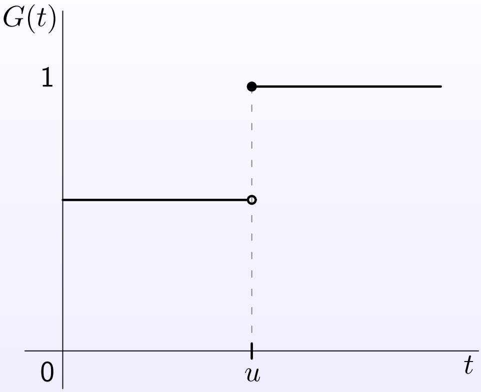
LARAKI, R., E. SOLAN, AND N. VIEILLE (2003): "Continuous-time Games of Timing," Discussion Paper 1363, Kellogg School of Management, Northwestern University.

REINGANUM, J. (1981): "On the Diffusion of a New Technology: A Game Theoretic Approach," *The Review of Economic Studies*, 48, 395–405.

...

THANKS

Proof of Lemma 1 (No jumps)



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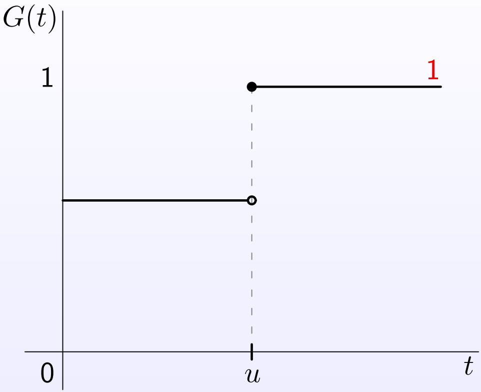
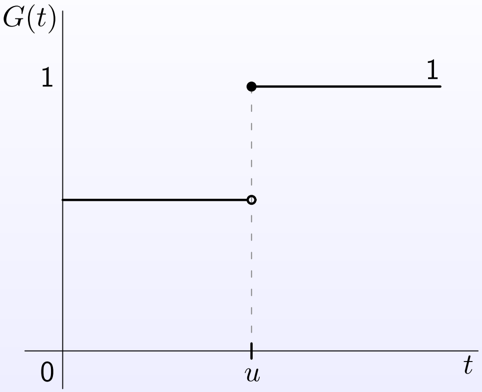


- Assume without loss of generality that 1 “jumps” at u

Return

Proof of Lemma 1 (No jumps)



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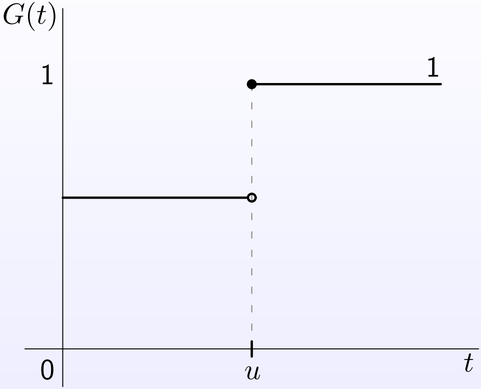


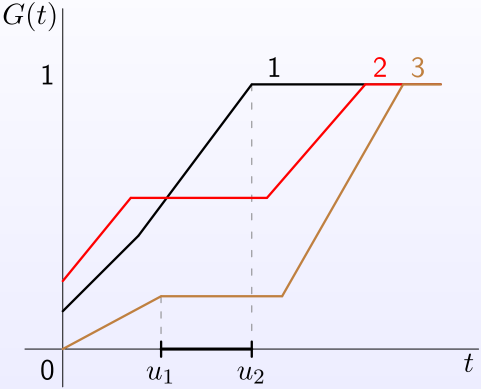
- Assume without loss of generality that 1 "jumps" at u
- $G_i(u^-) > 0$ for all i
- $\pi_i(G_{-i}, t) = \delta^t(\alpha_i + e\prod_{j\neq i} G_j(t^-))$ has a jump at u $(i \neq 1)$

Return

Proof of Lemma 1 (No jumps)

Return

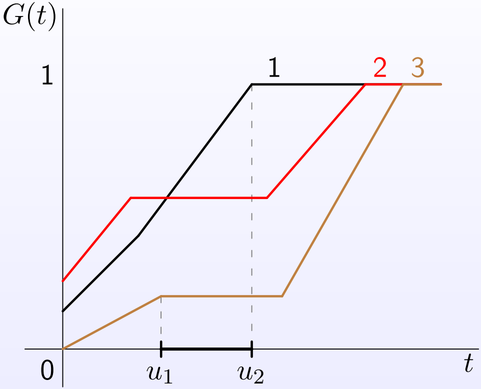
Proof of Lemma 2 (No one grows alone)



-No Jumps

◂ Return

Proof of Lemma 2 (No one grows alone)



-No Jumps

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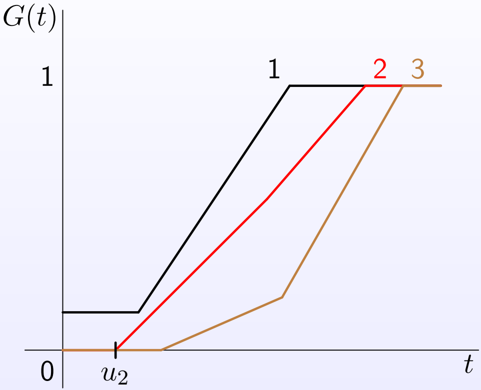

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◂ Return

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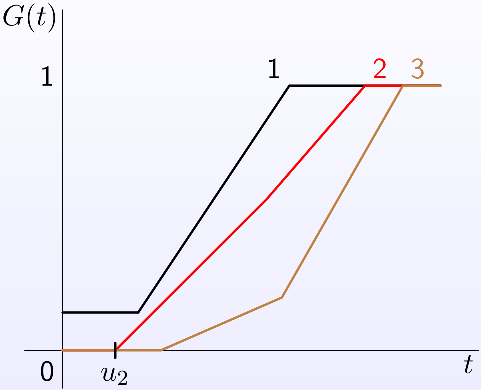


-No Jumps

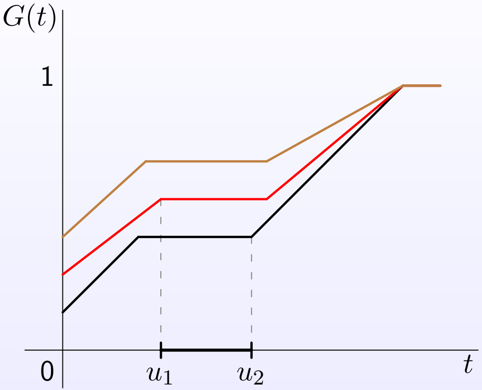
- $\pi_i(G_{-i}, t) = \delta^t(\alpha_i + e\prod_{j\neq i} G_j(t^-))$ decreasing in $[0, u_2)$

Return

Proof of Lemma 2 (No one grows alone)

Return

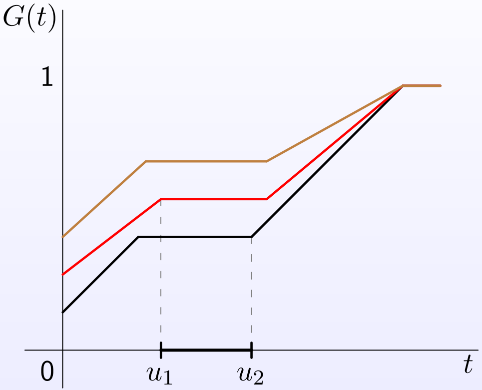
Proof of Lemma 3 (No stop&go)



-No Jumps
-No one grows alone
^{2}Common Support

Return

Proof of Lemma 3 (No stop&go)



-No Jumps
-No one grows alone
[2]Common Support

- $\pi_i(G_{-i},t) = \delta^t(\alpha_i + e\prod_{j\neq i} G_j(t^-))$ decreasing in $[u_1,u_2)$

Return

Proof of Lemma 3 (No stop&go)

Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing

Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1$, such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$

-No Jumps
-No one grows alone
^{2}Common Support
-No stop&go
^{2}Strictly Increasing

Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1, \bar{t}_2$ such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$
  - $\delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$

-No Jumps
-No one grows alone
^{2}Common Support
-No stop&go
^{2}Strictly Increasing

Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1, \bar{t}_2$ such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$
  - $\delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$
- $\frac{\delta^{\bar{t}_2}}{\delta^{\bar{t}_1}} = \frac{\alpha_2(\alpha_1+e)}{\alpha_1(\alpha_2+e)}$

-No Jumps
-No one grows alone
^{2}Common Support
-No stop&go
^{2}Strictly Increasing

Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1, \bar{t}_2$ such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$
  - $\delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$
- $\frac{\delta^{\bar{t}_2}}{\delta^{\bar{t}_1}} = \frac{\alpha_2(\alpha_1+e)}{\alpha_1(\alpha_2+e)} = \frac{\alpha_2(1-\alpha_2)}{\alpha_1(1-\alpha_1)}$

-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing

Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1, \bar{t}_2$ such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$
  - $\delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$
- $\frac{\delta^{\bar{t}_2}}{\delta^{\bar{t}_1}} = \frac{\alpha_2(\alpha_1+e)}{\alpha_1(\alpha_2+e)} = \frac{\alpha_2(1-\alpha_2)}{\alpha_1(1-\alpha_1)} > 1$

-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing

Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1, \bar{t}_2$ such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$
  - $\delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$
- $\frac{\delta^{\bar{t}_2}}{\delta^{\bar{t}_1}} = \frac{\alpha_2(\alpha_1+e)}{\alpha_1(\alpha_2+e)} = \frac{\alpha_2(1-\alpha_1)}{\alpha_1(1-\alpha_2)} > 1$
- $\bar{t}_1 > \bar{t}_2$

-No Jumps
-No one grows alone
^{2}Common Support
-No stop&go
^{2}Strictly Increasing

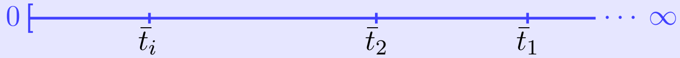
Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1, \bar{t}_2$ such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$
  - $\delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$
- $\frac{\delta^{\bar{t}_2}}{\delta^{\bar{t}_1}} = \frac{\alpha_2(\alpha_1+e)}{\alpha_1(\alpha_2+e)} = \frac{\alpha_2(1-\alpha_2)}{\alpha_1(1-\alpha_1)} > 1$
- $\bar{t}_1 > \bar{t}_2$

-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing



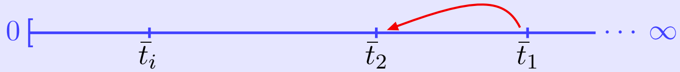
Return

Proof of Lemma 4 (Bounded Support)

How much are you willing to wait?

- Take $\bar{t}_1, \bar{t}_2$ such that
  - $\delta^{\bar{t}_1}(\alpha_1 + e) = \alpha_1$
  - $\delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$
- $\frac{\delta^{\bar{t}_2}}{\delta^{\bar{t}_1}} = \frac{\alpha_2(\alpha_1+e)}{\alpha_1(\alpha_2+e)} = \frac{\alpha_2(1-\alpha_2)}{\alpha_1(1-\alpha_1)} > 1$
- $\bar{t}_1 > \bar{t}_2$

-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing

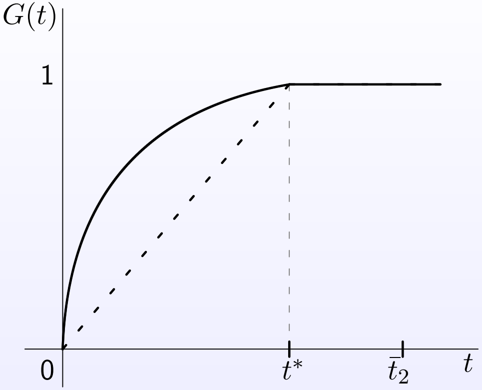


Return

Proof of Lemma 4 (Bounded Support)

Return

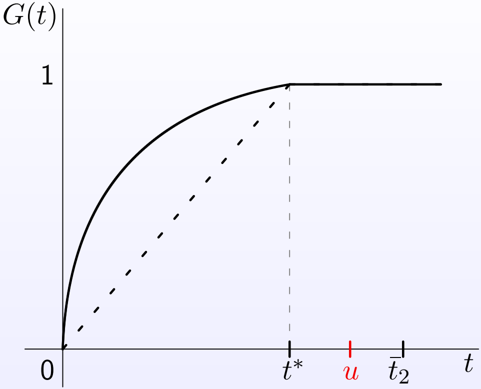
Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)



-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

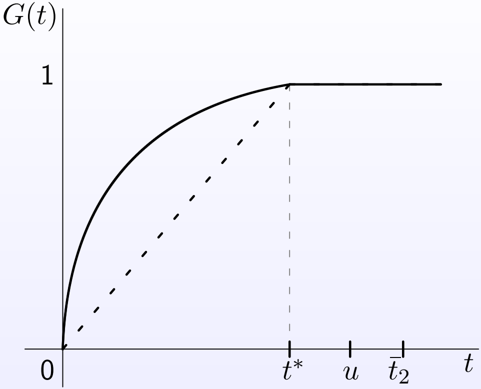


-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

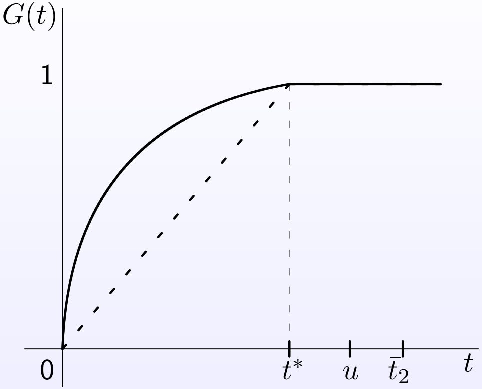


- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u))$

-No Jumps
-No one grows alone
^{2}Common Support
-No stop&go
^{2}Strictly Increasing
-Bounded support

Return

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

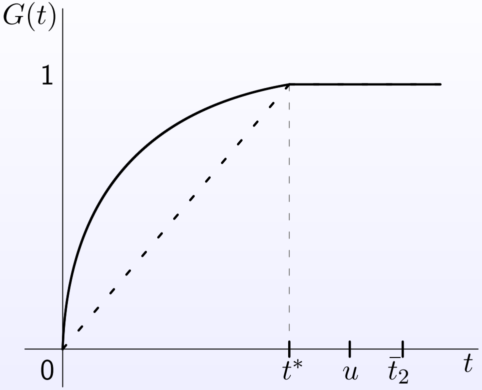


- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u)) = \delta^u(\alpha_1 + e)$

-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

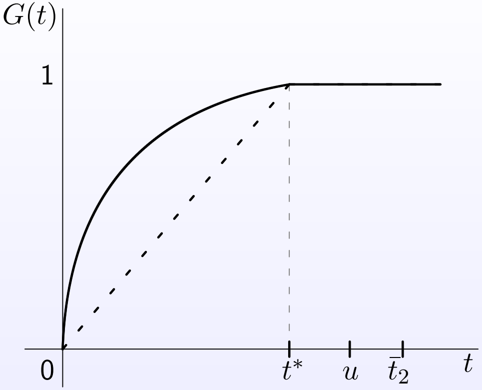


-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u)) = \delta^u(\alpha_1 + e) > \delta^{\bar{t}_2}(\alpha_1 + e)$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

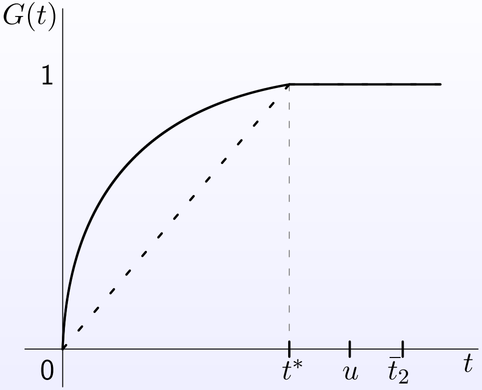


-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u)) = \delta^u(\alpha_1 + e) > \delta^{\bar{t}_2}(\alpha_1 + e) = \frac{\alpha_2(\alpha_1+e)}{\alpha_2+e} > \alpha_1$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)



-No Jumps
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[2]Common Support
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-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
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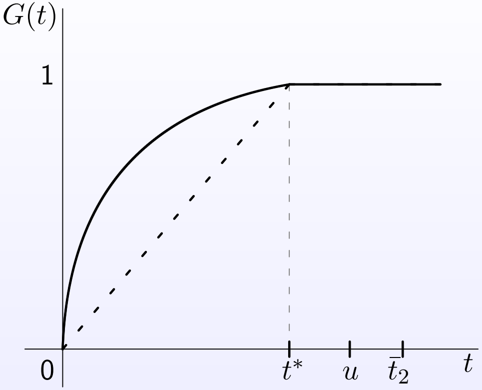
Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)



-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

◂ Return

- Take $u \in (t^*, \bar{t}_2)$
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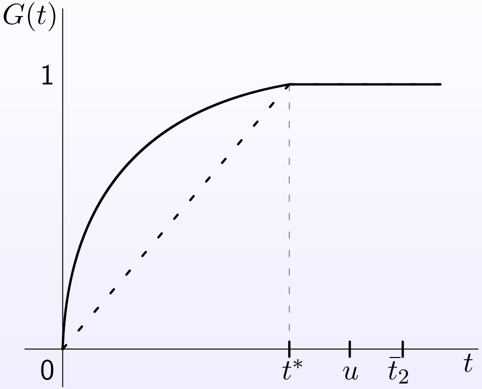
Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)



-No Jumps
-No one grows alone
[2]Common Support
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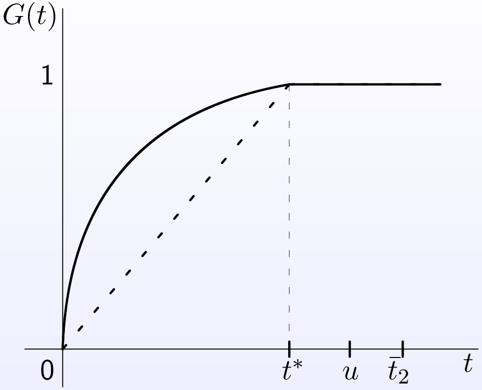
Return

- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u)) = \delta^u(\alpha_1 + e) > \delta^{\bar{t}_2}(\alpha_1 + e) = \frac{\alpha_2(\alpha_1+e)}{\alpha_2+e} > \alpha_1 = \pi_1(G_{-i}, 0)$. Player 1 puts probability 0 at 0
- Then $\pi_2(G_{-1}, t) = \delta^t(\alpha_i + eG_1(t))$ is continuous in $[0, \infty)$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)



-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
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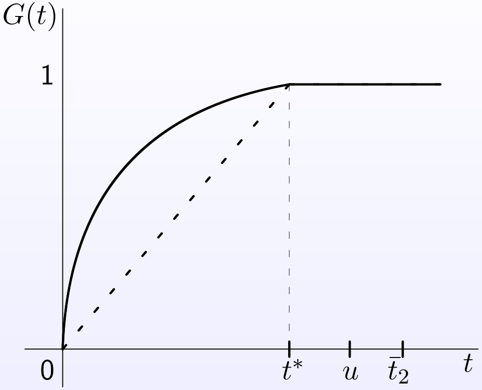
- Then $\pi_2(G_{-1}, t) = \delta^t(\alpha_i + eG_1(t))$ is continuous in $[0, \infty)$
- $\pi_2(u, G_1) = \delta^u(\alpha_2 + eG_1(u))$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)



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-No one grows alone
[2]Common Support
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[2]Strictly Increasing
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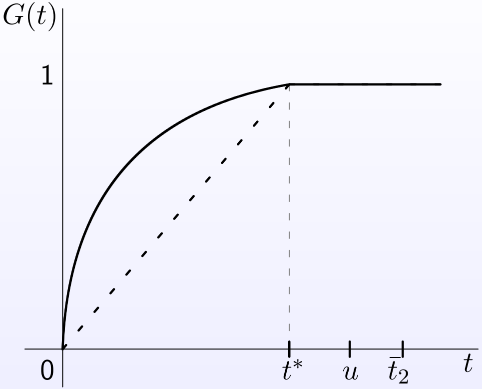
- Then $\pi_2(G_{-1}, t) = \delta^t(\alpha_i + eG_1(t))$ is continuous in $[0, \infty)$
- $\pi_2(u, G_1) = \delta^u(\alpha_2 + eG_1(u)) = \delta^u(\alpha_2 + e)$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

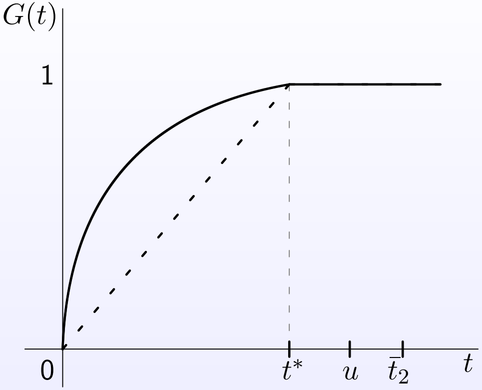


-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u)) = \delta^u(\alpha_1 + e) > \delta^{\bar{t}_2}(\alpha_1 + e) = \frac{\alpha_2(\alpha_1+e)}{\alpha_2+e} > \alpha_1 = \pi_1(G_{-i}, 0)$. Player 1 puts probability 0 at 0
- Then $\pi_2(G_{-1}, t) = \delta^t(\alpha_i + eG_1(t))$ is continuous in $[0, \infty)$
- $\pi_2(u, G_1) = \delta^u(\alpha_2 + eG_1(u)) = \delta^u(\alpha_2 + e) > \delta^{\bar{t}_2}(\alpha_2 + e)$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

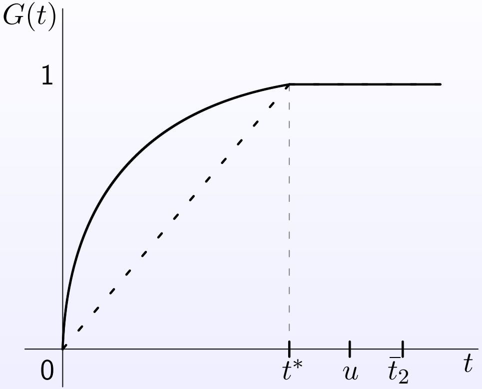


-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u)) = \delta^u(\alpha_1 + e) > \delta^{\bar{t}_2}(\alpha_1 + e) = \frac{\alpha_2(\alpha_1+e)}{\alpha_2+e} > \alpha_1 = \pi_1(G_{-i}, 0)$. Player 1 puts probability 0 at 0
- Then $\pi_2(G_{-1}, t) = \delta^t(\alpha_i + eG_1(t))$ is continuous in $[0, \infty)$
- $\pi_2(u, G_1) = \delta^u(\alpha_2 + eG_1(u)) = \delta^u(\alpha_2 + e) > \delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2$

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

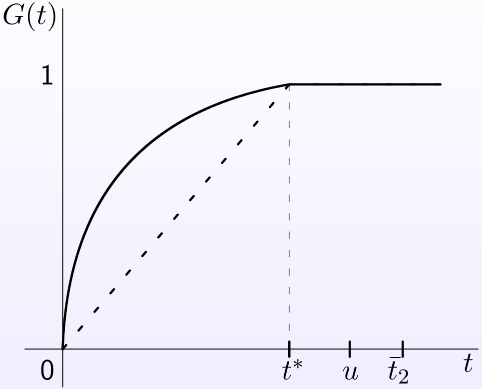


-No Jumps
-No one grows alone
[2]Common Support
-No stop&go
[2]Strictly Increasing
-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
- $\pi_1(u, G_2) = \delta^u(\alpha_1 + eG_2(u)) = \delta^u(\alpha_1 + e) > \delta^{\bar{t}_2}(\alpha_1 + e) = \frac{\alpha_2(\alpha_1+e)}{\alpha_2+e} > \alpha_1 = \pi_1(G_{-i}, 0)$. Player 1 puts probability 0 at 0
- Then $\pi_2(G_{-1}, t) = \delta^t(\alpha_i + eG_1(t))$ is continuous in $[0, \infty)$
- $\pi_2(u, G_1) = \delta^u(\alpha_2 + eG_1(u)) = \delta^u(\alpha_2 + e) > \delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2 = \pi_2(G_{-1}, 0)$.

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)



-No Jumps
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[2]Common Support
-No stop&go
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-Bounded support

Return

- Take $u \in (t^*, \bar{t}_2)$
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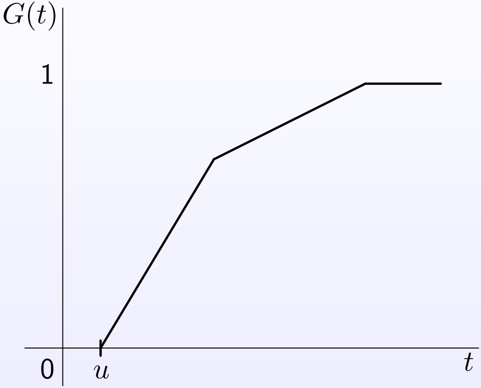
- Then $\pi_2(G_{-1}, t) = \delta^t(\alpha_i + eG_1(t))$ is continuous in $[0, \infty)$
- $\pi_2(u, G_1) = \delta^u(\alpha_2 + eG_1(u)) = \delta^u(\alpha_2 + e) > \delta^{\bar{t}_2}(\alpha_2 + e) = \alpha_2 = \pi_2(G_{-1}, 0)$.

Proof of Lemma 5 (n=2, Supports are $[0, \bar{t}_2]$)

Return

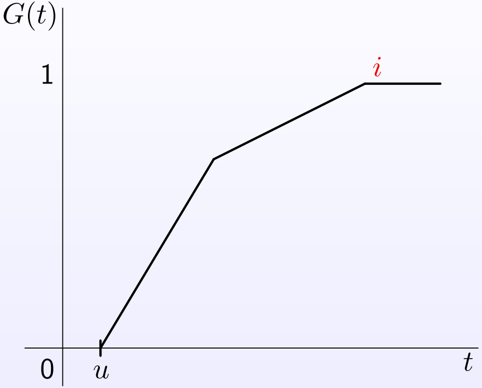
Proof of Lemma 6 (0 is in the support of every strategy)



-No Jumps
-No one grows alone
-No stop&go
-Bounded support

Return

Proof of Lemma 6 (0 is in the support of every strategy)

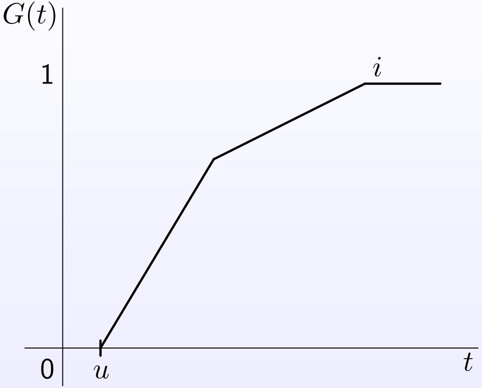


-No Jumps
-No one grows alone
-No stop&go
-Bounded support

- No j will put positive probability in $(0, u)$

Return

Proof of Lemma 6 (0 is in the support of every strategy)



-No Jumps
-No one grows alone
-No stop&go
-Bounded support

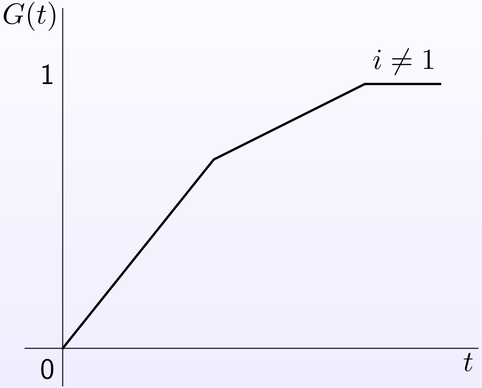
- No j will put positive probability in $(0, u)$
- $\pi_i(G_{-j}, t) = \delta^t(\alpha_i + e\prod_{j \neq i} G_j(t))$ decreasing in $(0, u)$

Return

Proof of Lemma 6 (0 is in the support of every strategy)

Return

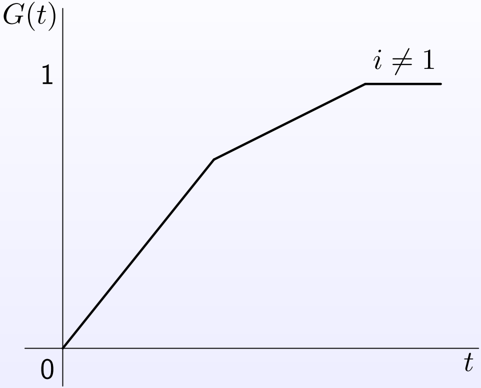
Proof of Lemma 7 (Every player but player 1 jumps at 0)



-No Jumps
-No one grows alone
-No stop&go
-Bounded support

Return

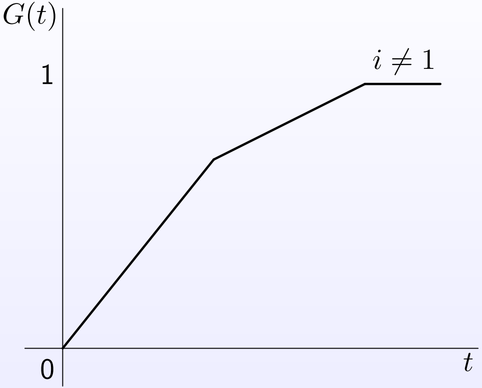
Proof of Lemma 7 (Every player but player 1 jumps at 0)



-No Jumps
-No one grows alone
-No stop&go
-Bounded support

- $\pi_1(G_{-1}, t) = \delta^t(\alpha_1 + e\prod_{j\neq 1} G_j(t))$ is continuous at 0

Proof of Lemma 7 (Every player but player 1 jumps at 0)



-No Jumps
-No one grows alone
-No stop&go
-Bounded support

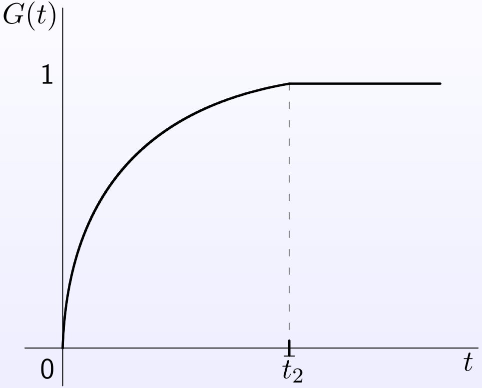
- $\pi_1(G_{-1}, t) = \delta^t(\alpha_1 + e \prod_{j \neq 1} G_j(t))$ is continuous at 0
- $\pi_1(G_{-1}, \bar{t}_2) = \delta^{\bar{t}_2}(\alpha_1 + e) > \alpha_1$

Return

Proof of Lemma 7 (Every player but player 1 jumps at 0)

Return

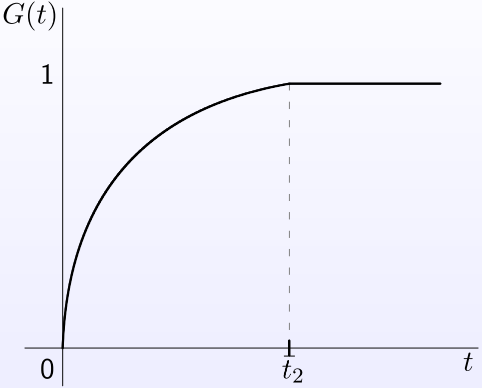
Proof of Lemma 8 (Nash Payoffs)



- No Jumps
- No one grows alone
- No stop&go
- Bounded support
- 0 is in the support
- $i \neq 1$ jumps at 0

Return

Proof of Lemma 8 (Nash Payoffs)

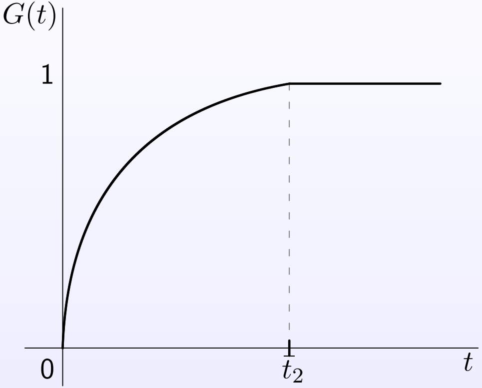


-No Jumps
-No one grows alone
-No stop&go
-Bounded support
-0 is in the support
-$i \neq 1$ jumps at 0

- Player 1 can ensure himself $\bar{\pi}_1$ by playing $\bar{t}_2$

Return

Proof of Lemma 8 (Nash Payoffs)



-No Jumps
-No one grows alone
-No stop&go
-Bounded support
-0 is in the support
-$i \neq 1$ jumps at 0

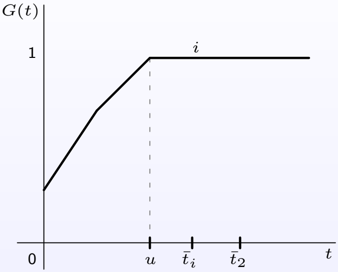
- Player 1 can ensure himself $\bar{\pi}_1$ by playing $\bar{t}_2$
- He cannot get more than that in equilibrium

Return

Proof of Lemma 8 (Nash Payoffs)

Return

Proof of Lemma 9 (Players $3, \ldots, n$ play $t = 0$)



- -No Jumps
- -No one grows alone
- -No stop&go
- -Bounded support
- -0 is in the support
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◂ Return

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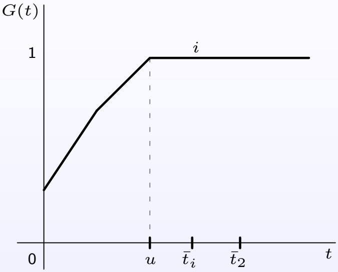

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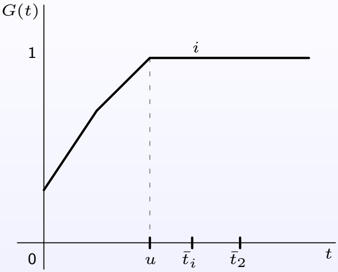

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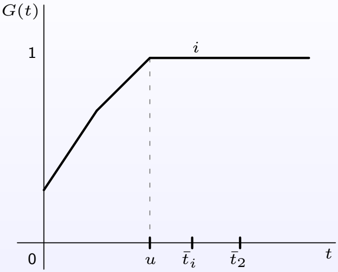

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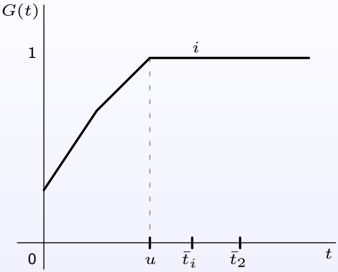

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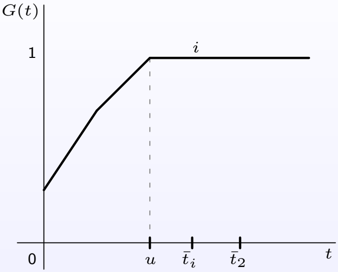

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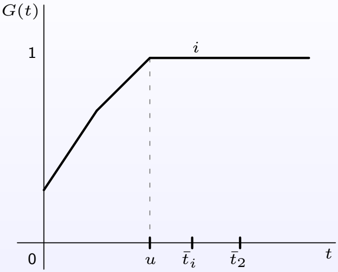

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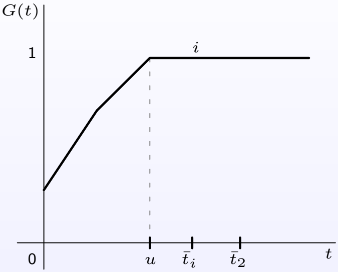
Return

Proof of Lemma 9 (Players $3, \dots, n$ play $t = 0$)



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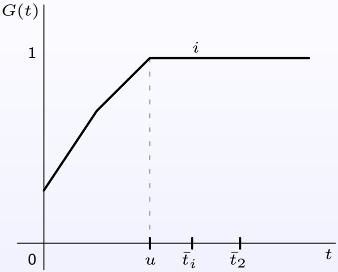
Then $G_2(u) > G_i(u) = 1$

Return

Proof of Lemma 9 (Players $3, \ldots, n$ play $t = 0$)



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Then $G_2(u) > G_i(u) = 1$, contradiction.

Return

Proof of Lemma 9 (Players $3, \ldots, n$ play $t = 0$)

Return

Index of results

- **Theorem 1:** 2-player result ▸ Proof
- **Theorem 2:** n-player result ▸ Proof
- **Lemma 1:** No jumps ▸ Proof
- **Lemma 2:** No one grows alone ▸ Proof
- **Lemma 3:** No stop&go ▸ Proof
- **Lemma 4:** Bounded support ▸ Proof
- **Lemma 5:** The supports are $[0, \bar{t}_2]$ ▸ Proof
- **Lemma 6:** 0 is in the support of every strategy ▸ Proof
- **Lemma 7:** Every player but player 1 jumps at 0 ▸ Proof
- **Lemma 8:** Nash Payoffs ▸ Proof
- **Lemma 9:** Players $3, \ldots, n$ play $t = 0$ ▸ Proof