Loss allocation in energy transmission networks

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Abstract

In this paper we study a cost allocation problem that is inherent to most energy networks: the allocation of losses. In particular, we study how to allocate gas losses between haulers in gas transmission networks. We discuss four allocation rules, two of them have already been in place in real networks and two others that are defined for the first time in this paper. We then present a comparative analysis of the different rules by studying their behavior with respect to a set of principles set forth by the European Union. This analysis also includes axiomatic characterizations of two of the rules. Finally, as an illustration, we apply them to the Spanish gas transmission network.

Keywords. Gas transmission networks, loss allocation, cost allocation, management

1 Introduction

We study the allocation of losses in energy transmission networks, in which the energy (gas, electricity,...) is sent through pipes from suppliers to consumers. A common problem is that, in virtually any network, there are losses whose sources are normally difficult to identify. Thus, one must anticipate them so that they do not lead to deficit in the system. In many cases the transmission network is owned by different agents and, typically, the authorities that manage the network decide how much energy each agent is allowed to lose. This decision should follow some general principles, which would then appear in the relevant regulations. For instance, one would like that the loss allocated

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to each agent takes into account characteristics of the agents, such as the size of its subnetwork or the amount of energy managed. In this paper we study rules for allocating losses among the agents. Even though our results could be applied to any transmission network with losses, we use gas transmission networks and, in particular, the Spanish network, as our leading example. Interestingly, we include a section in which the analysis developed in the rest of the paper is illustrated on this real gas network. Natural gas is an important energy resource, whose usage has increased very significantly over the last three decades. According to data from the EIA (United States Energy Information Administration), between 1980 and 2010, consumption of natural gas world wide rose from 53 million cubic feet to 113 million, leading to a 23.9% share of global primary energy consumption (British Petroleum, 2013). As a consequence of this, there is an increasing need for construction and expansion of gas transmission networks and, more importantly, an increasing need for its efficient management and operation.

Different networks have different estimates on the percentage of gas/electricity that is lost during transportation. In Spain, for instance, this estimate is 0.2% for the gas transported in the high pressure gas network and similar figures have been reported in other countries.[1] In order to prevent the ensuing monetary losses, a standard approach in energy networks is to withhold at the entry points a pre-set percentage of the gas/electricity entering the network; by doing this, the energy companies that use the network for transportation are the ones effectively assuming the associated cost in the first instance. In particular, in the Spanish high pressure gas network the pre-set percentage withhold to anticipate the estimated losses is precisely 0.2%. In monetary terms, the annual cost of the gas entering the Spanish gas network is around 1200 millions of Euro,[2] which results in approximately 25 millions of Euro in losses in the transmission network.

It is precisely at this point where the main question we try to address in this paper arises. Since a gas network is typically owned by different agents, called haulers, it must be decided how to share the withhold gas among them. More precisely, it must be decided, for each agent, the percentage of the gas entering his subnetwork that can lost. Note that it is not possible to let each agent lose the same percentage that has been withhold for the entire network. Since most gas entering the network crosses several subnetworks, this naive approach would result in allowing the agents to lose, in aggregate, more gas than the withhold amount.

To illustrate, consider the network depicted in Figure 1. There are two supply nodes, s_1 and s_2, and a demand node, c_1. There are three haulers in this network and v and f denote, respectively, a pipe's volume and the units of gas that flow through it. Since the network is transporting 1200 units of gas, according to the 0.2% mentioned above, it is estimated that 2.4 units of flow will be lost in the transmission process. The question is, how much of this loss is allowed to each hauler? We cannot assign to each of them 0.2%

[1] See Article 17.c) in Boletín Oficial del Estado (2013a) for the Spanish regulation and ERGEG (European Regularors Group for Electricity and Gas) (2008) and Comisión Nacional de la Energía (2006) for an overview of these estimates in different countries in both the gas and the electricity networks.

[2] Estimate based on the information provided by the Spanish Technical System Manager (Enagás GTS, 2013) and on a gas price of 30000 €/GWh.

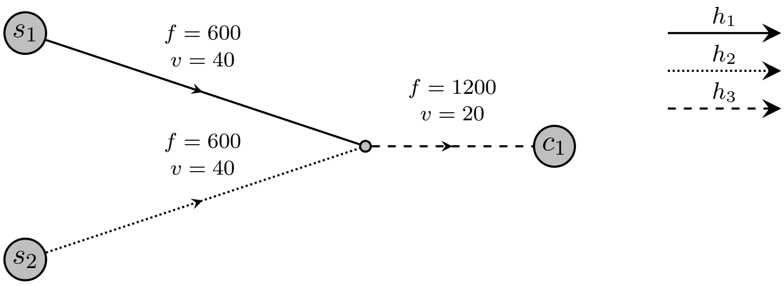


Figure 1: Example of the gas loss allocation problem.

of the gas he is carrying because that would result in 1.2, 1.2, and 2.4 being allocated to them, which results in a total of 4.8 units being allocated while the amount to allocate is just 2.4. A sensible alternative is to split the 2.4 units proportionally to the flows, so that hauler h_3 is assigned a loss of 1.2 units of flow and h_1 and h_2 a loss of 0.6 each. Yet, one can argue that the gas in the pipe of h_3 is covering half the distance (assuming that all pipes have the same diameter) and that the assigned loss should also reflect this fact, leading to 0.8 being allocated to each hauler. Even in a small network is not entirely obvious how the gas loss should be allocated, and other considerations arise for more general networks.

The European Union has already set forth some principles that should be pursued with the national and international regulations regarding the natural gas market. One of the main documents in this respect is Regulation (EC) (no. 55/2003), and some relevant principles mentioned there are non-discrimination, cost-reflectivity, and fostering competition. We formalize these principles with several properties, some of which we outline below, but first we build upon the Spanish system to be more explicit about the kind of mechanisms normally used regarding loss allocation. The Spanish regulations are designed trying to follow the principles of the European Union. In particular, Boletín Oficial del Estado (2013b, page 106656) presents the incentive mechanism to induce haulers to reduce the losses which we now outline. At the end of every period of one year the following values are computed:

- In view of the total amount of gas withhold in the network and following a well defined rule, the 'allowed' loss assigned to each hauler h, A_h.
- The real loss of each hauler h, L_h, is computed as the balance between entries and exits of gas in his subnetwork.
- Given a price p per unit of gas, the haulers pay $p(L_h - A_h)$ when $A_h - L_h < 0$ and, otherwise, they get $\frac{p}{2}(L_h - A_h)$.

Given this mechanism, it should be apparent that the definition of the rule to assign the 'allowed' losses is a relevant issue for the management of gas transmission networks.

There is an important observation regarding the impact of the rule on the hauler's behavior. One may argue that, when designing one such rule one should take into account not only the fairness considerations appearing in the European regulations, but also the impact the chosen rule might have on the hauler's incentives. Yet, we consider that in this specific setting there is not much room for the latter. Once the entries and exits of gas in the network are fixed, the network operation is essentially determined by the Technical System Manager, and there is not much a hauler can do to influence the final value of A_h.[3] Therefore, given that final payments due to losses depend on $L_h - A_h$ and the above arguments regarding the (lack of) control of haulers over A_h, the best a hauler can do, regardless of the chosen rule, is to try to reduce the losses in his subnetwork (as intended by the mechanism).

We now discuss the principles mentioned in the European regulation. We start with the principle of cost-reflectivity. *Independence of unused edges* says that if we remove from the network pipes without flow, then the loss allocated to the haulers does not change. Suppose that a hauler has a pipe from node i to node j and a pipe from j to k. Besides, node j does not belong to more pipes. Thus, we can represent the gas network with two edges, (i, j) and (j, k), or with one edge, (i, k). *Independence of edge sectioning* says that the rule does not depend on the chosen representation. Suppose that a hauler has several pipes from node i to node j, so we can represent the gas network with several edges from i to j or with a unique edge from i to j where the volume of this edge is the sum of the volumes of all pipes from i to j. *Independence of edge multiplication* says again that the rule does not depend on the chosen representation. *Independence by sales* says that if one hauler sells some of his edges to another hauler, then the loss allocated to the rest of the haulers does not change. *Independence of irrelevant changes* says that if there is a change in the gas network that does not affect to the gas that flows through the subnetwork of a hauler, then the loss allocated to such hauler should not change.

We move now to non-discrimination. The properties we define regarding this principle are based on the notion of proportionality. *Flow proportionality on edges* says that the loss allocated to two haulers, each one owning a unique edge with the same volume, should be proportional to the flow of their edges. Consider two haulers, each owning a unique edge and transporting the same amount of gas. *Volume proportionality on edges* says that the loss allocated to both haulers should be proportional to the volumes of their edges. This property does not take into account the position of the edges in the gas network. *Volume proportionality on paths* does it by asking for the proportionality on volumes when both edges have the same flow and the same "position" in the gas network. Namely, when the edges belong to the same paths (from supply nodes to demand nodes).

Finally, fostering competition is modeled by *merging-proofness*, which says that if two haulers merge in a new hauler, then the total loss allocated to the new hauler is not larger than the sum of the losses allocated to the original haulers. It is worth noting that this property also has a strong connection with the principle of non-discrimination, since a rule that does not satisfy merging-proofness discriminates small haulers.

In this paper we discuss four different loss allocation rules: the one used in Spain

[3] Changes in the network of a hauler that might influence A_h, such as changes in the volumes of the pipes, normally require an authorization of the corresponding national authorities and, therefore, are not easy to implement.

until 2013 (called *aggregate edge's rule*), the one used since 2014 (called *flow's rule*), and two other rules we define (called *edge's rule* and *proportional tracing rule*). As we have mentioned above, it is generally difficult to identify the sources of gas losses. There have been qualified experts arguing that losses depend mainly on the flows, while other experts have argued that they depend in a multiplicative way on the flow and volume of the pipes. Our rules cover both situations. The flow's rule divides the loss proportionally to the gas transported by each hauler. Thus, it only considers the flow. The other three rules divide the loss considering both flow and volume. The aggregate edge's rule divides the loss proportionally to the product of the volume and the gas transported by each hauler. The edge's rule is based on a similar idea. First, the loss is divided among the edges proportionally to the product of the flow and the volume of each edge. Then, the loss allocated to each hauler is the sum of the losses allocated to his edges. The proportional tracing rule is defined in a more elaborate way. First it allocates the loss among the different paths connecting supply nodes with demand nodes. Next, the loss assigned to each path is divided among the edges of the path proportionally to their volumes. Finally, the loss allocated to each hauler is the sum the loss assigned to his edges in each path.

Next, the paper presents a detailed analysis of the behavior of the rules with respect to the set of properties which are in turn related to the aforementioned European Union principles. One of the conclusions of our analysis is that the rule that exhibits worst behavior with respect to the EU principles is the aggregate edge's rule, the one that was in place in Spain until 2013. Interestingly, this rule was replaced by the flow's rule because of the strong opposition of most of the haulers (on the grounds that it favored big haulers). We find that the proportional tracing rule and the edge's rule are better than the flow's rule (in terms of the EU principles), with the former seeming slightly preferable.

We present axiomatic characterizations of two of the rules under study. The edge's rule is characterized with independence of edge sectioning, independence by sales and flow proportionality on edges. The proportional tracing rule is characterized with independence of unused edges, independence of edge multiplication, independence by sales, volume proportionality on paths and tracing additivity (a weak additivity property).

We also present an illustration of the different rules in the Spanish gas network, which is owned by a big hauler (owning 90% of the gas network) and six small haulers. We see that the change from the aggregate edge's rule to the flow's rule leaves the big hauler worse off and the small haulers better off. Actually, the rule was changed because the small haulers complained, arguing than the former Spanish rule unfairly favored the big hauler (and we agree with these arguments). We also note that there are significant differences in the allocations proposed by the rules, with the maximum gap we observed for a hauler having an annual monetary equivalent of almost 10 million Euro. Therefore, the issue of selecting a fair allocation rule can be very important for the haulers.

The paper is structured as follows. In Subsection 1.1 we review the literature more related to our work. In Section 2 we present a brief introduction to some relevant characteristics of the management and operation of a gas transmission network. In Section 3 we present the formal mathematical model. Sections 4 and 5 are devoted to the definitions of the rules and properties, respectively. In Section 6 we discuss the behavior of the rules with respect to the properties and EU principles. In Section 7 we present two axiomatic characterizations. Section 8 contains an illustration of the rules in the Spanish gas transmission network. In Section 9 we present some conclusions. For the sake of exposition, all proofs have been relegated to the Appendix.

1.1 Related literature

We study the problem of cost allocation in networks. See Sharkey (1995) for a survey on networks models in economics. We follow the axiomatic approach, where rules are compared in terms of the axioms (properties) they satisfy. See Thomson (2001) for an overview of the axiomatic method and Moulin (2002) for a survey of the axiomatic method in cost allocation problems.

Some papers like Moulin and Shenker (1992) and Sprumont (1998) study general cost allocation problems. Other papers study cost allocation problems associated with some network structure. We mention some of them. In Littlechild and Owen (1973) it is allocated the construction cost of a landing strip. In Ni and Wang (2007) it is allocated the cost of cleaning a polluted river. In Bergantiños and Vidal-Puga (2007) it is allocated the cost of connecting all agents to a source. In Bogomolnaia et al. (2010) it is allocated the cost of a network connecting a group of agents. In Moulin and Laigret (2011) it is allocated the cost of some resources among a set of agents. In Estevez-Fernandez (2012) it is allocated the penalty of delaying a project. In Bergantiños and Martínez (2014) it is allocated the maintenance cost of a network connecting some agents. In all these problems, and also in this paper, some cost allocation rules are studied in terms of the properties they satisfy. In some of these papers the rule studied is the Shapley value of an associated cooperative game. In Bergantiños et al. (2014), an extended version of this paper, we also consider a rule based on the Shapley value. This Shapley rule is technically more complicated and does not exhibit a specially good performance with respect to the properties we study.

Loss allocation has received a lot of attention in the electricity sector (see Kyung-Il et al. (2010), Conejo et al. (2002), Galiana et al. (2002), and references therein). However, most of the effort there concentrates on defining algorithms that allow to estimate the sources of the losses which would then make the “allocating task” straightforward. As far as we know, the former identification is much harder in gas networks and there are no such algorithms available. Maybe more importantly, we have found no paper developing a formal analysis of the properties of the different methods. The closest we have found to an axiomatic analysis is Lima and Padilha-Feltrin (2004), where the authors compare different allocation methods by means of their behavior in a series of examples. Interestingly, there are several papers that use game theoretical models to define new loss allocation methods, but do not build upon them to develop axiomatic analysis (Molina et al., 2010; Lima et al., 2008).

2 The underlying gas network problem

A gas network is formed by nodes and pipes. Some nodes are demand nodes, at which some gas leaves the network. Some nodes are supply nodes, from which the gas enters the network. The rest of the nodes are simply points at which two or more pipes intersect. Each pipe belongs to a hauler and each hauler may own several pipes. The gas network operation is decided by the Technical System Manager. Once the System Manager knows the demand of gas in each demand node he decides, following some criteria, the amount of gas that should be introduced in each supply node and how to route it so that the total demand is fulfilled.

Naturally, the most important element of our cost allocation model is the gas network, which we assume is in steady state, *i.e.*, the gas flowing through each pipe and the pressure at each node are constant.[4] Then, for the purposes of this paper, in order to have the network configuration completely specified we need to know, for each pipe, its volume and the amount of gas flowing through it. The flow represents the total amount of energy each pipe carries during a given period of time (which, when needed, we represent as GWh/d). In particular, it is worth noting that, as far as this paper is concerned, there is no direct connection between the volume of the pipe and the amount of gas that can flow through it. The volume of a pipe just depends on its length and its diameter and, since natural gas is a compressible fluid, the capacity of a pipe also depends on the construction materials and the maximum pressure they can support.

Ideally, the chosen flow configuration should be based on some realistic scenario of demands and operating regime. In energy networks it is customary to work with reference scenarios with high/peak demand and we will do so when working with the Spanish gas network in Section 8. Yet, this is not critical for the normative analysis in this paper. Indeed, once a methodology is chosen to allocate the losses, it can be run on a daily basis if needed to ensure that the final allocations stem from representative network configurations.

Given a network configuration and a percentage estimate for the gas loss, one can obtain an estimate for the total loss of the system during the given period. Suppose such a loss is L. Then, this total loss L has to be allocated among the haulers, conditioning on the current network configuration. Let A_h be the loss assigned to hauler h and let L_h be the real loss measured in the subnetwork of hauler h during this period. Then, the hauler is penalized if $A_h - L_h < 0$ and rewarded otherwise. As we already mentioned in the Introduction, L can be of the order of millions of Euro (around 25 million in the Spanish network) and so the way L is allocated is very important for the haulers.

In the allocation rules we study in this paper A_h is computed as a function of the flows and volumes of the pipes in the network. Thus, one may wonder to what extent a hauler can manipulate these parameters in order to influence the final allocation. As we mentioned above, the flow configuration is beyond their control, since it is determined by the Technical System Manager. Importantly, energy networks are under a strict control

[4] The steady state assumption is not realistic for real time analysis of the network operation but, since steady state modeling is much simpler, it is the standard approach for medium and long term analysis of energy networks.

of the corresponding national authorities, who must approve any change in the topology of the network. In particular, the modification of the volume of existing pipes or the construction of new ones is something a hauler cannot decide on his own.

3 The mathematical model

Let $U = \{1, 2, 3, \ldots\}$ be the (infinite) set of possible *nodes*. A *graph* is a pair $g = (N, E)$ where $N \subset U$ is the (finite) set of nodes and E is a collection of ordered pairs in N, *i.e.*, $E \subset \{(i, j) : (i, j) \in N \times N \text{ and } i \neq j\}$. The pairs (i, j) are called *edges*. More generally, a multigraph is also a pair $g = (N, E)$, but where the set of edges is a multiset $E \subset N \times N \times \mathbb{N}$. In particular, we say that two edges (i, j, n) and (i', j', n') are part of a multiedge if $i = i'$, $j = j'$, and $n \neq n'$. We say that E does not have multiedges if the projection of E on $N \times N$ is injective.

A *path* in g between i and j is a sequence of $l > 1$ nodes $\{k_1, \ldots, k_l\}$ such that $i = k_1$, $j = k_l$, and $(k_{q-1}, k_q) \in E$ for all $q \in \{2, \ldots, l\}$. A *simple path* in g between i and j is a path where all nodes are different. For the sake of notation we often identify a path with the set of edges $\{(k_{q-1}, k_q)\}_{q \in \{2,\ldots,l\}}$. A graph g is *connected* if for each pair of nodes i and j there is a path between i and j in the non-oriented version of g. We avoid the trivial extension of these definitions for multigraphs.

A *gas loss problem* G is a 5-tuple $(g, v, f, \mathcal{H}, \alpha)$ where

i) The multigraph $g = (N, E)$ represents the *gas network*.

   We assume that g is a connected graph without cycles modeling the way in which the gas flows. If $e = (i, j, l) \in E$, then there may be gas flowing from i to j.

ii) $v = (v_e)_{e \in E}$ where for each $e \in E$, $v_e > 0$ denotes the *volume* of e.

iii) $f = (f_e)_{e \in E}$ is the *flow configuration* where, for each $e \in E$, $f_e \geq 0$ denotes the instantaneous flow of gas through e. There is some flow of gas, *i.e.*, $\sum_{e \in E} f_e > 0$.

iv) $\mathcal{H} = (H, \{E_h\}_{h \in H})$ is the *hauler structure*, where H denotes the set of haulers and, for each $h \in H$, E_h denotes the (possibly empty) set of edges of hauler h. In particular, $E = \bigsqcup_{h \in H} E_h$.

v) $\alpha \in [0, 1]$ denotes the *proportion of gas* allowed to be lost by the set of haulers.

We present an example of a gas problem below but, before that, we make some observations and assumptions:

- For the sake of notation simplicity, we work with graphs instead of multigraphs, and explicitly refer to the later when they can make a difference.

- We assume that the set of haulers H is infinite, although only a finite number of them will effectively own some edge for each given problem. We do it because we want to be able to model situations in which a hauler sells one of its edges to another hauler in H having no edge. This assumption simplifies the notation. In the examples we only mention the haulers having some edges.

Example 1. Let G be the gas problem where

i) $g = (N, E)$ where $N = \{s_1, s_2, 1, c_1, c_2\}$ and $E = \{(s_1, 1), (1, c_1), (s_2, 1), (1, c_2)\}$.

ii) $v_{(s_1,1)} = v_{(s_2,1)} = v_{(1,c_1)} = v_{(1,c_2)} = 100$.

iii) $f_{(s_1,1)} = 20$, $f_{(s_2,1)} = 80$, $f_{(1,c_1)} = 60$, and $f_{(1,c_2)} = 40$.

iv) $\mathcal{H} = (H, \{E_h\}_{h\in H})$, where $H = \{h_1, h_2, h_3\}$ and $E_{h_1} = \{(s_1, 1), (1, c_1)\}$, $E_{h_2} = \{(s_2, 1)\}$, and $E_{h_3} = \{(1, c_2)\}$.

v) $\alpha = 0.1$.

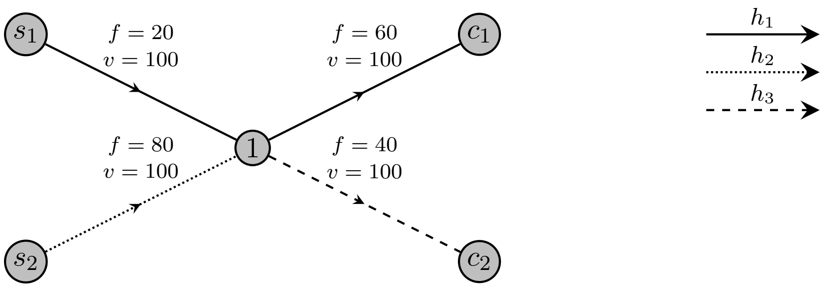


Figure 2: Representation of the gas problem in Example 1.

This gas problem is represented in Figure 2 and will be used extensively as a running example to illustrate the different concepts and definitions. ◇

We now introduce some terminology. For each $i \in N$, we denote by Q_i the gas balance at node i, *i.e.*, the amount of gas leaving node i minus the amount of gas arriving at node i. Formally,

$$Q_i = \sum_{(i,j)\in E} f_{(i,j)} - \sum_{(j,i)\in E} f_{(j,i)}.$$

The set of *suppliers* $S \subset N$ of the gas problem G is defined as the set of nodes $s \in N$ such that $Q_s > 0$. On the other hand, the set of *consumers* $C \subset N$ is defined as the set of nodes $c \in N$ such that $Q_c < 0$. For the rest of nodes $i \in N \setminus (S \cup C)$, we have that $Q_i = 0$. We make the natural assumption that total supply and total demand are balanced, namely,

$$\sum_{s\in S} Q_s = -\sum_{c\in C} Q_c \quad \text{or, equivalently,} \quad \sum_{i\in N} Q_i = 0.$$

The total loss allowed to the haulers is $L = \alpha \sum_{s\in S} Q_s$. The *flow carried* by each hauler $h \in H$, denoted by f_h, is defined as the gas that reaches one of the edges of hauler h from outside, that is, from some provider $s \in S$ or from an edge of another hauler. Formally, we first define, for each node $i \in N$ and each hauler $h \in H$, $Q_i^h =$

$\max\{\sum_{(i,j)\in E_h} f_{(i,j)} - \sum_{(j,i)\in E_h} f_{(j,i)}, 0\}$; if no edge of hauler h contains node i we define $Q_i^h = 0$. Then, for each $h \in H$,

$$f_h = \sum_{i\in N} Q_i^h.$$

In particular, $f_h = 0$ whenever $E_h = \emptyset$.[5]

Given a gas problem G and a pair $(s,c) \in S \times C$, we define $P(s,c)$ as the set of simple paths in g from s to c. We denote by $P(S,C)$ the set of all simple paths from suppliers to consumers. Namely,

$$P(S,C) = \bigcup_{(s,c)\in S\times C} P(s,c).$$

We now want to define an important notion for our analysis that we call *hauler's influence network*, which, given a hauler h, would contain all edges whose gas might either reach some edge in E_h or come from some edge in E_h. Formally, for each $h \in H$, we define $\mathcal{N}^h = (g^h, v^h, f^h)$, as the subnetwork of (g, v, f) where $g^h = (N^h, E^h)$ and

$$\begin{aligned}
E^h &= \{e \in E : \text{ there is } p \in P(S,C) \text{ with } e \in p \text{ and } p \cap E_h \neq \emptyset\},\\
N^h &= \{i \in N : i \in e \text{ for some } e \in E^h\},\\
v^h &= (v_e)_{e\in E^h},\\
f^h &= (f_e)_{e\in E^h}.
\end{aligned}$$

Sometimes we slightly abuse language and refer to an *edge's influence network*, to mean the influence network that would have a hauler who owned only that edge. Note that two edges with the same influence network belong to the same paths and, therefore, must carry the same flow.

Example 1. (cont.) Going back to the gas problem in Figure 2, we have that $Q_{s_1} = 20$, $Q_{s_2} = 80$, $Q_1 = 0$, $Q_{c_1} = -60$, and $Q_{c_2} = -40$. Thus, $S = \{s_1, s_2\}$ and $C = \{c_1, c_2\}$. If we compute Q_i^h we have the following table:

Q_i^h	s_1	s_2	1	c_1	c_2	f_h
h_1	20	0	40	0	0	60
h_2	0	80	0	0	0	80
h_3	0	0	40	0	0	40

The influence networks corresponding to this example are represented in Figure 3. ◇

3.1 Flow tracing methods

Given a gas problem G, we know the amount of gas flowing through each edge of the network. Ideally, we would also like to know how much of this gas comes from each supplier and how much goes to each consumer. Unfortunately, tracing the gas in a

[5] There are alternative ways to define the notion of "flow carried by a hauler", but, as far as our analysis is concerned, they would lead to similar results. Our formulation is the one implicit in the Spanish Regulations (Boletín Oficial del Estado, 2011, 2013b).

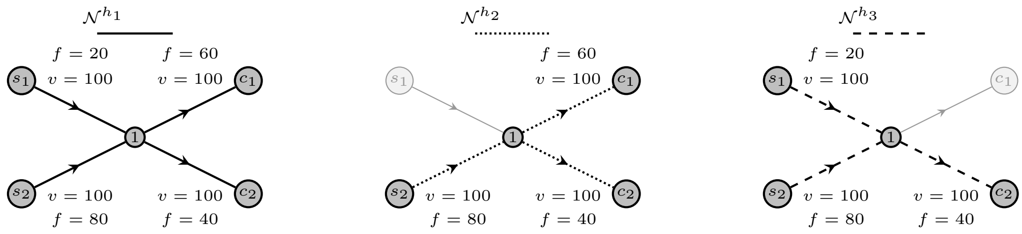


Figure 3: Illustration of the hauler's influence networks of Example 1.

network is far from being a trivial physical problem and, to the best of our knowledge, one has to settle for some approximations.

A *tracing method*, Γ, describes how the gas arriving at a given node is split towards the different outbound destinations. Once this is known, a *tracing method* can be used to obtain, for each pair $(s, c) \in S \times C$ and each $p \in P(s, c)$, an estimation of the amount of gas f_p^Γ going from s to c through path p.[6] For this last part, one has to build upon the natural assumption that the gas that enters a given pipe mixes to form a completely homogeneous gas. To illustrate, consider a situation where the gas of several (incoming) pipes meets at a given node and then is split in several outbound pipes. Let e_1 be one of the incoming pipes and e_2 be one of the outbound pipes. The tracing method delivers the proportion q of the gas flowing through e_2 that comes from e_1. Suppose that, somewhere else down the network, e_2 is an incoming pipe at some other node and its gas is split as well in several outbound pipes, one of them being e_3. Again, the tracing method pins down the proportion $\bar{q}$ of the gas flowing through e_3 that comes from e_2. The issue now would be to determine the proportion of the gas flowing through e_3 that comes from e_1. The homogeneity assumption on the gas flowing through e_2 immediately leads to the conclusion that $q\bar{q}$ is the proportion of the gas flowing through e_3 that comes from e_1.

Figure 4 represents the relevant information to define a tracing rule: inbound and outbound flows. In particular, it does not depend on the rest of the topology of the network, the haulers owning the different pipes, or the volumes of the pipes.

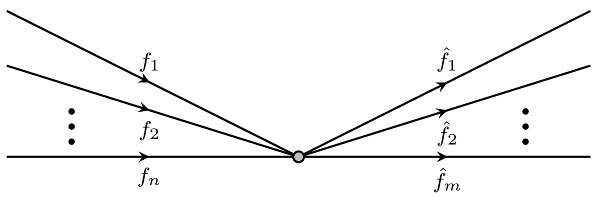


Figure 4: A tracing method only depends on the inbound and outbound flows.

Now we present a natural tracing method, referred to as the *proportional tracing method*, Γ^{pt}, introduced in Bialek (1996) and whose idea is that the incoming flow at a node is split on the outbound edges proportionally to their flows. Interestingly, this

[6]Then, for each $e \in E$, we would be able to recover f_e as $\sum_{p \in P(S,C),\, e \in p} f_p^\Gamma$.

method has already been used to study the allocation of losses in electricity networks (Conejo et al., 2002; Bialek and Kattuman, 2004).[7] In Bialek and Kattuman (2004) the authors write "This assumption can be neither proved nor disproved physically" and try to "rationalize" it. The proportional tracing method has also appeared in the context of gas networks (see, for instance, Alonso et al. (2010)).

Consider a node as the one in Figure 4. Denote by e_i the inbound "edge" with flow f_i and by $\hat{e}_j$ the outbound one with flow $\hat{f}_j$.[8] Then, for each $i \in \{1, \dots, n\}$ and each $j \in \{1, \dots, m\}$, the proportional tracing method computes the amount of gas coming through e_i that is leaving through $\hat{e}_j$ as

$$\frac{f_i}{\sum_{l=1}^{n} f_l} \hat{f}_j.$$

Example 1. (cont.) We illustrate the proportional tracing method using again our running example.

- If we consider the 60 units of flow of edge $(1, c_1)$, they are split so that $\frac{20}{20+80} 60 = 12$ come from $(s_1, 1)$ and $\frac{80}{20+80} 60 = 48$ come from edge $(s_2, 1)$.
- Similarly, the 40 units of edge $(1, c_2)$ are split so that $\frac{20}{20+80} 40 = 8$ come from edge $(s_1, 1)$ and $\frac{80}{20+80} 40 = 32$ come from edge $(s_2, 1)$. $\diamond$

Concerning how the flow is split in the different paths, we would have

(s, c)	$P(s, c)$	$f_p^{\Gamma^{\text{pt}}}$
(s_1, c_1)	$\{(s_1, 1), (1, c_1)\}$	12
(s_1, c_2)	$\{(s_1, 1), (1, c_2)\}$	8
(s_2, c_1)	$\{(s_2, 1), (1, c_1)\}$	48
(s_2, c_2)	$\{(s_2, 1), (1, c_2)\}$	32

4 Rules

The main question we study in this paper is how to allocate the loss allowed by the regulatory authority, L, among the haulers. We present several allocation rules, one of them in place in Spain. Another one was used in Spain from 2011 until 2013.

Identifying the source of such losses is a very complex physical problem. The loss may come from the different active elements of the network such as valves, compressors, regulation and measurement points. Indeed, even the measurement precision is

[7] Even though these papers apply the proportional tracing method for estimating the way in which the electricity flows, the approach in their setting is different from ours. Because of the physical differences between gas and electricity networks, in the later the tracing methods allow to pin down precisely where the losses take place and therefore can be used directly to allocate losses. In our setting the tracing method is not used to identify the sources of the losses, but to estimate how much each hauler is using each part of the network.

[8] Recall that here an "edge" may represent gas coming from outside the network or gas leaving the network.

a limitation since the precision of measurement instruments depends on gas pressure, temperature and other factors that may vary substantially across the network. Given these limitations, it is standard to assume that there is some proportionality connecting gas losses with gas flow and volume.[9] Most of the rules below build upon this idea.

A *rule* is a function assigning to each gas problem G a vector $R(G) \in \mathbb{R}^H_+$ such that $\sum_{h \in H} R_h(G) = L$, where $R_h(G)$ denotes the loss assigned to hauler h. In this paper we restrict attention to rules that divide the loss among the haulers present in a given problem, in the sense of owning some edges, *i.e.*, $R_h(G) = 0$ whenever $E_h = \emptyset$. Clearly, it makes no sense to assign losses to haulers that are not present in the network at hand. We consider four rules.[10] The first one is based on the flows, ignoring the volumes: the loss allocated to a hauler is proportional to the flow entering in the hauler's network.

Flow's rule, R^{flow}. For each gas problem G and each hauler $h \in H$,

$$R_h^{\text{flow}}(G) = L \frac{f_h}{\sum_{\hat{h} \in H} f_{\hat{h}}}.$$

This rule is the one in place in the Spanish gas transmission network since 2014. According to the official regulation published in Boletín Oficial del Estado (2013b): "the loss allocated to each hauler shall be computed sharing the total loss allocated to the transmission network proportionally to the gas entering the network of each hauler in the given year" (translated from Spanish).

The next three rules: aggregate edge's rule, edge's rule and proportional tracing rule offer different interpretations of the idea that the loss depends in a multiplicative way on both flow and volume. The first one computes, for each hauler, the product of his flow and his volume (the sum of the volumes of his edges) and allocates the total loss proportionally.

Aggregate edge's rule, R^{Aedge}. For each gas problem G and each hauler $h \in H$,

$$R_h^{\text{Aedge}}(G) = L \frac{f_h \sum_{e \in E_h} v_e}{\sum_{\hat{h} \in H} (f_{\hat{h}} \sum_{e \in E_{\hat{h}}} v_e)}.$$

The aggregate edge's rule was the one used in Spain since 2011 (Boletín Oficial del Estado, 2011), until it was replaced by the flow's rule. The next rule computes, for each

[9] We present a couple of brief intuitions for the role of flow and volume in gas losses. These intuitions come from basic physical principles combined with the opinion of experts in the field. First, in a given pipe, more gas flow leads to more pressure. Since higher pressures put more stress on the pipe walls, the amount of flow may be correlated with pipe breaches. On the other hand, for a given pipe and flow, pressure decreases with volume, so one could argue that the role of volume goes in the opposite direction. Yet, higher volume is associated with more surface in the pipe walls, which may increase the likelihood of some breach due to manufacturing defects or external factors. Maybe more importantly, volume is typically positively correlated with length, and longer pipes have more active elements in them such as valves and measurement points. Defects in the assembly process of these active elements with the pipes may lead to gas losses.

[10] In a previous version of this paper we followed a game theory approach to define a fifth rule, which we called the Shapley rule (see Bergantiños et al. (2014)). The idea is to associate a cooperative game to each gas problem and obtain the loss allocated to each hauler from the Shapley value of the game. The axiomatic analysis for this rule is more complex than for the other four and, further, its behavior is not good. Thus, we have decided not to include the Shapley rule in the present paper.

edge, the product of its flow and its volume and allocates losses to edges proportionally. Then, the loss allocated to a hauler is the sum of the losses allocated to his edges.

Edge's rule, R^{edge}. For each gas problem G and each hauler $h \in H$,

$$R_h^{\text{edge}}(G) = L\frac{\sum_{e\in E_h} f_e v_e}{\sum_{\hat{e}\in E} f_{\hat{e}} v_{\hat{e}}}.$$

The following rule incorporates to the calculation the way in which the gas flows through the network as given by the proportional tracing method.

Proportional tracing rule, $R^{\Gamma^{\text{pt}}}$. In general we say that a rule is a *tracing rule* R^{Γ} if there is a tracing method Γ such that

$$R_h^{\Gamma}(G) = \alpha \sum_{e\in E_h} \sum_{\substack{p\in P(S,C)\\ e\in p}} f_p^{\Gamma} \frac{v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}}.$$

A tracing rule can be seen as a two-step procedure. First, one allocates the loss L among the different paths, $p \in P(S, C)$, proportionally to their flows, f_p^{Γ}. Second, inside each path, the loss allocated to it is split among its edges proportionally to their volumes. Finally, the loss allocated to each hauler is the sum of the losses allocated to his edges. In the particular case of the proportional tracing method we have for each gas problem G and each hauler $h \in H$,

$$\begin{aligned} R_h^{\Gamma^{\text{pt}}}(G) &= L \sum_{p\in P(S,C)} \frac{f_p^{\Gamma^{\text{pt}}}}{\sum_{\hat{p}\in P(S,C)} f_{\hat{p}}^{\Gamma^{\text{pt}}}} \cdot \frac{\sum_{e\in E_h\cap p} v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}} \\ &= L \sum_{e\in E_h} \sum_{\substack{p\in P(S,C)\\ e\in p}} \frac{f_p^{\Gamma^{\text{pt}}}}{\sum_{\hat{p}\in P(S,C)} f_{\hat{p}}^{\Gamma^{\text{pt}}}} \cdot \frac{v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}}. \end{aligned}$$

Since $\sum_{p\in P(S,C)} f_p^{\Gamma^{\text{pt}}} = \sum_{s\in S} Q_s$ is the total amount of gas flowing through the network and $L = \alpha \sum_{s\in S} Q_s$, $R_h^{\Gamma^{\text{pt}}}(G)$ can be rewritten as

$$R_h^{\Gamma^{\text{pt}}}(G) = \alpha \sum_{e\in E_h} \sum_{\substack{p\in P(S,C)\\ e\in p}} f_p^{\Gamma^{\text{pt}}} \frac{v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}},$$

which corresponds with the general expression given above.

Example 1. (cont.) In our running example the loss to allocate is $L = 10$. If we compute the losses assigned to each hauler with the different rules, we would get the following results

h	f_h	R^{flow}	R^{Aedge}	R^{edge}	$R^{\Gamma^{\text{pt}}}$
h_1	60	3.33	5	4	4
h_2	80	4.44	3.33	4	4
h_3	40	2.22	1.66	2	2

Although R^{edge} and $R^{\Gamma^{\text{pt}}}$ lead to different allocations in general, this example belongs to a class of gas problems in which they coincide. This is shown in the result below. $\diamond$

Lemma 1. *Let G be a gas problem with the following properties:*

i) All edges have the same volume.

ii) All paths have the same number of edges.

Then, all tracing rules coincide with R^{edge}.

Proof. By i) we have

$$R_h^{\text{edge}}(G) = L\frac{\sum_{e\in E_h} f_e v_e}{\sum_{\hat{e}\in E} f_{\hat{e}} v_{\hat{e}}} = \frac{L}{\sum_{\hat{e}\in E} f_{\hat{e}}} \sum_{e\in E_h} f_e.$$

Hence, $R_h^{\text{edge}}(G)$ is proportional to the sum of the flows of the edges of hauler h. The same conclusion arises for any tracing rule R^Γ, just by noting that $v_e / \sum_{\hat{e}\in p} v_{\hat{e}}$ is independent of e and p (by i) and ii), respectively), and that $f_e = \sum_{p\in P(S,C),\, e\in p} f_p^\Gamma$. □

5 Properties

Once we have defined the different rules the next objective is to set up a benchmark that allows to compare them. Here we follow an axiomatic approach, defining several properties that a rule should satisfy; recall the discussion in the introduction regarding the (lack of) impact of the rules on hauler's incentives. Most of the properties try to formalize the general principles established in the European regulations. We assign each property to one of the principles of these regulations, although we acknowledge that this classification is arbitrary and that some properties respond to various of the principles. Other properties are inspired in well established principles of game theory and cost allocation theory.

In Directive 2003/55/EC of the European parliament and the council of 26 June 2003 (Regulation (EC), no. 55/2003), concerning common rules for the internal market in natural gas, establishes some general principles that must be pursued. Some of them are the following:

i) "tariffs are published **prior** to their entry into force".

ii) "**the provision of adequate economic incentives**, using, where appropriate, all existing national and Community tools. These tools may include liability mechanisms to guarantee the necessary investment".

iii) "national regulatory authorities should ensure that transmission and distribution tariffs are **non-discriminatory** and **cost-reflective**".

iv) "Progressive opening of markets towards **full competition** should as soon as possible remove differences between Member States."

The Spanish regulation related to the gas loss ensures that tariffs are published prior to their entry into force. Moreover, since the amount received or paid by each hauler depends monotonically on their loss (the larger is the loss, the larger is the amount the hauler pays) we can say that it provides the adequate economic incentives.

Regarding the principles of being non-discriminatory, cost-reflective, and foster competition we proceed as follows. We introduce some properties related to these principles. Next, we check whether or not the different rules satisfy these properties and present a discussion based on these properties.

5.1 Cost-reflective properties

The first property requires that haulers that do not transport gas do not have any assigned loss.

Null hauler (NH). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h \in H$ be such that for each $e \in E_h$, $f_e = 0$. Then, $R_h(G) = 0$.

The following property has a spirit similar to that of NH. If two gas problems only differ on edges without flow, then the losses assigned to each hauler should coincide.

Independence of unused edges (IUE). Let the gas problems $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be such that $H = \bar{H}$ and, for each $h \in H$, $\bar{E}_h = E_h \setminus \hat{E}$, where $\hat{E} \subset E$ satisfies that, for each $e \in E \setminus \hat{E}$, $\bar{f}_e = f_e$ and $\bar{v}_e = v_e$, and, for each $e \in \hat{E}$, $f_e = 0$. Then, $R(G) = R(\bar{G})$.

A cost-reflective rule should not be sensitive to "equivalent" representations of the same network. The next property captures this idea. Suppose that an edge is (transversely) sectioned in several edges. Then the rule should not be affected by this operation.

Independence of edge sectioning (IES). Let the gas problems $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be such that $H = \bar{H}$ and there are $\hat{h} \in H$ and $(i, j) \in E_{\hat{h}}$ satisfying

- $\bar{g} = (\bar{N}, \bar{E})$, where $\bar{N} = N \cup \{l\}$ and $l \notin N$, $\bar{E}_{\hat{h}} = (E_{\hat{h}} \backslash \{(i, j)\}) \cup \{(i, l), (l, j)\}$ and, for each $h \in H \backslash \{\hat{h}\}$, $\bar{E}_h = E_h$, and
- $\bar{f}_{(i,l)} = \bar{f}_{(l,j)} = f_{(i,j)}$, $\bar{v}_{(i,l)} + \bar{v}_{(l,j)} = v_{(i,j)}$, and, for each $e \in E \backslash \{(i, j)\}$, $\bar{f}_e = f_e$ and $\bar{v}_e = v_e$.[11]

Then, for each $h \in H$, $R_h(G) = R_h(\bar{G})$.

The next property goes along similar lines, but focusing on the longitudinal representation of the network instead of the transverse sectioning. For this property we need to explicitly consider that the gas network can be a multigraph: if a hauler duplicates one of his edges then, as long as the total flow carried by them is the same, the loss allocation should not change.

[11] The condition $\bar{v}_{(i,l)} + \bar{v}_{(l,j)} = v_{(i,j)}$ just reflects that, when a pipe is transversely cut (orthogonally to the direction of the flow), the volume of the resulting two pipes adds up to the volume of the original pipe (and the same flow that was crossing the original pipe is crossing the two pipes in which it has been divided $\bar{f}_{(i,l)} = \bar{f}_{(l,j)} = f_{(i,j)}$).

Independence of edge multiplication (IEM). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be such that $H = \bar{H}$ and there are $\hat{h} \in H$, $e = (i, j, m) \in E$, $\bar{e}_1 = (i, j, l_1) \in \bar{E}$, and $\bar{e}_2 = (i, j, l_2) \in \bar{E}$ satisfying

- $\bar{g} = (N, \bar{E})$, where $\bar{E}_{\hat{h}} = (E_{\hat{h}} \setminus \{e\}) \cup \{\bar{e}_1, \bar{e}_2\}$ and, for each $h \in H \backslash \{\hat{h}\}$, $\bar{E}_h = E_h$, and
- $f_e = \bar{f}_{e_1} + \bar{f}_{e_2}$, $v_e = \bar{v}_{e_1} = \bar{v}_{e_2}$, and, for each $e \in E \backslash \{e\}$, $\bar{f}_e = f_e$ and $\bar{v}_e = v_e$.[12]

Then, for each $h \in H$, $R_h(G) = R_h(\bar{G})$.

To prevent haulers from artificially distorting the final allocation of losses, if two haulers engage in some trades affecting their own edges, then the rest of the haulers should not be affected. This implies, in particular, that the loss allocated to a hauler does not depend on who owns the edges different from his own.

Independence by sales (IS). Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$, h_1 and h_2 in H, and $e \in E$ be such that $\bar{E}_{h_1} = E_{h_1} \backslash \{e\}$, $\bar{E}_{h_2} = E_{h_2} \cup \{e\}$, and, for each $h \in H \backslash \{h_1, h_2\}$, $\bar{E}_h = E_h$. Then, for each $h \in H \backslash \{h_1, h_2\}$, $R_h(G) = R_h(\bar{G})$.

The rules satisfying IS have an interesting property, that we call *edge decomposability* (ED). Namely, these rules can be computed in a two stage procedure. We first decide the allowed loss on each edge and later compute the allowed loss to each hauler adding the amount assigned to each of his edges. Formally, given a gas problem $G = (g, v, f, \mathcal{H}, \alpha)$, we define the *canonical gas problem* associated with G, $G^c = (g, v, f, \mathcal{H}^c, \alpha)$, by considering that each edge belongs to a different hauler; for each $h \in H^c$, $|E_h| = 1$ and we can identify H^c with the edge set E. Then, for each gas problem G and each $h \in H$,

$$R_h(G) = \sum_{e \in E_h} R_e(G^c).$$

We define below a property that deals with the way in which some changes in the gas network should affect the loss allocated to the different haulers. It says that the gas allocated to a hauler should not be affected by changes outside his influence network.

Independence of irrelevant changes (IIC). Consider the gas problems $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ and let $h \in H \cap \bar{H}$ be such that $\mathcal{N}^h = \bar{\mathcal{N}}^h$. Then, $R_h(G) = R_h(\bar{G})$.

5.2 Non-discriminatory properties

Next, we present properties related with the principle of non-discrimination. The most standard non-discriminatory principle says that we should offer an equal treatment to equal agents. Some of the following properties deal with formalizations of this general notion. We start with an illustrative example.

Example 2. Let G be the gas problem depicted in Figure 5. Assume that the volume of the three edges is 100 and $\alpha = 0.06$. Since there is only one supply node and 100 units of gas are entering through it, we have $L = \alpha \cdot 100 = 6$.

[12] In this case, the condition $v_e = \bar{v}_{e_1} = \bar{v}_{e_2}$ just reflects that the original pipe e is being replaced by two pipes identical to it: same volume and same endpoints. The total flow in the network remains unchanged, so these two new pipes, together, carry the same flow as e ($f_e = \bar{f}_{e_1} + \bar{f}_{e_2}$).

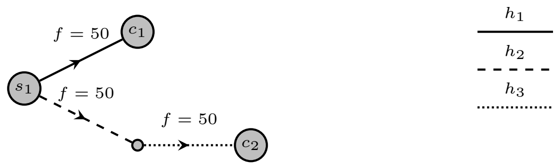


Figure 5: Allocating the loss among "symmetric" haulers.

How we allocate the loss among the three haulers? Two approaches seem reasonable:

i) We focus on edges. Three edges (haulers) are needed to send the 100 units of flow. All edges are "symmetric" because they have the same volume and the same flow. The loss allowed to each hauler is 2.

ii) We focus on flows. The flow is sent through two independent paths, each of them carrying 50 units of flow and so it seems natural to assign the same loss, 3, to both paths. The first path has a unique edge, thus the 3 units of loss go to h_1. The second path has two edges which are "symmetric" because they have the same flow and the same volume. Thus, we assign the same loss to each one. Then, the loss allocated to h_2 is 1.5 and the loss allocated to h_3 is 1.5. $\diamond$

The first symmetry property is related with the first approach. Thus, we focus only on flows and volumes.

Symmetry on edges (SE). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, $f_e = f_{\bar{e}}$, and $v_e = v_{\bar{e}}$. Then, $R_h(G) = R_{\bar{h}}(G)$.

The next symmetry property is related with the second approach and we have to consider also the rest of the graph. If two haulers own exactly one edge each and have the same influence network, then, provided that the two edges have the same volume, the rule should assign the same loss to both haulers.

Symmetry on paths (SP). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, $v_e = v_{\bar{e}}$, and $\mathcal{N}^h = \mathcal{N}^{\bar{h}}$. Then, $R_h(G) = R_{\bar{h}}(G)$.

Since two edges with the same influence network have the same volume and must carry the same flow, symmetry on edges implies symmetry on paths.

The following properties build upon the idea that there should be some kind of proportionality on flow and volume.

Flow proportionality on edges (FPE). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, and $v_e = v_{\bar{e}}$. Then, if $f_{\bar{e}} > 0$, we have

$$R_h(G) = \frac{f_e}{f_{\bar{e}}} R_{\bar{h}}(G).$$

We could also define a flow proportionality on paths property but, since all the edges with the same influence network would carry the same flow, such a property would be equivalent to SP.

Volume proportionality on edges (VPE). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, and $f_e = f_{\bar{e}}$. Then,

$$R_h(G) = \frac{v_e}{v_{\bar{e}}} R_{\bar{h}}(G).$$

Volume proportionality on paths (VPP). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, and $\mathcal{N}^h = \mathcal{N}^{\bar{h}}$. Then,

$$R_h(G) = \frac{v_e}{v_{\bar{e}}} R_{\bar{h}}(G).$$

5.3 Properties to foster competition

The way in which losses are allocated among haulers should not harm competition among agents. In particular, two haulers should not be better off by merging together.

Merging proofness (MP). Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$, $h_1, h_2 \in H$, and $h \in \bar{H}$ be such that $\bar{E}_h = E_{h_1} \cup E_{h_2}$ and, for each $\hat{h} \in H \setminus \{h_1, h_2\}$, $\bar{E}_{\hat{h}} = E_{\hat{h}}$. Then $R_h(\bar{G}) \leq R_{h_1}(G) + R_{h_2}(G)$.

It is important to emphasize the importance of the previous property. Not only it is important for its direct implications towards facilitating competition, but also because of its connection with non-discrimination. Clearly, a rule in which merging is profitable is also a rule that favors big haulers with respect to small haulers, which is *size discrimination*.[13] This is not saying that big haulers should not get assigned a higher loss, but that the way the assigned loss grows with size should obey some principles (which we have captured with the notion of merging proofness).

Because of the above comments, in the discussion in Section 6, the MP property will be considered both a property to foster competition and a non-discriminatory property.

5.4 Additivity properties

In this subsection we present two properties that deal with how a rule should react when we add gas problems defined on the same gas network. They are standard in game theory and cost allocation theory. The first property says that if a gas problem can be obtained as the sum of the flows of other problems, then the loss should be the sum of the losses.

Strong additivity (SA). For each $i \in \{1, \ldots, n\}$, let $G_i = (g, v, f_i, \mathcal{H}, \alpha)$ and let $G^* = (g, v, \sum_{i=1}^n f_i, \mathcal{H}, \alpha)$. Then, $R(G^*) = \sum_{i=1}^n R(G_i)$

It turns out that this property is very strong and quite incompatible with the properties we have discussed so far. In the example below we show that SA is incompatible with SP and NH. Recall that NH seems to be an essential cost-reflective requirement and, moreover, Proposition 1 in Section 5.5 shows that SP is the weakest non-discriminatory property.

Example 3. Let $G_1 = (g, v, f_1, \mathcal{H}, \alpha)$, $G_2 = (g, v, f_2, \mathcal{H}, \alpha)$, $\bar{G}_1 = (g, v, \bar{f}_1, H, \alpha)$, and $\bar{G}_2 = (g, v, \bar{f}_2, \mathcal{H}, \alpha)$, be as depicted in Figure 6 with $H = E$ and all the volumes being 100. Note that $G^* = (g, v, f_1 + f_2, \mathcal{H}, \alpha) = (g, v, \bar{f}_1 + \bar{f}_2, \mathcal{H}, \alpha)$.

[13] This is illustrated in Section 8 for the Spanish gas transmission network.

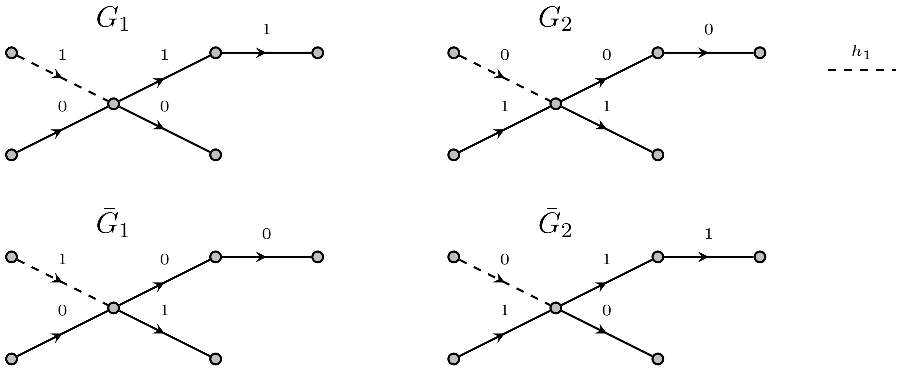


Figure 6: Incompatibility of SA with NH and SP.

Suppose that R is a rule satisfying SA, NH, and SP. Then, by SA

$$R_{h_1}(G^*) = R_{h_1}(G_1) + R_{h_1}(G_2) = R_{h_1}(\bar{G}_1) + R_{h_1}(\bar{G}_2).$$

By NH all haulers with flow 0 must receive 0. Thus, $R_{h_1}(G_2) = R_{h_1}(\bar{G}_2) = 0$. By SP, $R_{h_1}(G_1) = \frac{1}{3}L$ and $R_{h_1}(\bar{G}_1) = \frac{1}{2}L$, which leads to a contradiction. $\diamond$

It is worth noting that all the rules we have defined satisfy SP and NH, so they don't satisfy SA. For this reason we exclude SA from the rest of the analysis and define a weaker additivity property, which imposes a consistency condition between the flows of the gas problems to be combined. Consider the gas problems $G_1 = (g, v, f_1, \mathcal{H}, \alpha)$, $G_2 = (g, v, f_2, \mathcal{H}, \alpha)$,..., $G_n = (g, v, f_n, \mathcal{H}, \alpha)$, and $G^* = (g, v, f_1 + f_2 + \ldots + f_n, \mathcal{H}, \alpha)$, for some $n \in \mathbb{N}$, and let Γ be a tracing rule. We say that $G_1, G_2, \ldots, G_n$ are Γ-*compatible* if they have the same sets of suppliers and consumers and G^* is such that, for each $p \in P^*(S, C)$, $f_p^{\Gamma}(G^*) = \sum_{i=1}^{n} f_p^{\Gamma}(G_i)$.[14]

Tracing additivity (TA). Let Γ be a tracing method. Consider the set of Γ-compatible gas problems $\{G_i = (g, v, f_i, \mathcal{H}, \alpha)\}_{i \in \{1,\ldots,n\}}$ and let $G^* = (g, v, \sum_{i=1}^{n} f_i, \mathcal{H}, \alpha)$. Then, $R(G^*) = \sum_{i=1}^{n} R(G_i)$.

This property is weaker than SA since, given a tracing methodology Γ, requiring that a set of gas problems is Γ-compatible is typically quite demanding.

5.5 Relationships between the properties

The proposition below summarizes some connections between the different properties we have defined in this section.

Proposition 1. *The following individual relationships hold:*

[14] Note that $P_1(S, C) = \ldots = P_n(S, C) = P^*(S, C)$.

i) IIC implies IS.

ii) IS implies MP.

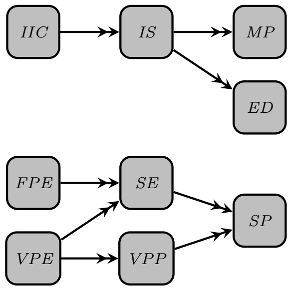
iii) SE implies SP.

iv) VPE implies SE and VPP.

v) VPP implies SP.

vi) FPE implies SE.

vii) IS implies ED.



Relationships in Proposition 1.

Proof. The first 6 statements are straightforward and we omit their proof.

We now prove statement *vii*). Let G be a gas problem and R a rule satisfying IS. Let G^c be the canonical gas problem associated with G. Let $E_h = \left\{e_h^1, ..., e_h^p\right\}$ denote the edges of hauler h in G. Let $G^{h,1}$ be the problem obtained from G when hauler h sells edge e_h^1 to hauler e_h^1. By IS, $R_{h'}(G) = R_{h'}\left(G^{h,1}\right)$ for each $h' \in H \backslash \{h\}$. Thus,

$$R_h(G) = R_{e_h^1}\left(G^{h,1}\right) + R_h\left(G^{h,1}\right).$$

Let $G^{h,2}$ be the problem obtained from $G^{h,1}$ when hauler h sells edge e_h^2 to hauler e_h^2. By IS, $R_{e_h^1}\left(G^{h,1}\right) = R_{e_h^1}\left(G^{h,2}\right)$ and $R_{h'}\left(G^{h,1}\right) = R_{h'}\left(G^{h,2}\right)$ for each $h' \in H \backslash \{h\}$. Thus, noting that $R_h\left(G^{h,1}\right) = R_{e_h^2}\left(G^{h,2}\right) + R_h\left(G^{h,2}\right)$, we have

$$R_h(G) = R_{e_h^1}\left(G^{h,2}\right) + R_{e_h^2}\left(G^{h,2}\right) + R_h\left(G^{h,2}\right).$$

Repeating this argument we can prove that

$$R_h(G) = \sum_{k=1}^{p} R_{e_h^k}\left(G^{h,p}\right).$$

Let $h' \in H \backslash \{h\}$. Let $E_{h'} = \left\{e_{h'}^1, ..., e_{h'}^{p'}\right\}$ denote the edges of hauler h' in G. Let $G^{h',1}$ be the problem obtained from $G^{h,p}$ when hauler h' sells edge $e_{h'}^1$ to hauler $e_{h'}^1$. By IS, $R_{h^*}\left(G^{h,p}\right) = R_{h^*}\left(G^{h',1}\right)$ for each $h^* \in E_h$. Thus,

$$R_h(G) = \sum_{k=1}^{p} R_{e_h^k}\left(G^{h',1}\right).$$

If we continue with this procedure until each hauler $h' \in H \backslash \{h\}$ sells each of its edges e to hauler e, then the gas problem obtained at the end is just G^c. By IS,

$$R_h(G) = \sum_{k=1}^{p} R_{e_h^k}\left(G^c\right),$$

which means that R satisfies ED. □

Interestingly, we also show in Appendix B that the combination of IES, IS, and FPE implies both NH and VPE (lemmas 2 and 3). This implication is crucial for the characterization of the edge's rule in Section 7.

6 Comparing the rules

In this section we study the behavior of the different rules with respect to the properties defined in the previous section. For the sake of exposition, we present the results in Table 1, where we have underlined those properties used in a characterization in Section 7. The proofs can be found in Appendix A.

EU Principles	Property \ Rule	Flow	Aedge	Edge	Prop. Tracing
Cost-reflective	Null hauler	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	Indep. Unused Edges	$\checkmark$		$\checkmark$	$\underline{\checkmark}$
	Indep. Edge Sectioning	$\checkmark$	$\checkmark$	$\underline{\checkmark}$	$\checkmark$
	Indep. Edge Mult.	$\checkmark$		$\checkmark$	$\underline{\checkmark}$
	Ind. Sales			$\underline{\checkmark}$	$\underline{\checkmark}$
	Indep. Irrelevant Changes				$\checkmark$
Non-discriminatory	Symmetry on Edges	$\checkmark$	$\checkmark$	$\checkmark$	
	Symmetry on Paths	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	Flow Proportionality Edges	$\checkmark$	$\checkmark$	$\underline{\checkmark}$	
	Volume Proportionality Edges		$\checkmark$	$\checkmark$	
	Volume Proportionality Paths		$\checkmark$	$\checkmark$	$\underline{\checkmark}$
	Merging Proofness	$\checkmark$		$\checkmark$	$\checkmark$
Competition	Merging Proofness	$\checkmark$		$\checkmark$	$\checkmark$
Additivity	Tracing Additivity				$\underline{\checkmark}$

Table 1: Behavior of the different rules with respect to the different properties.

6.1 Discussion

If we take a general look at the table, there are two rules that stand as the ones with a better behavior: the proportional tracing rule and the edge's rule. In the following lines we take a closer look, with the focus on the principles taken from the European regulations. Depending on the behavior with respect to the properties associated to each principle, we assign a "grade" or "degree of fulfillment" of each principle by each rule. These grades, on which we elaborate below are summarized in the following table:

Principle \ Rule	Flow	Aedge	Edge	Prop. tracing
Cost-reflective	*Normal*	*Low*	*High*	*Very high*
Non-discriminatory	*High*	*High*	*Very high*	*High*
Competition	*Very high*	*Low*	*Very high*	*Very high*

It is important to recall that one might have chosen a different classification of the properties into principles, so the grades and ensuing discussion might change. Thus, the arguments in this section partially respond to our subjective criteria when assigning properties to principles. In particular, we have chosen to include merging proofness in two categories, because we consider it is strongly related to non-discrimination and fosters competition. Yet, we consider that, overall, the main conclusions we draw in this section are quite objective.

Since the proportional tracing rule satisfies all cost-reflective properties, its grade is very high. Then, the edge's rule only violates IIC and gets a high grade. Flow's rule also violates IS, which we consider an important cost reflective property, so it gets a normal grade. Finally, aggregate edge's gets a low grade.

Concerning non-discrimination, the grades require some explanation. First, since the edge's rule satisfies all properties, it gets again a very high grade. The aggregate edge's rule only violates one of the properties, since it favors the haulers with large networks.[15] Thus, the grade for this rule is high. Flow's rule satisfies all properties but the ones related with the volume. The idea underlying this rule is that gas losses are much more related with flows than with volumes and, under this assumption, the properties related to volumes make no sense. Thus, we still classify the flow's rule as high. We move now to the proportional tracing rule. Most of our non-discriminatory properties build upon the principle of equal treatment of equals but, as we already argued when we introduced them, it is not clear when should we consider two agents equal. We can focus on flows and the paths they follow or on edges. In the first case the proportional tracing rule would be non-discriminatory and in the second it would be discriminatory. We believe that focusing on flows and paths is more reasonable, because the whole structure of the graphs is taken into account and not only the edges on isolation. Thus, we still give a high grade to the proportional rule.

The grades related to the competition principle are the natural ones.

We are in position of revisiting our initial comparison of rules in the light of the grade's table. According to it, if we had to provide a ranking of the rules we would have the proportional rule and the edge's rule on top. Interestingly both of them dominate the third one, the flow's rule, according to all principles and the flow's rule also dominates the aggregate's edges rule, which is the last one.

In the light of the previous discussion we can also draw some normative conclusions regarding the situation in the Spanish gas transmission network:

i) The flow's rule satisfies more principles than the aggregate edge's rule. Thus, the change in the Spanish law can be seen as an improvement.

ii) There are other rules that seem to exhibit a better behavior than the flow's rule with respect to those principles.

[15] Recall that this is not saying that big haulers should not get assigned a higher loss (see the discussion in Section 5.3 and the illustration in Section 8).

7 Axiomatic characterizations

In this section we present axiomatic characterizations of the edge's rule and the general family of tracing rules. We also present an independent characterization of the proportional tracing rule. The proofs can be found in Appendix B.

7.1 Edge's rule characterization

We first present a characterization of edge's rule using two cost-reflective properties (IES and IS), and a non-discriminatory property (FPE).

Theorem 1. *The edge's rule is the unique rule satisfying IES, IS, and FPE. Besides, the properties are independent.*

7.2 Tracing rules characterization

We present a characterization of the tracing rules using two cost reflective properties, IUE and IS, one non-discriminatory property, VPP, and one additivity property, TA.

Theorem 2. *The tracing rules are the unique rules satisfying IUE, IS, VPP, and TA. Besides, the properties are independent.*

In particular, the proportional tracing rule is characterized with TA with respect to the proportional tracing.

Corollary 1. *The proportional tracing rule is the unique rule satisfying IUE, IS, VPP, and TA with respect to* Γ^{pt}*. Besides, the properties are independent.*

The characterizations in Theorem 1 and Corollary 1 share property IS. Then, for the edge's rule we have added IES, which is also satisfied by the proportional tracing rule and FPE, which is not. For the proportional tracing rule we have added IUE and VPP, which are also satisfied by the edge's rule, and TA, which is not. Thus, the main difference comes from FPE vs. TA.

To conclude, we present another characterization of the proportional tracing rule.

Theorem 3. *The proportional tracing rule is the unique tracing rule satisfying IEM. In particular, it is the unique rule satisfying IUE, IS, VPP, TA, and IEM. Besides, the properties are independent.*

8 Illustration using the Spanish gas transmission network

In this section we illustrate the rules discussed in this paper by applying them to the Spanish transmission network, which has a total extension of around 11000 km.[16] The

[16]To be precise, what we are representing is the primary network, the high pressure one (operating pressures from 40 to 80 bar). The network representation is based on official documents and the ownership of the different pipes is based on the information provided by the Technical System Manager, where the area of operation of each hauler is specified.

computations build upon the optimal network operation in a hypothetical day of very high demand.[17]

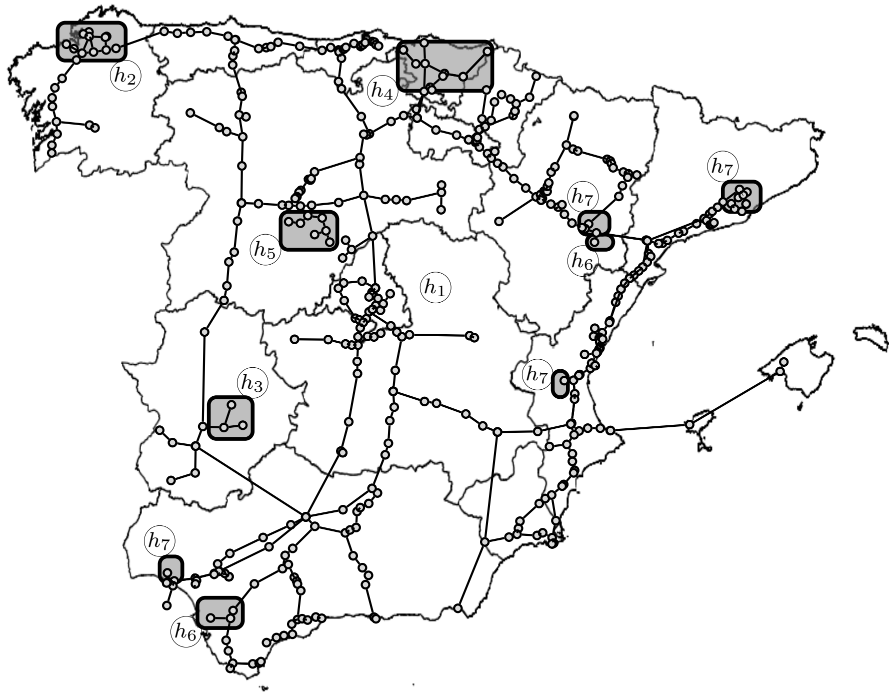


Figure 7: Haulers of the Spanish gas transmission network.

In Figure 7 we represent the Spanish gas transmission network. We have boxed the pipes belonging to each hauler, except for hauler h_1 who owns all the remaining ones. It is worth noting this hauler corresponds with Enagás, a former public body who initially owned the whole network and still owns around 10000 km of pipes, much more than any other hauler. The second largest one is Enagás Transporte del Norte with approximately 350 km and it is worth noting that 90% of this last company is also owned by Enagás.

In tables 2-4 we present the results of applying the different rules to the Spanish gas network. All of them are based on a parameter $\alpha = 0.002$, which is the parameter used in Spain (Boletín Oficial del Estado, 2013a).[18] Table 2 represents the allocated losses measured in gas units, Table 3 represents the percentage allocated to each hauler, and Table 4 contains an estimation of the annual monetary equivalent; under the assumption

[17]The main reason for taking a day with very high demand as reference instead of an average day is that, when studying energy networks for different purposes (capacity, expansion planning, security of supply,...), peak days are the norm and so finding realistic data on peak demand is easier. On the other hand, for the determination of the optimal network operation, we have relied on GANESO™, a software developed by researchers at the University of Santiago de Compostela for Reganosa Company (a hauler in the Spanish network).

[18]For the sake of clarity, each number is presented with the precision needed to show the first non-zero decimal digit and also the following one.

Gas losses in GWh/d	Network Owned (%)	Flow	Aedge	Edge	Prop. Tracing
Enagás (h_1)	91.44	4.55	5.32	5.27	4.72
Reganosa (h_2)	1.76	0.21	0.0024	0.031	0.21
Gas Extremadura (h_3)	0.61	0.0071	0.000010	0.00020	0.000073
Enagás Transporte del Norte (h_4)	3.54	0.31	0.0086	0.027	0.24
Transportista Regional Gas (h_5)	1.46	0.016	0.000051	0.0005	0.00052
Endesa Gas Transportista (h_6)	0.36	0.0045	0.0000019	0.000029	0.000035
Gas Natural (h_7)	0.82	0.24	0.00095	0.0062	0.17

Table 2: Gas loss allocated to the haulers (GWh/d) with $\alpha = 0.002$.

Percentage of gas losses (%)	Network Owned (%)	Flow	Aedge	Edge	Prop. Tracing
Enagás (h_1)	91.44	85.19	99.77	98.77	88.37
Reganosa (h_2)	1.76	3.97	0.046	0.59	3.95
Gas Extremadura (h_3)	0.61	0.13	0.00019	0.0037	0.0014
Enagás Transporte del Norte (h_4)	3.54	5.74	0.16	0.51	4.44
Transportista Regional Gas (h_5)	1.46	0.31	0.00096	0.0094	0.0098
Endesa Gas Transportista (h_6)	0.36	0.083	0.000035	0.00055	0.00066
Gas Natural (h_7)	0.82	4.58	0.018	0.12	3.23

Table 3: Percentage of gas loss allocated to the haulers.

Monetary equivalent in millions of €	Network Owned (%)	Flow	Aedge	Edge	Prop. Tracing
Enagás (h_1)	91.44	49.77	58.30	57.71	51.64
Reganosa (h_2)	1.76	2.32	0.027	0.34	2.31
Gas Extremadura (h_3)	0.61	0.077	0.00011	0.0022	0.00080
Enagás Transporte del Norte (h_4)	3.54	3.35	0.095	0.30	2.60
Transportista Regional Gas (h_5)	1.46	0.18	0.00056	0.0055	0.0057
Endesa Gas Transportista (h_6)	0.36	0.049	0.000020	0.00032	0.00039
Gas Natural (h_7)	0.82	2.68	0.010	0.068	1.89

Table 4: Annual monetary equivalent, assuming that 1 GWh = 30000 €.

that the given scenario repeats itself throughout the year. For this last table it should be taken into account that the peak day considered has nearly twice the demand of an average day, so dividing by two the amounts in Table 4 would deliver more realistic figures. In practice, in order to minimize the dependence of the final allocation on the chosen demands and network configuration, one might for instance apply the chosen rule on a daily basis and then add up the daily allocations to get the annual loss allocation.

We can readily see that all rules allocate the largest gas loss to Enagás, which agrees with the fact that Enagás is, by far, the biggest hauler. Yet, according to the aggregate edge's rule 99.77% of the allocated losses go to Enagás, which we believe is unreasonable even if we take into account that this hauler owns 91.44% of the network. This goes along the lines mentioned when discussing the properties of the rules, where we argued that the aggregate edge's rule size discriminates, penalizing small haulers and favoring mergers, which hurts competition. Indeed, the allocated loss under the flow's rule is, for most haulers, over 100 times larger than it was before; for instance, Gas Natural (h_7) gets more than two millions of Euro when, according to the aggregate edge's rule, it was barely getting ten thousand. This probably explains why most Spanish haulers strongly opposed to the aggregate edges rule until it was finally replaced by the flow's rule.

Pursuing the above discussion a bit further, one might wonder what would happen if the haulers could choose the rule by some voting mechanism. Although a general answer is beyond the scope of this paper, in the Spanish network all haulers except Enagás would have the flow's rule as their first choice and the aggregate edge's rule as the last one. Also, the tracing rule would be chosen in second place, since only Enagás and Gas Extremadura would rank the edge's rule on top of it.

9 Conclusions

We have addressed the issue of how to allocate losses between the agents of an energy transmission network. To the best of our knowledge, this is the first time this problem is formally studied for gas networks and the first time a formal axiomatic approach is developed for any kind of energy network.

We have discussed several allocation rules, two of them that have already been used in practice and two new ones we define. We have studied their behavior with respect to some principles set forth by the European Union such as non-discrimination, and cost-reflectivity. As a result, we have seen that one of the rules that has already been used in practice exhibits by far the worst behavior with respect to these principles. On the other side, the two rules we define seem to abide better by them.

From a more theoretical perspective, we have introduced several properties representing the EU principles. Besides, we have obtained axiomatic characterizations of two of the rules.

Finally, we have applied the developed methodology to the Spanish gas transmission network and noted that the allocated losses vary significantly depending on the chosen rule. Thus, confirming that the selection of a fair allocation scheme is an important issue for the haulers.

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References

Alonso, A., L. Olmos, and M. Serrano (2010): "Application of an entry-exit tariff model to the gas transport system in Spain," *Energy Policy*, 38, 5133–5140.

Bergantiños, G., J. González-Díaz, A. M. González-Rueda, and M. P. Fernández de Córdoba (2014): "Gas loss allocation in gas transmission networks," Tech. Rep. 14-02, Department of Statistics and Operations Research. University of Santiago de Compostela.

Bergantiños, G. and R. Martínez (2014): "Cost allocation in asymmetric trees," *European Journal of Operational Research*, 237, 975–987.

Bergantiños, G. and J. Vidal-Puga (2007): "A fair rule in minimum cost spanning tree problems," *Journal of Economic Theory*, 137, 326–352.

Bialek, J. (1996): "Tracing the flow of electricity," in *IEE Proceedings on Generation Transmission and Distribution*, vol. 143.

Bialek, J. W. and P. A. Kattuman (2004): "Proportional sharing assumption in tracing methodology," in *IEE Proceedings on Generation Transmission and Distribution*, vol. 151.

Bogomolnaia, A., R. Holzman, and H. Moulin (2010): "Sharing the cost of a capacity network," *Mathematics of Operations Research*, 35, 173–192.

Boletín Oficial del Estado (2011): "Incentivo a la reducción de mermas en la red de transporte," in *Orden ITC/3128/2011*, Spanish Government, vol. 278.

——— (2013a): "Coeficientes de mermas en las instalaciones gasistas," in *Orden IET/2446/2013*, Spanish Government, vol. 312.

——— (2013b): "Incentivo a la reducción de mermas en la red de transporte (Amendment)," in *Orden IET/2446/2013*, Spanish Government, vol. 312.

British Petroleum (2013): "BP Statistical Review of World Energy," Annual report.

Comisión Nacional de la Energía (2006): "Estudio de mermas y autoconsumos en las instalaciones de regasificación, almacenamiento subterráneo, transporte y distribución de gas natural," Official report.

Conejo, A. J., J. M. Arroyo, N. Alguacil, and A. L. Guijarro (2002): "Transmission Loss Allocation: A Comparison of Different Practical Algorithms," in *IEEE Transactions on Power Systems*, vol. 17.

Enagás GTS (2013): "El Sistema Gasista Español: Informe 2013," Annual report.

ERGEG (European Regularors Group for Electricity and Gas) (2008): "Treatment of Losses by Network Operators," Tech. Rep. E08-ENM-04-03, Brussels, Belgium.

Estevez-Fernandez, A. (2012): "A game theoretical approach to sharing penalties and rewards in projects," *European Journal of Operational Research*, 216, 647–657.

Galiana, F. D., A. J. Conejo, and I. Kockar (2002): "Incremental Transmission Loss Allocation Under Pool Dispatch," in *IEEE Transactions on Power Systems*, vol. 17.

Kyung-Il, M., H. Sang-Hyeon, L. Su-Won, and M. Young-Hyun (2010): "Transmission Loss Allocation Algorithm Using Path-Integral Based on Transaction Strategy," *IEEE Transactions on Power Systems*, 25, 195–205.

Lima, D. A., J. Contreras, and A. Padilha-Feltrin (2008): "Electric Power Systems Research," *Electric Power Systems Research*, 78, 264–275.

Lima, D. A. and A. Padilha-Feltrin (2004): "Allocation of the costs of transmission losses," *Electric Power Systems Research*, 72, 13–20.

Littlechild, S. C. and G. Owen (1973): "A simple expression for the Shapley value in a special case," *Management Science*, 20, 370–372.

Molina, Y. P., R. Prada, and O. R. Saavedra (2010): "Complex Losses Allocation to Generators and Loads Based on Circuit Theory and Aumann-Shapley Method," *IEEE Transactions on Power Systems*, 25, 1928–1936.

Moulin, H. (2002): "Axiomatic cost and surplus sharing," *Handbook of Social Choice and Welfare*, 1, 289–357.

Moulin, H. and F. Laigret (2011): "Equal-need sharing of a network under connectivity constraints," *Games and Economic Behavior*, 72, 314–320.

Moulin, H. and S. Shenker (1992): "Serial cost sharing," *Econometrica*, 60, 1009–1037.

Ni, D. and Y. Wang (2007): "Sharing a polluted river," *Games and Economic Behavior*, 60, 176–186.

Regulation (EC) (no. 55/2003): "Common rules for the internal market in natural gas and repealing Directive 98/30/EC," *Official Journal of the European Union (Legislation Series)*, 176, 57–77.

Sharkey, W. (1995): "Network models in economics," *Network Routing, Handbooks in Operations Research and Management Science*, 8, 713–765.

Sprumont, Y. (1998): "Ordinal cost sharing," *Journal of Economic Theory*, 81, 126–162.

Thomson, W. (2001): "On the axiomatic method and its recent applications to game theory and resource allocation," *Social Choice and Welfare*, 18, 327–386.

A Results concerning the properties of the different rules

Unless explicitly mentioned otherwise, in all the examples in this section we assume that the volume of the edges is 1, so the number on the edges represents flows.

Proposition 2. *i) R^{flow} satisfies NH, IUE, IES, IEM, SE, SP, FPE, and MP.*

ii) R^{flow} does not satisfy IS, VPE, VPP, IIC, and TA.

Proof. • NH. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h \in H$ be such that, for each $e \in E_h$, $f_e = 0$. Then,

$$f_h = \sum_{i \in N} Q_i^h = \sum_{i \in N} (\max\{\sum_{(i,j) \in E_h} f_{(i,j)} - \sum_{(j,i) \in E_h} f_{(j,i)}, 0\}) = 0,$$

and so $R_h^{\text{flow}}(G) = L \frac{f_h}{\sum_{\hat{h} \in H} f_{\hat{h}}} = 0$.

• IUE. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be as in the definition of IUE, that is, there is $\hat{E} \subset E$ such that, for each $h \in H$, $\bar{E}_h = E_h \setminus \hat{E}$ and, for each $e \in \hat{E}$, $f_e = 0$.

If $i \in N \setminus \bar{N}$, the edges of E of the form (i, j) or (j, i) belong to $\hat{E}$ and have flow zero. Thus, for each $h \in H$,

$$Q_i^h = \max\{\sum_{(i,j) \in E_h} f_{(i,j)} - \sum_{(j,i) \in E_h} f_{(j,i)}, 0\} = 0. \tag{1}$$

If $i \in \bar{N}$, since $f_e = 0$ for $e \in \hat{E}$ and $f_e = \bar{f}_e$ for $e \in E \setminus \hat{E}$, we have, for each $h \in H$,

$$\begin{aligned} Q_i^h &= \max\{\textstyle\sum_{(i,j) \in E_h} f_{(i,j)} - \sum_{(j,i) \in E_h} f_{(j,i)}, 0\} \\ &= \max\{\textstyle\sum_{(i,j) \in E_h \setminus \hat{E}} f_{(i,j)} - \sum_{(j,i) \in E_h \setminus \hat{E}} f_{(j,i)}, 0\} \\ &= \max\{\textstyle\sum_{(i,j) \in \bar{E}_h} \bar{f}_{(i,j)} - \sum_{(j,i) \in \bar{E}_h} \bar{f}_{(j,i)}, 0\} \\ &= \bar{Q}_i^h. \end{aligned} \tag{2}$$

Then, $R_h^{\text{flow}}(G) = R_h^{\text{flow}}(\bar{G})$, since, by (1) and (2) we have that, for each $h \in H$,

$$f_h = \sum_{i \in N} Q_i^h = \sum_{i \in \bar{N}} Q_i^h = \sum_{i \in \bar{N}} \bar{Q}_i^h = \bar{f}_h.$$

• IES. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $(i,j) \in E_{\hat{h}}$ satisfying that (i,j) is sectioned in two consecutive edges $(i,l),(l,j) \in \bar{E}_{\hat{h}}$.

Since $\bar{f}_{(i,l)} = \bar{f}_{(l,j)} = f_{(i,j)}$ we have that, for each $h \in H$ and each $k \in \bar{N} \setminus \{l\}$, $\bar{Q}_k^h = Q_k^h$. Further, it is easy to see that for each $h \in H$, $\bar{Q}_l^h = 0$. Thus, for each $h \in H$, $\bar{f}_h = f_h$ and, therefore, $R_h^{\text{flow}}(G) = R_h^{\text{flow}}(\bar{G})$.

• IEM. It is straightforward.

• SE and SP. Since SE and SP are weaker than FPE (Proposition 1), SE and SP follow from the fact that R^{flow} satisfies FPE (see below).

• FPE. Let G be as in the definition of FPE. Since $E_h = \{e\}$ and $E_{\bar{h}} = \{\bar{e}\}$, we have that $f_h = f_e$ and $f_{\bar{h}} = f_{\bar{e}} > 0$. Then, the definition of R^{flow} ensures that

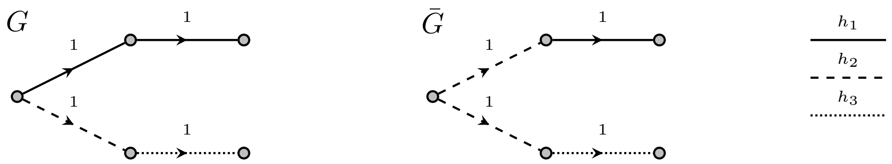
$$R_h^{\text{flow}}(G) = \frac{f_e}{f_{\bar{e}}} R_{\bar{h}}^{\text{flow}}(G).$$

• MP. Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$, h, h_1, and h_2 be as in the definition of MP. Now, to compute Q_i^h, the gas reaching i through edges of h_1 and exiting through edges of h_2 cancels out, whereas it does not cancel out to compute $Q_i^{h_1}$; a similar observation holds for $Q_i^{h_2}$. Then, for each $i \in N$, $Q_i^h \leq Q_i^{h_1} + Q_i^{h_2}$ and, hence, $\bar{f}_h \leq f_{h_1} + f_{h_2}$. Let $F = \sum_{\bar{h} \in \bar{H} \setminus \{h\}} \bar{f}_{\bar{h}} = \sum_{\bar{h} \in H \setminus \{h_1, h_2\}} f_{\bar{h}}$. Then, since $F \geq 0$, $\frac{x}{x+F}$ is an increasing function,

$$R_h^{\text{flow}}(\bar{G}) = L \frac{\bar{f}_h}{\bar{f}_h + F} \leq L \frac{f_{h_1} + f_{h_2}}{f_{h_1} + f_{h_2} + F} = R_{h_1}^{\text{flow}}(G) + R_{h_2}^{\text{flow}}(G).$$

Next, we present some counterexamples to prove statement ii) of Proposition 2.

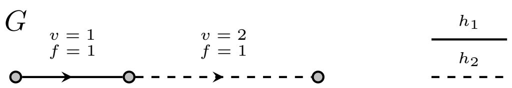
• IS. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$ as in the picture below.



Problems G and $\bar{G}$ satisfy the assumptions of the definition of IS. However, we have that $R_{h_3}^{\text{flow}}(G) = L\frac{1}{3} \neq L\frac{1}{4} = R_{h_3}^{\text{flow}}(\bar{G})$.

• VPE. Since VPE is stronger than VPP (Proposition 1) and R^{flow} does not satisfy VPP (see below), R^{flow} does not satisfy VPE.

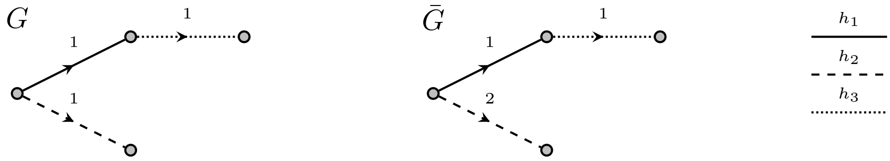
• VPP. Let $G = (g, v, f, \mathcal{H}, \alpha)$, h_1 and h_2 be as in the picture below.



Then,

$$R^{\text{flow}}_{h_2}(G) = R^{\text{flow}}_{h_1}(G) \neq 2R^{\text{flow}}_{h_1}(G) = \frac{v_{h_2}}{v_{h_1}} R^{\text{flow}}_{h_1}(G).$$

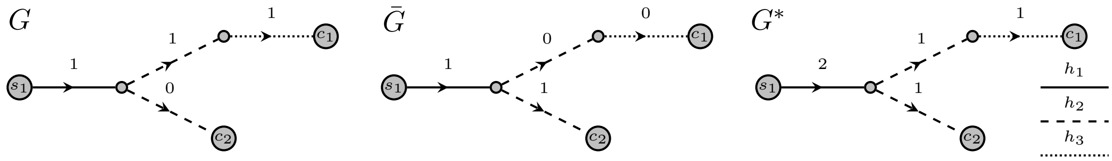
• IIC. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (g, v, \bar{f}, \mathcal{H}, \alpha)$ be as in the picture below.



The gas problems G and $\bar{G}$ are as in the definition of IIC. However,

$$R^{\text{flow}}_{h_1}(G) = \alpha\frac{1}{3} \neq \alpha\frac{1}{4} = R^{\text{flow}}_{h_1}(\bar{G}).$$

• TA. Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, \bar{f}, \mathcal{H}, \alpha)$, and $G^* = (g, v, f + \bar{f}, \mathcal{H}, \alpha)$ be as in the picture below.



Let p_1 be the path from s_1 to c_1 and p_2 the path from s_1 to c_2. Then, $\{p_1, p_2\} = P(S, C) = \bar{P}(S, C) = P^*(S, C)$. Moreover, for each tracing method Γ and each $i \in \{1, 2\}$, we have $f^{\Gamma}_{p_i}(G) + f^{\Gamma}_{p_i}(\bar{G}) = 1 = f^{\Gamma}_{p_i}(G^*)$. Thus, G, $\bar{G}$, and G^* satisfy the assumptions of the definition of TA with respect to any tracing method Γ. However,

$$R^{\text{flow}}_{h_3}(G) + R^{\text{flow}}_{h_3}(\bar{G}) = \frac{L}{3} + 0 = \alpha\frac{1}{3} \neq \alpha\frac{1}{5} = \frac{L^*}{5} = R^{\text{flow}}_{h_3}(G^*).$$ □

Proposition 3. *i) R^{Aedge} satisfies NH, IES, SE, SP, FPE, VPE, and VPP.*

ii) R^{Aedge} does not satisfy IUE, IEM, IS, IIC, MP, and TA.

Proof. • NH. Trivially, a hauler who does not carry flow through his edges gets 0 according to the aggregate edge's rule.

• IES. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $(i, j) \in E_{\hat{h}}$ satisfying that (i, j) is sectioned in two consecutive edges $(i, l), (l, j) \in \bar{E}_{\hat{h}}$.

Since $v_{(i,j)} = \bar{v}_{(i,l)} + \bar{v}_{(l,j)}$, we have that, for each $h \in H$, $\sum_{e \in E_h} v_e = \sum_{e \in \bar{E}_h} \bar{v}_e$. Moreover, in the proof of IES in Proposition 2 we showed that $\bar{f}_h = f_h$ for all h. Thus,

$$R^{\text{Aedge}}_h(G) = L\frac{f_h(\sum_{e \in E_h} v_e)}{\sum_{\hat{h} \in H} f_{\hat{h}}(\sum_{e \in E_{\hat{h}}} v_e)} = L\frac{\bar{f}_h(\sum_{e \in E_h} \bar{v}_e)}{\sum_{\hat{h} \in H} \bar{f}_{\hat{h}}(\sum_{e \in \bar{E}_{\hat{h}}} \bar{v}_e)} = R^{\text{Aedge}}_h(\bar{G}).$$

• SE and SP. Since SE and SP are weaker than FPE (Proposition 1), SE and SP follow from the fact that R^{Aedge} satisfies FPE (see below).

• FPE. Let G be as in the definition of FPE. Since $E_h = \{e\}$ and $E_{\bar{h}} = \{\bar{e}\}$ we have that $f_h = f_e$ and $f_{\bar{h}} = f_{\bar{e}}$. Thus, since $v_e = v_{\bar{e}}$, the definition of aggregate edge's rule implies that

$$R_h^{\text{Aedge}}(G) = \frac{f_e}{f_{\bar{e}}} R_{\bar{h}}^{\text{Aedge}}(G).$$

• VPE. Let G be as in the definition of VPE. Since $E_h = \{e\}$ and $E_{\bar{h}} = \{\bar{e}\}$ we have that $f_h = f_e = f_{\bar{e}} = f_{\bar{h}}$. Then, the definition of aggregate edge's rule implies that

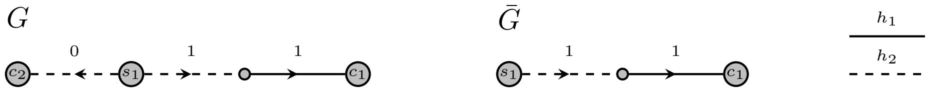
$$R_h^{\text{Aedge}}(G) = \frac{v_e}{v_{\bar{e}}} R_{\bar{h}}^{\text{Aedge}}(G).$$

• VPP. The aggregate edge's rule satisfies VPP, since we have seen that it satisfies VPE and, by Proposition 1, VPE implies VPP.

Next, we present some counterexamples to prove statement ii) of Proposition 3.

• IS. Since IS is stronger than MP (Proposition 1) and R^{Aedge} does not satisfy MP (see below), R^{Aedge} does not satisfy IS.

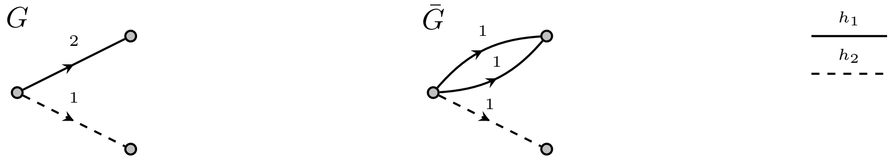
• IUE. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be as in the picture below.



Clearly, G and $\bar{G}$ are as in the definition of IUE. However,

$$R_{h_2}^{\text{Aedge}}(G) = L\frac{2}{3} \neq L\frac{1}{2} = R_{h_2}^{\text{Aedge}}(\bar{G})$$

• IEM. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be as in the picture below.

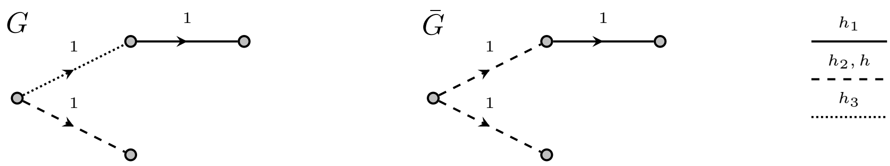


Clearly, G and $\bar{G}$ are as in the definition of IEM. However,

$$R_{h_1}^{\text{Aedge}}(G) = L\frac{2}{3} \neq L\frac{4}{5} = R_{h_1}^{\text{Aedge}}(\bar{G}).$$

• IIC. We can use the same counterexample used for R^{flow} in Proposition 2, since R^{flow} and R^{Aedge} coincide for the gas problems there.

• MP. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$ be as in the picture below.



Note that $H = \{h_1, h_2, h_3\}$ and $\bar{H} = \{h_1, h\}$ where h is the union of h_2 and h_3. Problems G and $\bar{G}$ are as in the definition of MP. However,

$$R_h^{\text{Aedge}}(\bar{G}) = L\frac{4}{5} > L\frac{1}{3} + L\frac{1}{3} = R_{h_2}^{\text{Aedge}}(G) + R_{h_3}^{\text{Aedge}}(G).$$

- TA. We can use the counterexample used for R^{flow} in Proposition 2, where

$$R_{h_3}^{\text{Aedge}}(G) + R_{h_3}^{\text{Aedge}}(\bar{G}) = \frac{L}{3} + 0 = \alpha\frac{1}{4} \neq \alpha\frac{1}{7} = \frac{L^*}{7} = R_{h_3}^{\text{Aedge}}(G^*). \qquad \square$$

Proposition 4. *i) R^{edge} satisfies NH, IUE, IES, IEM, IS, SE, SP, FPE, VPE, VPP, and MP.*

ii) R^{edge} does not satisfy IIC and TA.

Proof. • NH. Trivially, a hauler who does not carry flow through his edges gets 0 according to the edge's rule.

• IUE. It is obvious.

• IES. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $(i, j) \in E_{\hat{h}}$ satisfying that (i, j) is sectioned in two consecutive edges $(i, l), (l, j) \in \bar{E}_{\hat{h}}$. Since $v_{(i,j)} = \bar{v}_{(i,l)} + \bar{v}_{(l,j)}$ and $f_{(i,j)} = \bar{f}_{(i,l)} = \bar{f}_{(l,j)}$, we have

$$f_{(i,j)}v_{(i,j)} = f_{(i,j)}(\bar{v}_{(i,l)} + \bar{v}_{(l,j)}) = \bar{f}_{(i,l)}\bar{v}_{(i,l)} + \bar{f}_{(l,j)}\bar{v}_{(l,j)}.$$

Then, for each $h \in H$, $\sum_{e\in E_h} f_e v_e = \sum_{e\in \bar{E}_h} \bar{f}_e \bar{v}_e$ and $R_h^{\text{edge}}(G) = R_h^{\text{edge}}(\bar{G})$.

• IEM. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $e \in E_{\hat{h}}$ satisfying that e is duplicated in two multiedges $e_1, e_2 \in \bar{E}_{\hat{h}}$, with $v_e = \bar{v}_{e_1} = \bar{v}_{e_2}$. Since $f_e = \bar{f}_{e_1} + \bar{f}_{e_2}$, we have

$$f_e v_e = (\bar{f}_{e_1} + \bar{f}_{e_2})v_e = \bar{f}_{e_1}\bar{v}_{e_1} + \bar{f}_{e_2}\bar{v}_{e_2}.$$

Then, for each $h \in H$, $\sum_{e\in E_h} f_e v_e = \sum_{e\in \bar{E}_h} \bar{f}_e \bar{v}_e$ and $R_h^{\text{edge}}(G) = R_h^{\text{edge}}(\bar{G})$.

• IS. Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$, $h^1, h^2 \in H$ and $e \subset E_{h^1}$ be such that, $\bar{E}_{h^1} = E_{h^1}\backslash\{e\}$, $\bar{E}_{h^2} = E_{h^2} \cup \{e\}$ and, for each $h \in H\backslash\{h^1, h^2\}$, $\bar{E}_h = E_h$.

Note that $\sum_{e\in E} f_e v_e = \sum_{e\in E} \bar{f}_e \bar{v}_e$ and that, for each $h \in H \setminus \{h^1, h^2\}$, $\sum_{\bar{e}\in E_h} f_{\bar{e}} v_{\bar{e}} = \sum_{\bar{e}\in \bar{E}_h} \bar{f}_{\bar{e}} \bar{v}_{\bar{e}}$. Then, for each $h \in H \setminus \{h^1, h^2\}$, $R_h^{\text{edge}}(G) = R_h^{\text{edge}}(\bar{G})$.

• SE and SP. Since SE and SP are weaker than FPE (Proposition 1), SE and SP follow from the fact that R^{edge} satisfies FPE (see below).

• FPE. Let G be as in the definition of FPE. Since $E_h = \{e\}$, $E_{\hat{h}} = \{\hat{e}\}$ and $v_e = v_{\hat{e}}$ it is straightforward to see that

$$R_h^{\text{edge}}(G) = \frac{f_e}{f_{\hat{e}}} R_{\hat{h}}^{\text{edge}}(G).$$

• VPE. Let G be as in the definition of VPE. Since $E_h = \{e\}$ and $E_{\hat{h}} = \{\hat{e}\}$ with $f_e = f_{\hat{e}}$, it is straightforward to see that

$$R_h^{\text{edge}}(G) = \frac{v_e}{v_{\hat{e}}} R_{\hat{h}}^{\text{edge}}(G).$$

• VPP. The edge's rule satisfies VPP, since we have seen that it satisfies VPE and, by Proposition 1, VPE implies VPP.

• MP. The edge's rule satisfies MP, since we have seen that it satisfies IS and, by Proposition 1, IS implies MP.

Next, we present some counterexamples to prove statement ii) of Proposition 4.

• IIC. We can use the same counterexample used for R^{flow} in Proposition 2, since R^{edge} and R^{flow} coincide for the gas problems there.

• TA. We can use the counterexample used for R^{flow} in Proposition 2, assuming that all edges have the same volume.

$$R^{\text{edge}}_{h_3}(G) + R^{\text{edge}}_{h_3}(\bar{G}) = \frac{L}{3} + 0 = \alpha\frac{1}{3} \neq \alpha\frac{1}{5} = \frac{L^*}{5} = R^{\text{edge}}_{h_3}(G^*). \qquad \square$$

Proposition 5. *i) $R^{\Gamma^{pt}}$ satisfies NH, IUE, IES, IEM, IS, SP, VPP, IIC, MP, and TA.*

ii) $R^{\Gamma^{pt}}$ does not satisfy SE, FPE, and VPE.

Proof. • NH. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h \in H$ be such that, for each $e \in E_h$ $f_e = 0$. Then, for each $p \in P(S, C)$ with $e \in p$, $f_p^{\Gamma^{\text{pt}}} = 0$, since all $f_p^{\Gamma^{\text{pt}}}$ flows are nonnegative numbers and $0 = f_e = \sum_{p\in P(S,C), e\in p} f_p^{\Gamma^{\text{pt}}}$. Then, the definition of $R^{\Gamma^{\text{pt}}}$ immediately implies that $R_h^{\Gamma^{\text{pt}}}(G) = 0$.

• IUE. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be as in the definition of IUE, that is, there is $\hat{E} \subset E$ such that, for each $h \in H$, $\bar{E}_h = E_h \setminus \hat{E}$ and, for each $e \in \hat{E}$, $f_e = 0$.

Reasoning as above, we again have that, for each $p \in P(S, C)$ such that $p \cap \hat{E} \neq \emptyset$, $f_p^{\Gamma^{\text{pt}}} = 0$. Moreover, $\bar{P}(S, C) = P(S, C) \setminus \{p \in P(S, C) : p \cap \hat{E} \neq \emptyset\}$. Thus,

$$R_h^{\Gamma^{\text{pt}}}(\bar{G}) = \alpha \sum_{e\in\bar{E}_h} \sum_{\substack{p\in\bar{P}(S,C)\\ e\in p}} \bar{f}_p^{\Gamma^{\text{pt}}}(\frac{\bar{v}_e}{\sum_{\hat{e}\in p}\bar{v}_{\hat{e}}}) = \alpha \sum_{e\in E_h\setminus\hat{E}} \sum_{\substack{p\in P(S,C)\\ e\in p,\, p\cap\hat{E}=\emptyset}} f_p^{\Gamma^{\text{pt}}}(\frac{v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}}) = R_h^{\Gamma^{\text{pt}}}(G).$$

• IES. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $(i, j) \in E_{\hat{h}}$ satisfying that (i, j) is sectioned in two consecutive edges $(i, l), (l, j) \in \bar{E}_{\hat{h}}$.

Since (i, l) and (l, j) are the only two edges containing node l, then, given $p \in \bar{P}(S, C)$, $(i, l) \in p$ if and only if $(l, j) \in p$. On the other hand,

$$P(S, C) \setminus \{p \in P(S, C) : (i, j) \in p\} = \bar{P}(S, C) \setminus \{p \in P(S, C) : (i, l) \in p\}.$$

Thus, there is a natural bijection between $\{p \in P(S, C) : (i, j) \in p\}$ and $\{p \in \bar{P}(S, C) : (i, l) \in p\}$, so, hereafter, we identify $\bar{P}(S, C)$ with $P(S, C)$. Then, for each $p \in P(S, C)$, $f_p^{\Gamma^{\text{pt}}} = \bar{f}_p^{\Gamma^{\text{pt}}}$.

Since $v_{(i,j)} = \bar{v}_{(i,l)} + \bar{v}_{(l,j)}$ we have that, for each $p \in P(S, C)$ with $(i, j) \in p$, $\sum_{\hat{e}\in p} v_{\hat{e}} = \sum_{\hat{e}\in p} \bar{v}_{\hat{e}}$ and

$$f_p^{\Gamma^{\text{pt}}} \frac{v_{(i,j)}}{\sum_{\hat{e}\in p} v_{\hat{e}}} = \bar{f}_p^{\Gamma^{\text{pt}}} \frac{\bar{v}_{(i,l)}}{\sum_{\hat{e}\in p} \bar{v}_{\hat{e}}} + \bar{f}_p^{\Gamma^{\text{pt}}} \frac{\bar{v}_{(l,j)}}{\sum_{\hat{e}\in p} \bar{v}_{\hat{e}}}.$$

Therefore, for each $h \in H$,

$$R_h^{\Gamma^{\mathrm{pt}}}(G) = \alpha \sum_{e\in E_h} \sum_{\substack{p\in P(S,C)\\ e\in p}} f_p^{\Gamma^{\mathrm{pt}}} \frac{v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}} = \alpha \sum_{e\in \bar{E}_h} \sum_{\substack{p\in \bar{P}(S,C)\\ e\in p}} \bar{f}_p^{\Gamma^{\mathrm{pt}}} \frac{\bar{v}_e}{\sum_{\hat{e}\in p} \bar{v}_{\hat{e}}} = R_h^{\Gamma^{\mathrm{pt}}}(\bar{G}).$$

• IEM. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $e \in E_{\hat{h}}$ satisfying that e is duplicated in two multiedges $e_1, e_2 \in \bar{E}_{\hat{h}}$, with $v_e = \bar{v}_{e_1} = \bar{v}_{e_2}$.

For each path $p \in P(S,C)$ with $e \in p$, there are two paths $p_1, p_2 \in \bar{P}(S,C)$ such that $e_1 \in p_1$, $e_2 \in p_2$, and $p_1 \setminus \{e_1\} = p_2 \setminus \{e_2\} = p \setminus \{e\}$. Further, the proportional tracing method ensures that $\bar{f}_{p_1}^{\Gamma^{\mathrm{pt}}} + \bar{f}_{p_2}^{\Gamma^{\mathrm{pt}}} = f_p^{\Gamma^{\mathrm{pt}}}$. Hence, for each $p \in P(S,C)$ with $e \in p$,

$$f_p^{\Gamma^{\mathrm{pt}}} \frac{v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}} = (\bar{f}_{p_1}^{\Gamma^{\mathrm{pt}}} + \bar{f}_{p_2}^{\Gamma^{\mathrm{pt}}}) \frac{v_e}{\sum_{\hat{e}\in p} v_{\hat{e}}} = \bar{f}_{p_1}^{\Gamma^{\mathrm{pt}}} \frac{\bar{v}_{e_1}}{\sum_{\hat{e}\in p_1} \bar{v}_{\hat{e}}} + \bar{f}_{p_2}^{\Gamma^{\mathrm{pt}}} \frac{\bar{v}_{e_2}}{\sum_{\hat{e}\in p_2} \bar{v}_{\hat{e}}}.$$

On the other hand, for each $p \in \{p \in P(S,C) : e \notin p\} = \{p \in \bar{P}(S,C) : e_1, e_2 \notin p\}$, we have that $f_p^{\Gamma^{\mathrm{pt}}} = \bar{f}_p^{\Gamma^{\mathrm{pt}}}$ and for each $\hat{e} \in p$, $v_{\hat{e}} = \bar{v}_{\hat{e}}$. Therefore, for each $h \in H$, $R_{\hat{h}}^{\Gamma^{\mathrm{pt}}}(G) = R_{\hat{h}}^{\Gamma^{\mathrm{pt}}}(\bar{G})$.

• IS. Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$, $h^1, h^2 \in H$ and $e \subset E_{h^1}$ be such that, $\bar{E}_{h^1} = E_{h^1} \backslash \{e\}$, $\bar{E}_{h^2} = E_{h^2} \cup \{e\}$ and, for each $h \in H \backslash \{h^1, h^2\}$, $\bar{E}_h = E_h$.

Obviously, $P(S,C) = \bar{P}(S,C)$ and, for each $p \in P(S,C)$, $\bar{f}_p^{\Gamma^{\mathrm{pt}}} = f_p^{\Gamma^{\mathrm{pt}}}$. Now, for each $h \in H \backslash \{h^1, h^2\}$, we have that $\bar{E}_h = E_h$ and the definition of $R^{\Gamma^{\mathrm{pt}}}$ implies that $R_h^{\Gamma^{\mathrm{pt}}}(G) = R_h^{\Gamma^{\mathrm{pt}}}(\bar{G})$.

• SP. Since SP is weaker than VPP (Proposition 1), SP follows from the fact that $R^{\Gamma^{\mathrm{pt}}}$ satisfies VPP (see below).

• VPP. Let G be as in the definition of VPP. Since $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, and $\mathcal{N}^h = \mathcal{N}^{\hat{h}}$, we have that, for each $p \in P(S,C)$, $e \in p$ if and only if $\hat{e} \in p$, . Then, the definition of $R^{\Gamma^{\mathrm{pt}}}$ implies that

$$R_h^{\Gamma^{\mathrm{pt}}}(G) = \frac{v_e}{v_{\bar{e}}} R_{\bar{h}}^{\Gamma^{\mathrm{pt}}}(G).$$

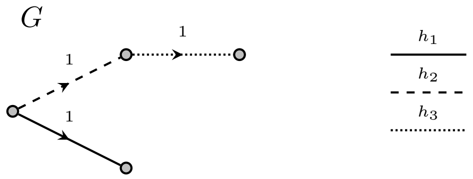
• IIC. Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$, and $h \in H \cap \bar{H}$ be such that $\mathcal{N}^h = \bar{\mathcal{N}}^h$. Then $\{p \in P(S,C) : p \cap E_h \neq \emptyset\} = \{p \in \bar{P}(S,C) : p \cap \bar{E}_h \neq \emptyset\}$ and the proportional method assigns to all these paths the same flow in both problems. Since $\mathcal{N}^h = \bar{\mathcal{N}}^h$, we have that for each $e \in E^h = \bar{E}^h$, $v_e = \bar{v}_e$. Thus, $R_h^{\Gamma^{\mathrm{pt}}}(G) = R_h^{\Gamma^{\mathrm{pt}}}(\bar{G})$.

• MP. The proportional tracing rule satisfies MP, since we have seen that it satisfies IS and, by Proposition 1, IS implies MP.

• TA. Let $G_1 = (g, v, f_1, \mathcal{H}, \alpha)$, $G_2 = (g, v, f_2, \mathcal{H}, \alpha)$,..., $G_n = (g, v, f_n, \mathcal{H}, \alpha)$ be n Γ^{pt}-compatible problems, and let $G^* = (g, v, f_1 + f_2 + \ldots + f_n, \mathcal{H}, \alpha)$. Recall that, by definition, for each $h \in H$, $E_h^i = E_h^*$ and $P_i(S,C) = P^*(S,C)$, for each $i \in \{1, \ldots, n\}$. Moreover, for each $p \in P^*(S,C)$, $f_p^{\Gamma^{\mathrm{pt}}}(G^*) = \sum_{i=1}^n f_p^{\Gamma^{\mathrm{pt}}}(G_i)$. Then, the definition of $R^{\Gamma^{\mathrm{pt}}}$ implies that $R^{\Gamma^{\mathrm{pt}}}(G^*) = \sum_{i=1}^n R^{\Gamma^{\mathrm{pt}}}(G_i)$.

Next, we present some counterexamples to prove statement ii) of Proposition 5.

• SE. Let $G = (g, v, f, \mathcal{H}, \alpha)$ as in the picture below.



Problem G is as in the definition of SE, since $h_1 = \{e_1\}$ and $h_2 = \{e_2\}$ with $f_{e_1} = f_{e_2}$ and $v_{e_1} = v_{e_2}$. However,

$$R^{\Gamma^{\text{pt}}}_{h_1}(G) = \frac{L}{2} \neq \frac{L}{4} = R^{\Gamma^{\text{pt}}}_{h_2}(G).$$

• FPE and VPE. Since FPE and VPE are stronger than SE (Proposition 1) and $R^{\Gamma^{\text{pt}}}$ does not satisfy SE, $R^{\Gamma^{\text{pt}}}$ satisfies neither FPE nor VPE. □

B Proofs of the axiomatic characterizations

B.1 Edge's rule

Before proving Theorem 1, we present two lemmas.

Lemma 2. *Let R be a rule satisfying IES, IS, and FPE, then R satisfies NH.*

Proof. Let $(g, v, f, \mathcal{H}, \alpha)$ be a gas problem. Since IS implies ED we can assume that G is a canonical gas problem, *i.e*, for each $h \in H$, $|E_h| = 1$. Thus, we can identify H and E.

We assume that there are $e, \hat{e} \in E$ with $f_e = 0$ and $f_{\hat{e}} > 0$ and show that $R_e(G) = 0$ (the case where the flow of each edge is 0 is obvious). Let $n > 1$ be such that $\frac{v_e}{n} < v_{\hat{e}}$. Consider the gas problem $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ obtained from G by dividing edge e in n consecutive edges $e_1, e_2, \ldots, e_n$ (as in the definition of IES) with $\bar{v}_{e_i} = \frac{v_e}{n}$ and by dividing the edge $\hat{e}$ in two consecutive edges $\hat{e}_1, \hat{e}_2$ such that $\bar{v}_{\hat{e}_1} = \frac{v_e}{n}$.

We can construct a sequence of problems starting in G and finishing in $\bar{G}$ by sectioning at each step of the sequence only one edge. Applying sequentially IES we get that

$$R_e(G) = R_{e_1}(\bar{G}) + \ldots + R_{e_n}(\bar{G}). \tag{3}$$

Note that, for each $i \in \{1, \ldots, n\}$, $\bar{f}_{e_i} = f_e = 0$ and $\bar{f}_{\hat{e}_1} = \bar{f}_{\hat{e}_2} = f_{\hat{e}} > 0$. Thus, for each $i \in \{1, \ldots, n\}$, since $\bar{v}_{e_i} = \bar{v}_{\hat{e}_1}$, FPE implies that

$$R_{e_i}(\bar{G}) = \frac{\bar{f}_{e_i}}{\bar{f}_{\hat{e}_1}} R_{\hat{e}_1}(\bar{G}) = 0,$$

which, combined with (3), leads to $R_e(G) = 0$. □

Lemma 3. *Let R be a rule satisfying IES, IS, and FPE, then R satisfies VPE.*

Proof. Let $(g, v, f, \mathcal{H}, \alpha)$ be a gas problem. By IS we can assume that G is a canonical gas problem, *i.e*, for each $h \in H$, $|E_h| = 1$. Thus, we can identify H and E.

Let $e, \hat{e} \in E$ be two edges such that $f_e = f_{\hat{e}}$, we have to prove that $R_e(G) = \frac{v_e}{v_{\hat{e}}} R_{\hat{e}}(G)$. By Lemma 2, R satisfies NH and, hence, if $f_e = f_{\hat{e}} = 0$ we have $R_e(G) = \frac{v_e}{v_{\hat{e}}} R_{\hat{e}}(G) = 0$.

On the other hand, if $v_e = v_{\hat{e}}$, then, by FPE, $R_e(G) = \frac{f_e}{f_{\hat{e}}} R_{\hat{e}}(G) = R_{\hat{e}}(G) = \frac{v_e}{v_{\hat{e}}} R_{\hat{e}}(G)$. Thus, we can assume that $f_e = f_{\hat{e}} > 0$ and, for instance, that $v_e > v_{\hat{e}}$.

Consider the gas problem $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ obtained from G by dividing the edge e in two consecutive edges e_1, e_2 (as in the definition of IES) where $\bar{v}_{e_1} = v_{\hat{e}}$, $\bar{v}_{e_2} = v_e - v_{\hat{e}}$ and $\bar{f}_{e_1} = \bar{f}_{e_2} = f_e$. By IES,

$$R_{\hat{e}}(G) = R_{\hat{e}}(\bar{G}) \quad \text{and} \quad R_e(G) = R_{e_1}(\bar{G}) + R_{e_2}(\bar{G}). \tag{4}$$

Since $\bar{v}_{e_1} = \bar{v}_{\hat{e}}$, by FPE,

$$R_{e_1}(\bar{G}) = \frac{\bar{f}_e}{\bar{f}_{\hat{e}}} R_{\hat{e}}(\bar{G}) = \frac{f_e}{f_{\hat{e}}} R_{\hat{e}}(\bar{G}) = R_{\hat{e}}(\bar{G}). \tag{5}$$

From (4) and (5) we have

$$\frac{R_e(G)}{R_{\hat{e}}(G)} = \frac{R_{e_1}(\bar{G}) + R_{e_2}(\bar{G})}{R_{e_1}(\bar{G})} = 1 + \frac{R_{e_2}(\bar{G})}{R_{e_1}(\bar{G})}.$$

Now, if we were able to prove that $R_{e_2}(\bar{G}) = \frac{\bar{v}_{e2}}{\bar{v}_{e_1}} R_{e_1}(\bar{G})$ we would have

$$\frac{R_e(G)}{R_{\hat{e}}(G)} = 1 + \frac{\bar{v}_{e_2}}{\bar{v}_{e_1}} = \frac{\bar{v}_{e_1} + \bar{v}_{e_2}}{\bar{v}_{e_1}} = \frac{v_e}{v_{\hat{e}}}, \text{ and so } R_e(G) = \frac{v_e}{v_{\hat{e}}} R_{\hat{e}}(G),$$

obtaining the desired result.

Thus, it suffices to prove that, when an edge e is sectioned in two edges e_1, e_2, then $R_{e_2}(\bar{G}) = \frac{\bar{v}_{e2}}{\bar{v}_{e_1}} R_{e_1}(\bar{G})$. We consider two cases, when $\frac{\bar{v}_{e_1}}{v_e}$ is a rational number and when it is not.

- **Case 1:** $\boldsymbol{\frac{\bar{v}_{e_1}}{v_e}} \in \mathbb{Q}$. Thus, $\bar{v}_{e_1} = \frac{p}{q} v_e$ with $p, q \in \mathbb{N}$ and so $\bar{v}_{e_2} = \frac{q-p}{q} v_e$. Consider the gas problem $\hat{G} = (\hat{g}, \hat{f}, \hat{v}, \hat{\mathcal{H}}, \alpha)$ obtained from G by sequentially dividing the edge e in 2 consecutive edges (as in the definition of IES) so that in the end we have $\hat{e}_1, \ldots, \hat{e}_q$, where, for each $i \in \{1, \ldots, q\}$, $\hat{v}_{\hat{e}_i} = \frac{1}{q} v_e$ and $\hat{f}_{\hat{e}_i} = f_e$.

Now, for each pair $i, j \in \{1, \ldots, q\}$, we can apply FPE to get that $R_{\hat{e}_i}(\hat{G}) = \frac{\hat{f}_{\hat{e}_i}}{\hat{f}_{\hat{e}_j}} R_{\hat{e}_i}(\hat{G}) = R_{\hat{e}_j}(\hat{G})$.

Now,

$$R_{e_2}(\bar{G}) = R_{\hat{e}_{p+1}}(\hat{G}) + \ldots + R_{\hat{e}_q}(\hat{G}) = (q-p)x.$$

where $x = R_{\hat{e}_{p+1}}(\hat{G})$. Besides

$$R_{e_1}(\bar{G}) = R_{\hat{e}_1}(\hat{G}) + \ldots + R_{\hat{e}_p}(\hat{G}) = px$$

Thus,

$$R_{e_2}(\bar{G}) = (q-p)x = \frac{\frac{q-p}{q} v_e}{\frac{p}{q} v_e} px = \frac{\bar{v}_{e_2}}{\bar{v}_{e_1}} R_{e_1}(\bar{G}).$$

• **Case 2:** $\frac{\bar{v}_{e_1}}{v_e} \notin \mathbb{Q}$. Thus, $\bar{v}_{e_1} = sv_e$ with $s \notin \mathbb{Q}$ and $0 < s < 1$. Then, $\bar{v}_{e_2} = (1-s)v_e$. Consider two sequences $\{q_n\}$ and $\{p_n\}$, both converging to s, with $0 < q_n < s < p_n < 1$ and $q_n, p_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, consider two gas problems $\hat{G}_n = (\hat{g}, \hat{f}, \hat{v}, \hat{\mathcal{H}}, \alpha)$ and $G'_n = (g', f', v', \mathcal{H}', \alpha)$ obtained from G by dividing the edge e in two consecutive edges (as in the definition of IES) $e^1_{q_n}, e^2_{q_n}$ and $e^1_{p_n}, e^2_{p_n}$ respectively, where $\hat{v}_{e^1_{q_n}} = q_n v_e$ and $v'_{e^1_{p_n}} = p_n v_e$.

We are in the hypothesis of Case 1, so we have

$$\frac{R_{e^2_{q_n}}(\hat{G}_n)}{R_{e^1_{q_n}}(\hat{G}_n)} = \frac{\hat{v}_{e^2_{q_n}}}{\hat{v}_{e^1_{q_n}}} = \frac{1-q_n}{q_n} \quad \text{and} \quad \frac{R_{e^2_{p_n}}(G'_n)}{R_{e^1_{p_n}}(G'_n)} = \frac{v'_{e^2_{p_n}}}{v'_{e^1_{p_n}}} = \frac{1-p_n}{p_n}. \tag{6}$$

Note that, for each $n \in \mathbb{N}$, $R_{e^1_{q_n}}(\hat{G}_n) \leq R_{e_1}(\bar{G}) \leq R_{e^1_{p_n}}(G'_n)$ and $R_{e^2_{p_n}}(G'_n) \leq R_{e_2}(\bar{G}) \leq R_{e^2_{q_n}}(\hat{G}_n)$, since each edge is a section of the next one. Then, by (6), we have

$$\frac{1-p_n}{p_n} = \frac{R_{e^2_{p_n}}(G')}{R_{e^1_{p_n}}(G')} \leq \frac{R_{e_2}(\bar{G})}{R_{e_1}(\bar{G})} \leq \frac{R_{e^2_{q_n}}(\hat{G})}{R_{e^1_{q_n}}(\hat{G})} = \frac{1-q_n}{q_n}.$$

Finally, when n goes to infinity, both $\frac{1-q_n}{q_n}$ and $\frac{1-p_n}{p_n}$ converge to $\frac{1-s}{s}$ and we have

$$\frac{R_{e_2}(\bar{G})}{R_{e_1}(\bar{G})} = \frac{1-s}{s} = \frac{\bar{v}_{e_2}}{\bar{v}_{e_1}} \quad \text{and so} \quad R_{e_2}(\bar{G}) = \frac{\bar{v}_{e_2}}{\bar{v}_{e_1}} R_{e_1}(\bar{G}). \qquad \square$$

Proof of Theorem 1. By Proposition 4 we already know that the edge's rule satisfies IS, IES, and FPE. Now, we we prove that no other rule does. Let R be a rule satisfying IES, IS, and FPE. By lemmas 2 and 3 we know that R also satisfies NH and VPE.

Let $(g, v, f, \mathcal{H}, \alpha)$ be a gas problem. By IS we can assume that G is a canonical gas problem, *i.e*, for each $h \in H$, $|E_h| = 1$. Thus, we can identify H and E. By NH, For each $e \in E$ with $f_e = 0$, $R_e(G) = 0$, so $R_e(G) = R^{\text{edge}}_e(G)$. Below we prove the equality for $e \in E$ with $f_e > 0$.

Let $\lambda > 0$ be such that, for each $e \in E$, $v_e > \lambda$. Let $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be the problem obtained from G by dividing each edge e in two consecutive edges (as in the definition of IES) e_1, e_2, where $\bar{v}_{e_1} = \lambda$, $\bar{v}_{e_2} = v_e - \lambda$ and $\bar{f}_{e_1} = \bar{f}_{e_2} = f_e$. We can construct a sequence of problems starting in G and finishing in $\bar{G}$ by changing, at each step of the sequence, an edge e by the two edges e_1 and e_2.

By sequentially applying IES we get that, for each $e \in E$, $R_e(G) = R_{e_1}(\bar{G}) + R_{e_2}(\bar{G})$. Moreover, if $f_e > 0$, we have that $\bar{f}_{e_1} = \bar{f}_{e_2} > 0$ and, by VPE, $R_{e_2}(\bar{G}) = \frac{v_e - \lambda}{\lambda} R_{e_1}(\bar{G})$. Combining the above two equalities we get that

$$R_e(G) = \frac{v_e}{\lambda} R_{e_1}(\bar{G}).$$

Since R is such that $\sum_{h \in H} R_h(G) = L > 0$, there is $e \in E$ such that $R_e(G) > 0$. Now, for each $\hat{e} \in E$ with $f_{\hat{e}} > 0$,

$$\frac{R_{\hat{e}}(G)}{R_e(G)} = \frac{\frac{v_{\hat{e}}}{\lambda} R_{\hat{e}_1}(\bar{G})}{\frac{v_e}{\lambda} R_{e_1}(\bar{G})} = \frac{v_{\hat{e}}}{v_e} \frac{R_{\hat{e}_1}(\bar{G}).}{R_{e_1}(\bar{G})} \tag{7}$$

Since $\bar{v}_{e_1} = \bar{v}_{\hat{e}_1}$, by FPE, $R_{\hat{e}_1}(\bar{G}) = \frac{\bar{f}_{\hat{e}_1}}{f_{e_1}} R_{e_1}(\bar{G}) = \frac{f_{\hat{e}}}{f_e} R_{e_1}(\bar{G})$, and by (7) we have

$$R_{\hat{e}}(G) = \frac{f_{\hat{e}}}{f_e} \frac{v_{\hat{e}}}{v_e} R_e(G). \tag{8}$$

Note that NH implies that (8) also holds for $\hat{e} \in E$ with $f_{\hat{e}} = 0$. Then,

$$L = \sum_{\hat{e} \in E} R_{\hat{e}}(G) = \sum_{\hat{e} \in E} \frac{f_{\hat{e}}}{f_e} \frac{v_{\hat{e}}}{v_e} R_e(G) = \sum_{\hat{e} \in E} f_{\hat{e}} v_{\hat{e}} \frac{R_e(G)}{f_e v_e}.$$

Therefore, $R_e(G) = L \frac{f_e v_e}{\sum_{\hat{e} \in E} f_{\hat{e}} v_{\hat{e}}} = R_e^{edge}(G)$.

To conclude the proof we show the independence of the properties.

• IES. For each edge e let $w_e = \lceil v_e \rceil$, that is, the smallest integer greater than v_e. Let R be the rule defined as

$$R_h(G) = L \frac{\sum_{e \in E_h} f_e w_e}{\sum_{\hat{e} \in E} f_{\hat{e}} w_{\hat{e}}}.$$

It is not difficult to prove that R satisfies IS, and FPE, but violates IES.

• IS. By Proposition 3 the aggregate edge's rule satisfies IES and FPE, but violates IS.

• FPE. By Proposition 5, the proportional tracing rule satisfies IES and IS, but violates FPE. □

B.2 Tracing rules

Proof of Theorem 2. By Proposition 5 we already know that the proportional tracing rule satisfies IS, IUE, VPP, and TA. Using the same arguments it can be shown that all tracing rules satisfy IS, IUE, VPP, and TA.

We now prove the uniqueness. Let R be a rule satisfying IS, IUE, VPP, and TA. We claim that $R = R^{\Gamma}$ for some tracing method Γ. Let $G = (g, v, f, \mathcal{H}, \alpha)$ be a gas problem. By IS we can assume that G is a canonical problem, *i.e*, for each $h \in H$, $|E_h| = 1$. Thus, we can identify H and E.

Since R satisfies TA, R is additive with respect to a flow tracing method Γ. For each $p \in P(S, C)$ we define the problem $G^p = (g, f^p, v, \mathcal{H}, \alpha)$ obtained from G by assuming that the only gas in G^p that flows through p according to Γ. Formally, $f_e^p = f_p^{\Gamma}(G)$ if $e \in p$ and $f_e^p = 0$ if $e \notin p$. Note that $G = \sum_{p \in P(S,C)} G^p$ and we are in the assumptions of TA because for each $\hat{p} \in P(S, C)$,

$$f_{\hat{p}}^{\Gamma}(G) = \sum_{p \in P(S,C)} f_{\hat{p}}^{\Gamma}(G^p) = f_{\hat{p}}^{\Gamma}(G^{\hat{p}}).$$

Since R satisfies TA with respect to Γ, we have that, for each $e \in E$,

$$R_e(G) = R_e(\sum_{p \in P(S,C)} G^p) = \sum_{p \in P(S,C)} R_e(G^p).$$

Let $\hat{e} \notin p$ and let $G^{p-\hat{e}}$ be obtained from G^p by removing edge $\hat{e}$. By IUE, for each $e \in E \setminus \{\hat{e}\}$, $R_e(G^p) = R_e(G^{p-\hat{e}})$. Since

$$\sum_{e \in E \setminus \hat{e}} R_e(G^p) = \sum_{e \in E \setminus \hat{e}} R_e(G^{p-\hat{e}}) = \alpha f_p^\Gamma(G) = \sum_{e \in E} R_e(G^p),$$

we get that $R_{\hat{e}}(G^p) = 0$. Then, for each $e \in E$,

$$R_e(G) = \sum_{p \in P(S,C),\, e \in p} R_e(G^p). \tag{9}$$

Let G^{p-E} be obtained from G^p by removing all edges not belonging to p. Let $e, \hat{e} \in p$. We have that $\mathcal{N}^e(G^{p-E}) = \mathcal{N}^{\hat{e}}(G^{p-E}) = p$. By VPP,

$$R_{\hat{e}}(G^{p-E}) = \frac{v_{\hat{e}}}{v_e} R_e(G^{p-E}).$$

By IUE, for all $e \in p$, $R_e(G^p) = R_e(G^{p-E})$. Hence, $R_{\hat{e}}(G^p) = \frac{v_{\hat{e}}}{v_e} R_e(G^p)$. Thus,

$$\alpha f_p^\Gamma(G) = \sum_{\hat{e} \in E} R_{\hat{e}}(G^p) = \sum_{\hat{e} \in p} R_{\hat{e}}(G^p) = \sum_{\hat{e} \in p} (\frac{v_{\hat{e}}}{v_e}) R_e(G^p)$$

and we get that, for each $e \in p$,

$$R_e(G^p) = \alpha f_p^\Gamma(G)(\frac{v_e}{\sum_{\hat{e} \in p} v_{\hat{e}}}). \tag{10}$$

Finally, combining (9) and (10), we have that

$$R_e(G) = \sum_{\substack{p \in P(S,C) \\ e \in p}} \alpha f_p^\Gamma(G)(\frac{v_e}{\sum_{\hat{e} \in p} v_{\hat{e}}}) = R_e^\Gamma(G).$$

To conclude the proof we show the independence of the properties.

- IS. Given $e \in E$, let h^e denote the hauler owning edge e. For each $p \in P(S,C)$ and each $\hat{e} \in p$, let $n(h^{\hat{e}}, p)$ be the number of edges that $h^{\hat{e}}$ owns in p. Let R^1 be defined as

$$R_h^1(G) = \alpha \sum_{e \in E_h} \sum_{\substack{p \in P(S,C) \\ e \in p}} f_p^{\Gamma^{\mathrm{pt}}} \frac{v_e n(h^e, p)}{\sum_{\hat{e} \in p} v_{\hat{e}} n(h^{\hat{e}}, p)}$$

It is not difficult to prove that R^1 satisfies IUE, VPP, and TA, but violates IS.

- VPP. Let $|p|$ denote the number of edges of a path p and let R be defined as

$$R_h^2(G) = \alpha \sum_{e \in E_h} \sum_{\substack{p \in P(S,C) \\ e \in p}} f_p^{\Gamma^{\mathrm{pt}}} \frac{1}{|p|}.$$

It is not difficult to prove that R^2 satisfies IS, IUE, and TA, but violates VPP.

• IUE. Let P^1 be the set of problems such that no two edges have the same influence network (so any rule trivially satisfies VPP for all problems in P^1). We define R^3 such that $R^3(G) = R^2(G)$ when $G \in P^1$ and $R^3(G) = R^{\Gamma^{pt}}(G)$ otherwise. It is not difficult to prove that R^3 satisfies IS, VPP, and TA, but violates IUE.

• TA. The edge's rule satisfies IS, IUE, and VPP but violates TA. □

Proof of Corollary 1. In the proof of Theorem 2 we proved that if a rule satisfies IS, IUE, VPP, and TA with respect to a tracing method Γ, then $R = R^{\Gamma}$. Consequently, $R^{\Gamma^{pt}}$ is the unique rule satisfying IS, IUE, VPP and TA with respect to Γ^{pt}. □

B.3 Proportional tracing rule

We start introducing a last property that will be useful in the proof of Theorem 3.

Equally treatment of equals (ETE). Let $G = (g, v, f, \mathcal{H}, \alpha)$ be such that there are two haulers $h, \bar{h} \in H$, and two edges $e = (i, j, l_1) \in E$ and $\bar{e} = (i, j, l_2) \in E$ satisfying that $E_h = \{e\}$ and $E_{\bar{h}} = \{\bar{e}\}$ with $v_e = v_{\bar{e}}$ and $f_e = f_{\bar{e}}$. Then, $R_h(G) = R_{\bar{h}}(G)$.

Lemma 4. *Let R be a rule satisfying IEM and IS, then R satisfies ETE.*

Proof. Let $G = (g, v, f, \mathcal{H}, \alpha)$ be as in the definition of ETE. Consider the gas problem $\hat{G} = (\hat{g}, \hat{v}, \hat{f}, \mathcal{H}, \alpha)$ obtained from G by duplicating e and $\bar{e}$ in two multiedges $e_1 = (i, j, l_3)$, $e_2 = (i, j, l_4)$, and $\bar{e}_1 = (i, j, l_5)$, $\bar{e}_2 = (i, j, l_6)$ respectively, with $\hat{v}_{e_i} = \hat{v}_{\bar{e}_i} = v_e$ and $\hat{f}_{e_i} = \hat{f}_{\bar{e}_i} = \frac{1}{2} f_e$ for $i \in \{1, 2\}$. By IEM, $R_h(G) = R_h(\hat{G})$.

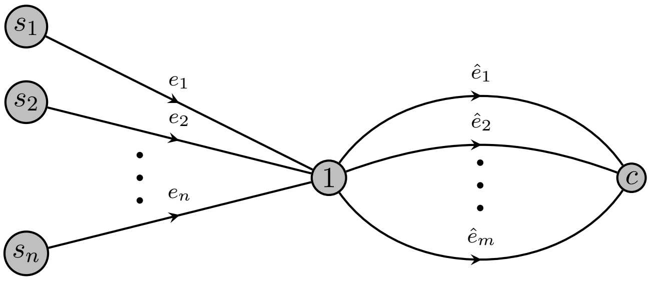
Now, consider $G^* = (g^*, v^*, f^*, \mathcal{H}^*, \alpha)$ obtained from G by duplicating e and $\bar{e}$ in two multiedges $e_1 = (i, j, l_5)$, $e_2 = (i, j, l_6)$, and $\bar{e}_1 = (i, j, l_3)$, $\bar{e}_2 = (i, j, l_4)$ respectively, with $v^*_{e_i} = v^*_{\bar{e}_i} = v_e$ and $f^*_{e_i} = f^*_{\bar{e}_i} = \frac{1}{2} f_e$ for $i \in \{1, 2\}$. By IEM, $R_{\bar{h}}(G) = R_{\bar{h}}(G^*)$.

Let $\hat{G}^{12}$ (respectively G^{*12}) be obtained from $\hat{G}$ (respectively G^*) when hauler h sells his edges to hauler h_1 and hauler $\bar{h}$ sells his edges to hauler h_2 (we assume that haulers h_1 and h_2 have not edges in G). By IS, $R_h(\hat{G}) = R_{h_1}(\hat{G}^{12})$ and $R_{\bar{h}}(G^*) = R_{h_1}(G^{*12})$

Since $\hat{G}^{12}$ and G^{*12} are the same problem, $R_{h_1}(\hat{G}^{12}) = R_{h_1}(G^{*12})$. Thus, $R_h(G) = R_{\bar{h}}(G)$.

□

Proof of Theorem 3. By Proposition 5 we already know that the proportional tracing rule satisfies IS, IUE, IEM, VPP, and TA. Further, by Lemma 4, it also satisfies ETE. By Theorem 2, it suffices to show that $R^{\Gamma^{pt}}$ is the unique tracing rule satisfying IEM. More precisely, we want to show that if a tracing rule R^{Γ} satisfies ETE and IEM, then the gas arriving at a given node is split towards the different outbound destinations using the proportional method. In order to characterize the underlying tracing method it suffices to consider a canonical gas problem $G = (g, v, f, \mathcal{H}, \alpha)$, where g is as in the picture below.



Given $i \in \{1,\ldots,n\}$ and $j \in \{1\ldots,m\}$, we denote by $p_{ij} = \{e_i, \hat{e}_j\}$ the path from s_i to c containing $\hat{e}_j$ and by $f_{ij} = f^{\Gamma}_{p_{ij}}(G)$ the amount of gas that flows through p_{ij}. We denote by F the gas entering in the network, that is $F = \sum_{i=1}^{n} f_{e_i} = \sum_{j=1}^{m} f_{\hat{e}_j} = \sum_{i,j} f_{ij}$. We want to prove that, for each $i \in \{1,\ldots,n\}$ and each $j \in \{1\ldots,m\}$,

$$f_{ij} = \frac{f_{e_i} f_{\hat{e}_j}}{F}.$$

We consider three cases: in the first one we assume that the outbound edges have the same flow, in the second one their flows may be different but are rational numbers, and in the last one we consider the general case where outbound flows can be different and irrational. Since a tracing method is independent of the volumes, we can assume that $v_{\hat{e}_1} = \ldots = v_{\hat{e}_m} = v$.

Case 1. Assume that $f_{\hat{e}_1} = \ldots = f_{\hat{e}_m} = \frac{F}{m}$. By ETE, $R^{\Gamma}_{\hat{e}_1}(G) = \ldots = R^{\Gamma}_{\hat{e}_m}(G)$. Thus, from the definition of the Γ-tracing rule we get

$$\sum_{i=1}^{n} f_{i1} \frac{v}{v_{e_i} + v} = \ldots = \sum_{i=1}^{n} f_{im} \frac{v}{v_{e_i} + v}.$$

The above equalities hold independently of the values of v_{e_i} and this implies that, for each $i \in \{1,\ldots,n\}$, $f_{i1} = \ldots = f_{im}$. On the other hand, since $f_{e_i} = \sum_{j=1}^{m} f_{ij} = m f_{ij}$, we have that

$$f_{ij} = \frac{f_{e_i}}{m} = \frac{f_{e_i} f_{\hat{e}_j}}{m f_{\hat{e}_j}} = \frac{f_{e_i} f_{\hat{e}_j}}{m \frac{F}{m}} = \frac{f_{e_i} f_{\hat{e}_j}}{F}.$$

Case 2. Assume that $f_{\hat{e}_1}, \ldots, f_{\hat{e}_m}$ are (maybe different) rational numbers. Then, there are natural numbers $n_j \in \mathbb{N}$ such that $\frac{f_{\hat{e}_1}}{n_1} = \ldots = \frac{f_{\hat{e}_m}}{n_m} = r$ for some $r > 0$.

Consider the gas problem $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ obtained by multiplying each edge $\hat{e}_j$ in n_j multiedges $\{\hat{e}_{j1}, \ldots, \hat{e}_{jn_j}\}$ with $\bar{v}_{\hat{e}_{jl}} = v_{\hat{e}_j} = v$. Further, according to $\bar{f}$, the flow of the original edge is equally divided among the multiedges, that is, for each $j \in \{1,\ldots,m\}$ and each $l \in \{1,\ldots,n_j\}$, $\bar{f}_{\hat{e}_{jl}} = \frac{f_{\hat{e}_j}}{n_j} = r$.

From $\bar{G}$ we obtain the canonical problem $\bar{G}_{\bar{E}}$ where each edge is a hauler. Now, for each $i \in \{1,\ldots,n\}$, each $j \in \{1,\ldots,m\}$, and each $l \in \{1,\ldots,n_j\}$, let $\bar{f}_{ijl}$ be the flow through the path in $\bar{G}_{\bar{E}}$ from s_i to c through edge $\hat{e}_{jl}$. Then, since $\bar{G}_{\bar{E}}$ satisfies the assumptions of Case 1, we have

$$\bar{f}_{ijl} = \frac{\bar{f}_{e_i} \bar{f}_{\hat{e}_{jl}}}{F} = \frac{f_{e_i} \cdot f_{\hat{e}_j}}{n_j F}. \qquad (11)$$

On the other hand, by IEM and IS, we have that, for each $h \in H$,

$$R_h^\Gamma(G) = R_h^\Gamma(\bar{G}) = \sum_{e \in \bar{E}_h} R_e^\Gamma(\bar{G}_{\bar{E}}). \tag{12}$$

Moreover, by ETE,

$$R_{\hat{e}_{j1}}^\Gamma(\bar{G}_{\bar{E}}) = \ldots = R_{\hat{e}_{jn_j}}^\Gamma(\bar{G}_{\bar{E}}). \tag{13}$$

Now, combining (12) and (13), we have that $R_{\hat{e}_j}^\Gamma(G) = n_j R_{\hat{e}_{j1}}^\Gamma(\bar{G}_{\bar{E}})$, that is,

$$\sum_{i=1}^n f_{ij} \frac{v}{v + v_{e_i}} = n_j \sum_{i=1}^n \bar{f}_{ij1} \frac{\bar{v}}{\bar{v} + \bar{v}_{e_i}} = \sum_{i=1}^n n_j \bar{f}_{ij1} \frac{v}{v + v_{e_i}}.$$

The above equalities hold, for each j, independently of the v_{e_i} values and this implies that, for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, m\}$, $f_{ij} = n_j \bar{f}_{ij1}$. Thus, we can conclude by (11) that

$$f_{ij} = n_j \bar{f}_{ij1} = \frac{f_{e_i} \cdot f_{\hat{e}_j}}{F}.$$

Case 3. Assume that the flows $f_{\hat{e}_1}, \ldots, f_{\hat{e}_m}$ may be different and irrational. For each $j \in \{1, \ldots, m\}$, take a sequence $\{q_j^t\}_{t \in \mathbb{N}}$ such that $\lim_{t \to \infty} q_j^t = f_{\hat{e}_j}$, with $q_j^t \in \mathbb{Q}$ and $q_j^t < f_{\hat{e}_j}$ for each $t \in \mathbb{N}$. Let $\varepsilon_j^t = f_{\hat{e}_j} - q_j^t$. Then, for each $t \in \mathbb{N}$, there are natural numbers $n_j^t \in \mathbb{N}$ for $j \in \{1, \ldots, m\}$ such that $\frac{q_1^t}{n_1^t} = \ldots = \frac{q_m^t}{n_m^t} = r^t$ for some $r^t > 0$.

For each $t \in \mathbb{N}$, consider the gas problem $G^t = (g^t, f^t, v^t, \mathcal{H}^t, \alpha)$ obtained from G by multiplying each edge $\hat{e}_j$ in $n_j^t + 1$ multiedges $\{\hat{e}_{j1}, \ldots, \hat{e}_{jn_j^t+1}\}$ with the same volume v and such that, for each $j \in \{1, \ldots, m\}$ and each $l \in \{1, \ldots, n_j^t\}$, $f_{\hat{e}_{jl}}^t = \frac{q_j^t}{n_j^t} = r^t$ and $f_{\hat{e}_{jn_j^t+1}}^t = \varepsilon_j^t$.

From G^t we obtain the canonical problem where each edge is a hauler. For the sake of notation, hereafter we assume that G^t itself is canonical. For each $i \in \{1, \ldots, n\}$, each $j \in \{1, \ldots, m\}$, and each $l \in \{1, \ldots, n_j^t + 1\}$, let f_{ijl}^t denote the flow inside the path in G^t from s_i to c through edge $\hat{e}_{jl}$. By ETE, for each $t \in \mathbb{N}$,

$$R_{\hat{e}_{11}}^\Gamma(G^t) = \ldots = R_{\hat{e}_{1n_1^t}}^\Gamma(G^t) = \ldots = R_{\hat{e}_{m1}}^\Gamma(G^t) = \ldots = R_{\hat{e}_{mn_m^t}}^\Gamma(G^t).$$

By the definition of the Γ-tracing rule, we have that

$$\sum_{i=1}^n f_{i11}^t \frac{v}{v_{e_i}^t + v} = \ldots = \sum_{i=1}^n f_{i1n_1^t}^t \frac{v}{v_{e_i}^t + v} = \ldots = \sum_{i=1}^n f_{im1}^t \frac{v}{v_{e_i}^t + v} = \ldots = \sum_{i=1}^n f_{imn_m^t}^t \frac{v}{v_{e_i}^t + v}.$$

Since the above equalities hold independently of the $v_{e_i}^t$ values, we have that, for each $t \in \mathbb{N}$ and each $i \in \{1, \ldots, n\}$, there is r_i^t such that

$$f_{i11}^t = \ldots = f_{i1n_1^t}^t = \ldots = f_{im1}^t = \ldots = f_{imn_m^t}^t = r_i^t. \tag{14}$$

Combining (14) with IEM and IS, we have that, for each $j \in \{1, \ldots, m\}$ and each $l \in \{1, \ldots, n_j^t\}$,

$$R^{\Gamma}_{\hat{e}_j}(G) = \Big(\sum_{l=1}^{n_j^t} R^{\Gamma}_{\hat{e}_{jl}}(G^t)\Big) + R^{\Gamma}_{\hat{e}_{jn_j^t+1}}(G^t) = n_j^t R^{\Gamma}_{\hat{e}_{jl}}(G^t) + R^{\Gamma}_{\hat{e}_{jn_j^t+1}}(G^t).$$

Therefore, for each $i \in \{1, \ldots, n\}$, each $j \in \{1, \ldots, m\}$, and each $l \in \{1, \ldots, n_j^t\}$,

$$f_{ij} = n_j^t f_{ijl}^t + f_{ijn_j^t+1}^t = n_j^t r_i^t + f_{ijn_j^t+1}^t. \tag{15}$$

On the other hand, $f_{e_i} = \sum_{j=1}^m f_{ij} = \sum_{j=1}^m (n_j^t r_i^t + f_{ijn_j^t+1}^t) = (n_1^t + \ldots + n_m^t) r_i^t + \sum_{j=1}^m f_{ijn_j^t+1}^t$. Then,

$$r_i^t = \frac{f_{e_i} - \sum_{j=1}^m f_{ijn_j^t+1}^t}{n_1^t + \ldots + n_m^t}. \tag{16}$$

Moreover, combining (15) and (16) we have

$$f_{ij} = \frac{n_j^t}{n_1^t + \ldots + n_m^t}(f_{e_i} - \sum_{j=1}^m f_{ijn_j^t+1}^t) + f_{ijn_j^t+1}^t \tag{17}$$

Taking into account that, as t goes to infinity, $n_j^t r^t = q_j^t$ converges to $f_{\hat{e}_j}$ and $f_{ijn_j^t+1}^t \leq f_{jn_j^t+1}^t = \varepsilon_j^t$ converges to 0, we have that

$$f_{ij} = \lim_{t\to\infty} \frac{q_j^t}{q_1^t + \ldots + q_1^t}(f_{e_i} - \sum_{j=1}^m f_{ijn_j^t+1}^t) + f_{ijn_j^t+1}^t = \frac{f_{\hat{e}_j} f_{e_i}}{F}.$$

To conclude the proof we show the independence of the properties.

• IEM. By Theorem 2, it suffices to find a tracing rule different from $R^{\Gamma^{\mathrm{pt}}}$. Consider the tracing method defined as follows: the flow inbound edge with the highest flow goes to the outbound edge with the highest flow. If after "filling" it there is some flow left, it goes to the one with the second largest flow and so on. If the flow of an inbound edge is finished before the outbound edge at hand is "filled", the inbound edge with the highest flow among the remaining ones is used to continue. If several edges have the same flow, they are taken simultaneously, that is, the flows of inbound edges with the same flow is divided proportionally among the outbound edge(s) at hand.

• IS, VPP, IUE, and TA. We can take the same rules used to establish the independence of these properties in the proof of Theorem 2, since all of them also satisfy IEM. □