Cores of Convex Games and Tartaglia's Triangle*

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Abstract: We follow the path initiated in Shapley (1971) and study the geometry of the core of convex and strictly convex games. Our main contribution is to establish a nice and instructive relation between Tartaglia's triangle and the combinatorial complexity of the core of a strictly convex game.

Keywords: cooperative TU games, convex games, combinatorial complexity, Tartaglia's triangle.

1 Preliminaries

A cooperative n-player game with transferable utility, shortly, a TU game, is a pair (N, v), where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function assigning, to each coalition $S \in 2^N$, its worth $v(S)$; by convention $v(\emptyset) := 0$. Let G^n be the set of n-player games. Given $S \subseteq N$, let $|S|$ be the number of players in S. For the sake of notation, we denote $\{i\}$ by i.

Let $(N, v) \in G^n$. The *core* (Gillies, 1953), is defined by $C(N, v) := \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)$ and, for each $S \subseteq N$, $\sum_{i \in S} x_i \geq v(S)\}$. Let BG^n be the set of n-player games with nonempty core. We say $C(N, v)$ is *full dimensional* if it has dimension $n - 1$.

Let $S \subseteq N$ and let $\Pi(S)$ be the set of orderings (permutations) of the elements in S. For each $\sigma_S \in \Pi(S)$ and each $i \in S$, $\sigma_S(i)$ denotes i's position. We denote σ_N by σ. For each $i \in N$ and each $\sigma \in \Pi(N)$, let $P_\sigma(i) := \{j \in N : \sigma(j) < \sigma(i)\}$ be the set of *predecessors* of i with respect to σ. Let $(N, v) \in G^n$ and $\sigma \in \Pi(N)$. The *marginal vector* associated with (N, v) and σ, $m^\sigma(N, v)$, is defined, for each $i \in N$, by $m_i^\sigma(N, v) := v(P_\sigma(i) \cup i) - v(P_\sigma(i))$.

A game (N, v) is *convex* if, for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N \setminus \{i\}$, $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$. Let CG^n be the set of n-player convex games. A game (N, v) is *strictly convex* if, for each $i \in N$ and each S and T such that $S \subsetneq T \subseteq N \setminus \{i\}$, $v(S \cup i) - v(S) < v(T \cup i) - v(T)$.

A (convex) polytope P is the convex hull of a finite set of points. A polytope P is an *m-polytope* if its dimension is m. A hyperplane H is a *supporting hyperplane for P* if $H \cap P \neq \emptyset$ and the halfspace below H contains P. A *face* of a polytope P is defined as (i) P itself, (ii) the empty set, or (iii) the intersection of P with some supporting hyperplane; faces of dimension m are called m-faces (with the convention that $\dim(\emptyset) = -1$). The 0-faces, 1-faces, and $(m-1)$-faces of an m-polytope P are respectively its vertices, edges, and facets. Let $\mathcal{F}(P)$ denote the set of all faces of P.

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2 Core Complexity and Tartaglia's Triangle

We discuss now the geometry of the cores of convex and strictly convex games. In our exposition we mainly use the terminology in Shapley (1971) and, also, some of the results included there. Let $(N,v) \in BG^n$. For each $\emptyset \neq T \subseteq N$, let H_T be the hyperplane $H_T := \{x \in \mathbb{R}^n : \sum_{i \in T} x_i = v(T)\}$ and let $F_T := C(N,v) \cap H_{N \backslash T}$. Clearly, $F_\emptyset = C(N,v)$; also, let $F_N := C(N,v)$.[1] Shapley (1971) and Ichiishi (1981) showed that a game is convex if and only if the vertices of the core are the marginal vectors, *i.e.*, $C(N,v) = \text{co}\{m^\sigma(N,v) : \sigma \in \Pi(N)\}$, where $\text{co}(A)$ denotes the convex hull of A. Thus, for convex games, each F_T is a nonempty face of $C(N,v)$ and we refer to F_T as a T*-face* of $C(N,v)$. By definition, in each allocation in F_T, coalition T receives $v(N) - v(N\backslash T)$. Clearly, for each $\emptyset \neq T \subsetneq N$, since both F_T and $F_{N\backslash T}$ lie in H_N, they are parallel to each other. Now we define, for each coalition $T \subseteq N$, a game (N, v_{F_T}) that is closely related to F_T.

Definition 1. *Let $(N,v) \in BG^n$ and $T \subseteq N$. The T-*face game *(N, v_{F_T}) is defined, for each $S \subseteq N$, by $v_{F_T}(S) := v((S \cap T) \cup (N\backslash T)) - v(N\backslash T) + v(S \cap (N\backslash T))$.*

Note that, if $T = \emptyset$ or $T = N$, then $(N, v_{F_T}) = (N,v)$. Besides, if $S \cap T = \emptyset$, then $v_{F_T}(S) = v(S)$. If $(N,v) \in CG^n$, then, in the game (N, v_{F_T}), the worth of coalition T coincides with his maximum possible payoff in $C(N,v)$, *i.e.*, $v(N) - v(N\backslash T)$ and, on the contrary, the worth of coalition $N\backslash T$ is its minimum payoff in $C(N,v)$, *i.e.*, $v(N\backslash T)$.

Lemma 1. *Let $(N,v) \in CG^n$ and $T \subseteq N$. Let $\sigma = (\sigma_{N\backslash T}, \sigma_T)$ and let $\bar\sigma \in \Pi(N)$ be such that it induces the orders σ_T and $\sigma_{N\backslash T}$ in T and $N\backslash T$, respectively. Then,*

(i) $(N, v_{F_T}) \in CG^n$.

(ii) $m^\sigma(N, v_{F_T}) = m^\sigma(N,v)$.

(iii) $m^\sigma(N, v_{F_T}) = m^{\bar\sigma}(N, v_{F_T})$.

(iv) $m^{\bar\sigma}(N,v) \in F_T$ *if and only if* $m^\sigma(N,v) = m^{\bar\sigma}(N,v)$. *Hence,* $F_T = \text{co}\{m^\sigma(N,v) : \sigma = (\sigma_{N\backslash T}, \sigma_T)\}$.

Proof. Since $(N, v_{F_\emptyset}) = (N, v_{F_N}) = (N,v)$, the result is trivial for $T = \emptyset$ and $T = N$. Hence, let $\emptyset \neq T \subsetneq N$.

(i) We show that, for each $R \subseteq S \subseteq N\backslash i$, $v_{F_T}(R \cup i) - v_{F_T}(R) \leq v_{F_T}(S \cup i) - v_{F_T}(S)$. Suppose that $i \in N\backslash T$. Since $v_{F_T}(S \cup i) - v_{F_T}(S) = v((S \cup i) \cap (N\backslash T)) - v(S \cap (N\backslash T))$, $v_{F_T}(R \cup i) - v_{F_T}(R) = v((R \cup i) \cap (N\backslash T)) - v(R \cap (N\backslash T))$, and (N,v) is convex, the desired inequality holds. Suppose that $i \in T$. Since $v_{F_T}(S \cup i) - v_{F_T}(S) = v((S \cap T) \cup (N\backslash T) \cup i) - v((S \cap T) \cup (N\backslash T))$, $v_{F_T}(R \cup i) - v_{F_T}(R) = v((R \cap T) \cup (N\backslash T) \cup i) - v((R \cap T) \cup (N\backslash T))$, and (N,v) is convex, the desired inequality holds.

(ii) Let $\sigma = (\sigma_{N\backslash T}, \sigma_T)$. We show that, for each $i \in N$, $m_i^\sigma(N,v) = m_i^\sigma(N, v_{F_T})$. Suppose that $i \in N\backslash T$. Since $P_\sigma(i) \subset P_\sigma(i) \cup i \subseteq N\backslash T$, then $v(P_\sigma(i) \cup i) = v_{F_T}(P_\sigma(i) \cup i)$ and $v(P_\sigma(i)) = v_{F_T}(P_\sigma(i))$. Suppose that $i \in T$. In this case, $N\backslash T \subseteq P_\sigma(i)$ and it is easy to check that, again, $v_{F_T}(P_\sigma(i) \cup i) = v(P_\sigma(i) \cup i)$ and $v_{F_T}(P_\sigma(i)) = v(P_\sigma(i))$.

(iii) If $\bar\sigma = (\sigma_{N\backslash T}, \sigma_T)$ the result is trivial. Let $\bar\sigma \neq (\sigma_{N\backslash T}, \sigma_T)$. Then, $\bar\sigma$ can be written as $(\sigma_{R_1}, \sigma_{T_1}, \sigma_{R_2}, \sigma_{T_2}, \ldots, \sigma_{R_p}, \sigma_{T_q})$, where $T_1, \ldots, T_q \subset T$, $R_1, \ldots, R_p \subset N\backslash T$, and T_1 and R_2 are nonempty. Let $\sigma^* := (\sigma_{R_1}, \sigma_{R_2}, \sigma_{T_1}, \sigma_{T_2}, \ldots, \sigma_{R_p}, \sigma_{T_q})$, *i.e.*, R_2 and T_1 are swapped. We show that $m^{\bar\sigma}(N, v_{F_T}) = m^{\sigma^*}(N, v_{F_T})$. Once the latter is proved, we get, after a finite

[1] Shapley (1971) defines F_T as $C(N,v) \cap H_T$. Although Shapley's definition might seem more natural, ours is more convenient for the exposition below.

number of swaps, that $m^{\bar\sigma}(N,v_{F_T})=m^{\sigma}(N,v_{F_T})$. Clearly, the marginal vectors associated with $\bar\sigma$ and σ^* can only differ for the players in T_1 or R_2. We distinguish two cases. **Case 1:** $i\in T_1$. Clearly, $P_{\bar\sigma}(i)=R_1\cup P_{\sigma_{T_1}}(i)$ and $P_{\sigma^*}(i)=R_1\cup R_2\cup P_{\sigma_{T_1}}(i)$. By the definition of v_{F_T}, $v_{F_T}(P_{\bar\sigma}(i)\cup i)-v_{F_T}(P_{\bar\sigma}(i))=v(P_{\sigma_{T_1}}(i)\cup(N\backslash T)\cup i)-v(P_{\sigma_{T_1}}(i)\cup(N\backslash T))=v_{F_T}(P_{\sigma^*}(i)\cup i)-v_{F_T}(P_{\sigma^*}(i))$. **Case 2:** $i\in R_2$. Clearly, $P_{\bar\sigma}(i)=R_1\cup T_1\cup P_{\sigma_{R_2}}(i)$ and $P_{\sigma^*}(i)=R_1\cup P_{\sigma_{R_2}}(i)$. By the definition of v_{F_T}, $v_{F_T}(P_{\bar\sigma}(i)\cup i)-v_{F_T}(P_{\bar\sigma}(i))=v(R_1\cup P_{\sigma_{R_2}}(i)\cup i)-v(R_1\cup P_{\sigma_{R_2}}(i))=v_{F_T}(P_{\sigma^*}(i)\cup i)-v_{F_T}(P_{\sigma^*}(i))$.

(iv) Since $v\in CG^n$, $m^{\sigma}(N,v)\in F_T$ and the necessity is trivial. We prove the sufficiency. Since $m^{\bar\sigma}(N,v)\in F_T$, then $\sum_{i\in T}m_i^{\bar\sigma}=v(N)-v(N\backslash T)=\sum_{i\in T}m_i^{\sigma}$. By convexity, for each $i\in T$, $m_i^{\bar\sigma}(N,v)\le m_i^{\sigma}(N,v)$ and, since $\sum_{i\in T}m_i^{\bar\sigma}=\sum_{i\in T}m_i^{\sigma}$, we have that, for each $i\in T$, $m_i^{\bar\sigma}(N,v)=m_i^{\sigma}(N,v)$. Similarly, for each $i\in N\backslash T$, $m_i^{\bar\sigma}(N,v)=m_i^{\sigma}(N,v)$. □

Proposition 1. *Let $(N,v)\in CG^n$ and $T\subseteq N$. Then, $C(N,v_{F_T})=F_T$. Therefore, $C(N,v)=\operatorname{co}\{C(N,v_{F_T}):\emptyset\neq T\subsetneq N\}$.*

Proof. The equality $C(N,v_{F_T})=F_T$ is trivial for $T=\emptyset$ and $T=N$. Let $\emptyset\neq T\subsetneq N$. By Lemma 1 (i), for each $\sigma\in\Pi(N)$, $m^{\sigma}(N,v_{F_T})$ is a vertex of $C(N,v_{F_T})$ and $C(N,v_{F_T})=\operatorname{co}\{m^{\sigma}(N,v_{F_T}):\sigma\in\Pi(N)\}$. Now, by Lemma 1 (ii) and (iii), $C(N,v_{F_T})\subseteq F_T$ and, by Lemma 1 (ii) and (iv), $F_T\subseteq C(N,v_{F_T})$. □

The following result is a compilation of different results in Shapley (1971).

Lemma 2. *Let (N,v) be a strictly convex game. Then,*

(i) *$m^{\sigma}(N,v)=m^{\bar\sigma}(N,v)$ if and only if $\sigma=\bar\sigma$. Hence, $C(N,v)$ has $n!$ vertices.*

(ii) *$C(N,v)$ is full dimensional and has 2^n-2 facets, one for each $\emptyset\neq T\subsetneq N$.*

(iii) *Let $\emptyset\neq T\subsetneq N$. Then, $m^{\sigma}(N,v)\in F_T$ if and only if σ is of the form $(\sigma_{N\backslash T},\sigma_T)$.*

Remark. From the previous result, for each strictly convex game and each $\emptyset\neq T\subsetneq N$, F_T is a facet of $C(N,v)$, *i.e.*, an $(n-2)$-polytope. Moreover, (i) and (iii) imply that F_T has $|T|!\,(n-|T|)!$ vertices and, hence, (N,v_{F_T}) is not strictly convex. Recall that, for each $t\in\{0,\dots,n\}$, the number of t-player coalitions is $\binom{n}{t}$. Hence, $\binom{n}{t}$ is also the number of faces of $C(N,v)$ that are associated with a coalition of size t.

Now, we introduce one more concept from Shapley (1971). Let $\mathcal{P}=\{N_1,\dots,N_p\}$ be a partition of N, with $p\ge 2$. The game (N,v) is *decomposable with respect to $\mathcal{P}$* if, for each $T\subseteq N$, $v(S)=v(S\cap N_1)+\ldots+v(S\cap N_p)$. That is, v is the addition of p smaller games; each of them is referred to as a *component*. The following result is also a compilation of different results in Shapley (1971).

Lemma 3. (i) *A strictly convex game is indecomposable.*

(ii) *A decomposable game is convex if and only if each component is convex.*

(iii) *The core of a decomposable convex game is the cartesian product of the cores of the components of any decomposition.*

Proposition 2. *Let $(N,v)\in BG^n$ and $\emptyset\neq T\subsetneq N$. Then, the game (N,v_{F_T}) is decomposable with respect to $\mathcal{P}=\{T,N\backslash T\}$.*

Proof. Let $S\subseteq N$. Then, $v_{F_T}(S\cap T)=v((S\cap T)\cup(N\backslash T))-v(N\backslash T)+v(\emptyset)$ and $v_{F_T}(S\cap(N\backslash T))=v(N\backslash T)-v(N\backslash T)+v(S\cap(N\backslash T))$. Hence, $v_{F_T}(S\cap T)+v_{F_T}(S\cap(N\backslash T))=v_{F_T}(S)$. □

Let (T, v^T) and $(N\backslash T, v^{N\backslash T})$ denote the two components of the decomposition in Proposition 2. Next result is now completely straightforward.

Corollary 1. *Let (N, v) be a strictly convex game and $\emptyset \neq T \subsetneq N$. Then, (T, v^T) and $(N\backslash T, v^{N\backslash T})$ are strictly convex games such that $C(N, v_{F_T}) = C(T, v^T) \times C(N\backslash T, v^{N\backslash T})$. The cores in the cartesian product have dimensions $|T| - 1$ and $|N\backslash T| - 1$, respectively.*

We move now to core complexity. Two polytopes P and P' are *combinatorially equivalent* if there is a one-to-one map $f : \mathcal{F}(P) \to \mathcal{F}(P')$ that is inclusion preserving, *i.e.*, $F \subseteq F'$ if and only if $f(F) \subseteq f(F')$. We define the *combinatorial complexity* of the core of a game as the number of different equivalence classes there are among its facets according to the above relation. Given a strictly convex game and a coalition $\emptyset \neq T \subsetneq N$, all the $|T|$-faces are combinatorially equivalent and, moreover, the faces F_T and $F_{N\backslash T}$ are also combinatorially equivalent. Let $\lfloor \cdot \rfloor$ be the floor function, *i.e.*, for each $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the largest integer not larger than r. The following corollaries are immediate from Lemma 2 and the remark below it.

Corollary 2. *Let (N, v) be a strictly convex game. Then, for each $t \in \{0, \ldots, n\}$, $C(N, v)$ has $2\binom{n}{t}$ combinatorially equivalent facets and each of them can be decomposed as the cartesian product of the cores of two strictly convex games with t and $n - t$ players, respectively.*

Corollary 3. *Let (N, v) be a strictly convex game. Then, the combinatorial complexity of $C(N, v)$ is $\lfloor \frac{n}{2} \rfloor$.*

In Figure 1 we illustrate, with the aid of Tartaglia's triangle, the last three corollaries. Given n, below $\binom{n}{0}$ and $\binom{n}{n}$, we draw the $F_\emptyset$ and F_N faces, *i.e.*, $C(N, v)$. Then, for each $t \in \{1, \ldots, n-1\}$, the polytope below $\binom{n}{t}$ represents one of the $\binom{n}{t}$ combinatorially equivalent T-facets (with $|T| = t$) of $C(N, v)$ and, since $F_T = C(N, v_{F_T})$, it also represents the cores of the T-face games. Remarkably, Figure 1 contains a lot of information about the geometry of the cores of strictly convex games and, moreover, does it in a noteworthy visual way.

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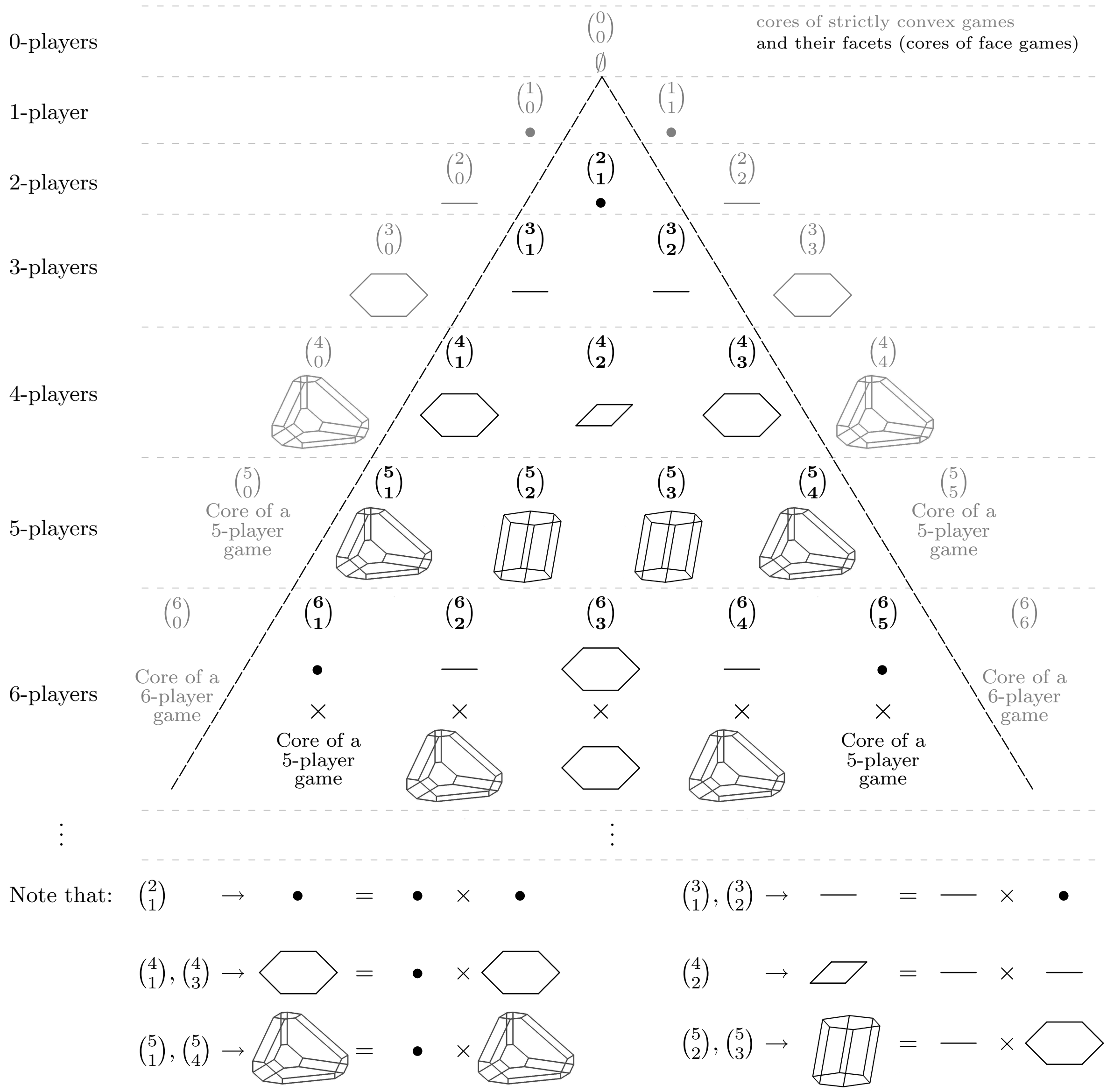

Figure 1: Tartaglia's triangle and core complexity