A Unifying Model for Contests: Effort-Prize Games*

Julio González-Díaz

Department of Statistics and OR,
University of Santiago de Compostela,
15782, Santiago de Compostela, Spain.
E-mail address: julkin@usc.es

Abstract

In this paper we introduce a unifying model of contests. We present a class of non-cooperative games, that we call effort-prize games, and show that they can be used to model many well known economic situations. First price auctions, all-pay auctions, politically contestable rents and transfers, Bertrand competition, R&D,... are examples of the models that fall within our framework. Then, we characterize the set of Nash equilibria of general effort-prize games and discuss the implications of our results in each one of the specific models. Finally, we briefly discuss about the potential of our unifying framework to model new economic situations.

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1 Introduction

Models of contests are pervasive in economic literature. Among them, maybe the models of auctions are the most studied ones, probably because of the simplicity of the model and the richness of its applications. Also the models of Bertrand competition among different firms fall within the literature on contests; being the set of consumers the contestable good. But there are many more situations where different agents are engaged in some competition with the common objective of getting some prize: rent-seeking, political campaigns, patent races, R&D, political lobbying, war of attrition models, taxation competition between districts to attract capital investment...

In this paper we present a unifying model that encompasses many of the situations quoted above. More specifically, the contests we pretend to unify have the following structure. There is a set of agents that want to get a prize and each of them has to choose the effort (investment) he wants to make in order to get the prize. Efforts are chosen independently and the prize is awarded to the agent that has made the highest effort, provided that

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there are no ties. In case of a tie, some tie-breaking rule has to be fixed to "share" the prize among the winners. So note that, once the game is over, we can split the players among losers and winners. Now, we present one more property that characterizes the models of contests we deal with in this paper. The payoff of each agent at the end of the game only depends on two things: i) his own effort and ii) if he is a winner or a loser. We model the contests satisfying this property through what we call *effort-prize games*. It is important to note that with the previous property we have excluded, for instance, both war of attrition models and second-price auctions. This is because in both of them, the payoff of the winners depends on the effort made by the losers and not on their own effort.

Next, we describe, in contrast with war of attrition models, the following feature of effort-prize games: an effort-prize game is always a normal form game and, hence, we only study Nash equilibrium. Suppose that we have a contest that can be modeled as a timing game, namely, as a war of attrition. In these models, each agent has to choose, at each moment in time, whether to stay in the game (keep investing effort) or to quit. Hence, we have a sequential game and subgame perfection is one of the solution concepts to be studied. This is not the case in an effort-prize game. Whereas our effort-prize games can also model the latter kind of contests, they implicitly assume that the effort choices of the players are made once and for all. That is, the players have to say when do they want to quit regardless of what the other players do in the meantime. A motivation for this assumption would be a situation in which players do not receive information about what the other players have done as the time passes; *i.e.*, they are playing a *silent* timing game. Hence, because of the previous observations, we do not have an extensive form game anymore. Patent races are a natural setting for silent timing games. In such a scenario, it might be the case that a firm, after deciding to quit the patent race, does it in a silent way; depriving its rival of the possibility of reallocating its resources in the other areas in which they might also be competitors.

In Section 2 we present several models that have already been studied in the literature along with the existing results concerning Nash equilibrium in each of them. The reader will note that many of the results are very similar to each other. Hence, the natural question is to wonder what are the common features underlying these models that lead to these similarities. That question is the one that motivated this paper and effort-prize games are our answer. We show that all these models are special effort-prize games and that the specific results of each model are special cases of our results for general effort-prize games. Up to the best of our knowledge, there has been no formal attempt to unify these models and results under the same framework before.

Summarizing, in this paper we do the following. First, we define a unifying model of contests: the effort-prize games. Second, we characterize the set of Nash equilibria of any effort-prize game and discuss the implications of these results within the specific models we generalize. Third, we discuss how far we can push our model to analyze new economic situations. Since all the existing results in the specific models can be incorporated to our framework, it is also the case that some of the proofs can also be adapted, *mutatis mutandis*, to our general situation. Nonetheless, our reasons for having carefully written all the proofs are i) our model is much more general than the existing ones and, hence, although the underlying arguments are very similar, new proofs are needed, ii) this is the first time that all the results are proved under the same notation; up to now, the different results were spread around different papers, with different notations, and with quite different degrees of mathematical rigor.

The paper is structured as follows. In Section 2 we present various models of contests. In Section 3 we present our unifying model and formally define the effort-prize games. In Sections 4 and 5 we present and discuss the main results. Finally, in Section 6 we briefly discuss about other possible applications of effort-prize games.

2 Some Models of Contests

In this Section we present an overview of a series of models of contests that have been studied in economic literature. We assume complete information throughout the paper. The underlying idea of all the models we present here is the same. There is a set of agents that want to get a prize. In order to do so, each of them has to make some effort (investment). These efforts are chosen simultaneously and independently. Finally, the prize is awarded to the agent that has made the higher effort. In case of a tie, the prize is shared according to some rule. The results we present in Sections 4 and 5 show that, with a lot of generality, the selected tie-breaking rule is not relevant for the equilibrium analysis. Hence, the basic elements of all the models we present below are the same. For deeper motivations and applications of each specific model we refer the reader to the specific literature.

Primitives: The set of players is $N := \{1, \dots, n\}$. Moreover, for each model, we have some parameters and functions that determine the payoff functions.

Strategies: There is an interval E whose elements represent the levels of effort available to each player in the model at hand. Let $\sigma = (e_1, \dots, e_n) \in E^n$ be a profile of efforts. Then, $w^\sigma := \operatorname{argmax}_{i \in N} e_i$ denotes the set of winners under σ and $|w^\sigma|$ denotes its cardinality.

Payoff Functions: The payoff of a player depends on his effort and on whether he has received the prize or not. In most of the models below it is implicitly assumed that, in the case of a tie, the prize is awarded to each one of the winners with equal probability.

Equilibrium Results: For each model, we briefly present the already existing results concerning Nash Equilibrium.

- **Model 1.** First Price Auction (FPA).
Players have to bid in order to get an object. Each player has a valuation v_i of the object. Players submit their bids simultaneously and independently. The player with the highest bid gets the object and pays his bid. The losers pay 0.

Primitives: For each $i \in N$, v_i is i's valuation of the object.
Strategies: $E = [0, \infty)$. To be interpreted as bids.
Payoff Functions: Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} v_i - e_i & \{i\} = w^\sigma \\ \frac{v_i - e_i}{|w^\sigma|} & \{i\} \in w^\sigma \\ 0 & \text{otherwise.} \end{cases}$$

Equilibrium Results: i) If there are at least two players with the highest valuation, then it is a pure Nash equilibrium for them to bid such valuation whereas the other players bid 0. Moreover, players get payoff 0. ii) If there is a unique player with the highest valuation, then there is no Nash equilibrium in pure strategies.[1]

- **Model 2.** Politically Contestable Rents and First Price All-Pay Auctions (APA).
Hilman and Riley [10], Baye et al. [1, 3, 4].

[1] It is well known that for the first price auction this problem of non-existence of the Nash equilibrium can be overcome by discretizing the sets of strategies. Nonetheless, this can not be done, for instance, in the all-pay auction below.

In this model, all players get a disutility equal to their effort (bid), regardless of whether get the prize or not. Both politically contestable rents and all-pay auctions are completely analogous models. For instance, the results in [1] for all-pay auctions are used in [3] to analyze the lobbying process, a well known representative of the applications of the politically contestable rents' models.

Primitives: For each $i \in N$, his valuation of the prize v_i.
Strategies: $E = [0, \infty)$. To be interpreted as bids.
Payoff Functions: Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} v_i - e_i & \{i\} = w^\sigma \\ \frac{v_i}{|w^\sigma|} - e_i & \{i\} \in w^\sigma \\ -e_i & \text{otherwise.} \end{cases}$$

Equilibrium Results: i) There is no Nash equilibrium in pure strategies. ii) A complete characterization of the set of mixed Nash equilibria and the corresponding payoffs is provided for the different configurations of the valuations.

- **Model 3.** Politically Contestable Transfers (PCT).
Hilman and Samet [11], Hilman and Riley [10].

In this model, each loser, apart from paying his investment e_i, has to make a transfer L_i. Then, the payoffs W_i of the winners are determined by both the prize and the transfers of the losers. In case of a tie, the transfers of the losers are shared equally among the agents.

Primitives: For each $i \in N$, we have his transfer L_i, his prize payoff is W_i and his "aggregate" valuation of the prize $v_i = W_i + L_i$.
Strategies: $E = [0, \infty)$. To be interpreted as bids.
Payoff Functions: Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} W_i - e_i & \{i\} = w^\sigma \\ \frac{W_i}{|w^\sigma|} - e_i & \{i\} \in w^\sigma \\ -L_i - e_i & \text{otherwise.} \end{cases}$$

Equilibrium Results: i) There is no Nash equilibrium in pure strategies. ii) Assume, without loss of generality, that $v_1 \geq v_2 \geq \ldots \geq v_n$. Then, if $v_2 > v_3$, the mixed Nash equilibrium exists and is unique. Only players 1 and 2 put some effort in equilibrium and the payoffs are $\eta_1 = v_1 - v_2$ and, for each $i \neq 1$, $\eta_i = -L_i$.

- **Model 4.** Standard Price Competition: Bertrand Model (BM).
Tirole [17, Chapter 5].

There are several firms that are immersed in a price competition. The firms are assumed to produce a homogeneous product with cost functions that exhibit weakly decreasing average costs. Firms competition is in prices and the winner takes all the market. For simplicity, we assume that the price cannot exceed an exogenously given value $\bar{p}$; think, for instance, that there is 0 demand for prices above $\bar{p}$. Moreover, the demand function is continuous and decreasing in price and such that, for each $i \in N$, the function $(\bar{p}-e)D(\bar{p}-e)-c_i(D(\bar{p}-e))$ is strictly decreasing in e.[2]

Primitives: The demand function $D(\cdot)$. For each player $i \in N$, his cost function $c_i(\cdot)$. The price upper bound $\bar{p}$.

[2]The latter assumption is not standard in general Bertrand competition models. We discuss a little bit more on it after introducing assumption A1′ in Sections 3 and 6.

Strategies: $E = [0, \infty)$. To be interpreted as discounts with respect to $\bar{p}$.
Payoff Functions: Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} (\bar{p} - e_i)D(\bar{p} - e_i) - c_i(D(\bar{p} - e_i)) & \{i\} = w^\sigma \\ (\bar{p} - e_i)\frac{D(\bar{p}-e_i)}{|w^\sigma|} - c_i(\frac{D(\bar{p}-e_i)}{|w^\sigma|}) & \{i\} \in w^\sigma \\ 0 & \text{otherwise.} \end{cases}$$

Equilibrium Results: Bertrand Paradox: If cost functions functions are constant and equal across firms, then the unique equilibrium is the one in which all the firms price their marginal cost and they make no profits.

- **Model 5.** Varian's Model of Sales: Price Competition with Loyal Customers (MS).
Varian [18], Narasimham [14], Deneckere et al. [5], Baye et al. [2].

We have the same setting as above, but now the consumers are divided in loyal and strategic. Moreover, the total demand is assumed to be constant. Each firm i has n_i loyal consumers. Loyal consumers will buy the product whenever it is below some maximum price $\bar{p}$. Strategic consumers buy from the cheapest firm. A firm cannot price-discriminate among consumers. The firm with the lowest price gets the m strategic consumers.

Primitives: For each player $i \in N$, his cost function $c_i(\cdot)$ and i's loyal customers n_i. The number of strategic consumers m. The price upper bound $\bar{p}$.
Strategies: $E = [0, \infty)$. To be interpreted as discounts with respect to $\bar{p}$.
Payoff Functions: Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} (\bar{p} - e_i)(n_i + m) - c_i(n_i + m) & \{i\} = w^\sigma \\ (\bar{p} - e_i)(n_i + \frac{m}{|w^\sigma|}) - c_i(n_i + \frac{m}{|w^\sigma|}) & \{i\} \in w^\sigma \\ (\bar{p} - e_i)n_i - c_i(n_i) & \text{otherwise.} \end{cases}$$

Equilibrium Results: If the cost function is common for all the firms and average costs are strictly decreasing: i) There is no Nash equilibrium in pure strategies. ii) Under the assumption that all the firms have the same number of loyal consumers, [18] finds a symmetric mixed Nash equilibrium and, later on, [2] characterizes the whole set of equilibria under the same assumption. iii) In [14], the equilibrium is shown to be unique when $n = 2$; even if the share of loyal consumers is not symmetric.[3]

- **Model 6.** Federalism and Economic Growth (FEG).
Hatfield [9].

This model compares the equilibrium taxation schemes when under centralized and decentralized governments. They consider a situation in which the capital investment depends on the chosen tax policies and show that, in the centralized case, there are equilibria in which the government does not implement a tax policy that maximizes economic growth. On the other hand, they show that this it not so when the different districts can choose their own tax policies to attract investors. The reason is that the latter leads to a Bertrand-like competition between the (homogeneous) districts to choose the tax policy that is more engaging for investors. A tax policy consists of a capital tax and a labor tax. Given a labor tax, there is a unique capital tax that makes the joint policy efficient, and *vice versa*. Moreover, under efficiency, both taxes are negatively related.

Here, we present a simplified version of the decentralized (federal) model included in [9]. Each district has to chose a labor tax policy, namely e, so as to attract investors (this

[3] In [5], price-leadership is studied using a 2-stage game. Then, the two-player result obtained in [14] is used to find the subgame perfect equilibria.

selection also pins down the capital tax). Investors will go to the district with the best rate of return on capital $\rho(e)$. The payoff of a district is calculated as the net wage *per capita* in such district, $\omega(\rho(e))$. We assume that there is a labor tax $m \in \mathbb{R}$ such that no investment is made in districts with labor taxes below m (high capital taxes). On the other hand, under the assumptions [9], there is $\bar{e} > m$ such that $\rho(\cdot)$ has a global maximum at $\bar{e}$ and $\omega(\rho(\bar{e})) > 0$. In [9], labor taxes above $\bar{e}$ are dominated strategies. Intuitively, the rate of return above $\bar{e}$ is worse than in $\bar{e}$ and the wage *per capita* decreases. Finally, in the interval $[m, \bar{e}]$, $\rho(\cdot)$ is strictly increasing whereas $\omega(\cdot)$ is strictly decreasing. Function $\omega(\cdot)$ is decreasing because it reflects the trade off between the rate of return offered to the investors and the wage *per capita* for the citizens in case of investment.[4]

Primitives: The functions $\omega(\cdot)$ and $\rho(\cdot)$. The threshold $m \in \mathbb{R}$.
Strategies: $E = [m, \bar{e}]$. To be interpreted as the labor tax charged to the citizens.
Payoff Functions: Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} \omega(\rho(e)) & \{i\} = w^\sigma \\ \frac{\omega(\rho(e))}{|w^\sigma|} & \{i\} \in w^\sigma \\ 0 & \text{otherwise.} \end{cases}$$

Equilibrium Results: In equilibrium, all the investment is made in districts with labor tax $\bar{e}$. Moreover, $\bar{e}$ maximizes economic growth (similar to marginal cost pricing in BM). Indeed, the results are shown under two different tie breaking rules, not only the one we presented above.

- **Model 7.** Price Competition between Market Makers (MM).
  Dennert [6].

  Here we present a rough simplification of the model introduced in [6, Section 2] to model price competition among market makers who want to trade a risky asset.[5] There are n firms that want to sell a product. There are no production costs. Its real value v can be either 1 or -1, each of them with equal probability. Each firm can only sell one unit of the product to each buyer. First, each firm chooses a price. Then, each buyer has probability μ of being informed about v. The uninformed buyers split evenly among those who think that $v = -1$ and those who thing that $v = 1$; the former do not want to buy the product and the latter buy it at the lowest price (provided it is not above 1). Informed buyers, that are thought of as speculators, buy one unit to each firm whenever $v = 1$ and the price is not above 1. To accommodate the model in our effort approach, we think of the strategy of a seller as a discount with respect to 1 (no trade takes place for higher prices). Suppose that firm i chooses a discount e_i. Then, with probability one half, $v = 1$ and the informed buyers buy a unit of the product from i and, hence, i looses e_i against each informed buyer. On the other hand, if i has set the lowest price, the $(1-\mu)/2$ uninformed buyers get the product from him, regardless of the real value of v. If $v = 1$, i looses e_i against each uninformed buyer and, if $v = -1$, he wins $2 - e_i$.

  Primitives: $\mu \in (0, 1)$.
  Strategies: $E = [0, \infty)$. To be interpreted as discounts with respect to 1.

[4] In [9], these functions are obtained from a more complex structure than the one we present here.
[5] Note that, in our simplification of the model, we change the underlying economic setting. Hence, some of the assumptions implicit in the model we present might seem unreasonable. We refer the reader to [6] for the motivations within the real scenario.

Payoff Functions: Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} -\frac{\mu}{2}e_i + \frac{1-\mu}{2}(1-e_i) & \{i\} = w^\sigma \\ -\frac{\mu}{2}e_i + \frac{1-\mu}{2|w^\sigma|}(1-e_i) & \{i\} \in w^\sigma \\ -\frac{\mu}{2}e_i & \text{otherwise.} \end{cases}$$

Equilibrium Results: i) There is no Nash equilibrium in pure strategies. ii) There is a unique symmetric Nash equilibrium in mixed strategies.

- **Model 8.** (Silent) Timing Games: A Silent Duel/Battle over a Cake (TG).
Hamers [8], González-Díaz et al. [7].
There is a divisible good of size 1 that has to be shared among the agents.[6] Each agent has an initial right α_i over the good and $\sum_{i \in N} \alpha_i < 1$. Each player has to choose the moment in time at which he claims his share. The latter choice is made simultaneously and independently. All the agents get their discounted initial right at the time in which they claim it. Moreover, the most patient one gets also the discounted extra share $1 - \sum_{i \in N} \alpha_i$ (discounted at the time in which he has claimed his share).[7]

Primitives: For each $i \in N$, the initial right α_i. The discount parameter $\delta \in (0,1)$.
Strategies: $E = [0, \infty)$. To be interpreted as the time the players wait before quitting (claiming his share).
Payoff Functions: Let $P = 1 - \sum_{i \in N} \alpha_i$ denote the undiscounted value of the prize. Now, for each $i \in N$ and each $\sigma \in E^n$,

$$u_i(e_1) = \begin{cases} (\alpha_i + P)\delta^{e_i} & \{i\} = w^\sigma \\ (\alpha_i + \frac{P}{|w^\sigma|})\delta^{e_i} & \{i\} \in w^\sigma \\ \alpha_i \delta^{e_i} & \text{otherwise.} \end{cases}$$

Equilibrium Results: i) There is no Nash equilibrium in pure strategies. ii) [8] shows existence and uniqueness of the mixed Nash equilbrium when $n = 2$ and [7] shows the same result for arbitrary number of players under the assumption $\alpha_1 < \alpha_2 < \ldots < \alpha_n$.

Despite of the outstanding similarities between some models and results above, it is not the case that we can take the results for one model and plug them into some of the others. Sometimes the new result will also be true, but the adaptations are not immediate. The model introduced in next Section will formally state the extent up to which the different results can be adapted across models.

3 The Unifying Model

Now we present a general model for which all the ones discussed above become very specific cases. Despite of the outstanding extra generality of the model, we show that under very mild assumptions the results of the papers quoted above carry out. Recall that we have assumed complete information. The idea of our model is the same one that underlies the above situations. There are several players that want to get a prize and it will be awarded to the one that makes the higher effort. In case of a tie, we can think of many different

[6] For this model we stick to the notations in [8], although the differences between the models in [8] and [7] do not affect the results.
[7] Silent timing games have already been applied to different economic situations. See [15] for an application to R&D.

shares of the prize among the tied players. Nonetheless, we show that is not relevant as far as, roughly speaking, at least one of the tied players does not receive the whole prize (we do even allow to share "more than one prize" when ties appear). Indeed, the later requirement is the source of the discontinuity in the payoffs and the main reason for the non-existence of a Nash equilibrium in pure strategies in most of the models introduced in Section 2.

The set N is assumed to be fixed throughout the paper. Let 2^N denote the set of all possible subsets of N and, for each $S \subseteq N$, $|S|$ denotes its cardinality. Informally, in a *effort-prize game* each player chooses the *effort* he wants to make, that is, a number in $[m, M]$, where $m \in \mathbb{R}$, $M \in \mathbb{R} \cup \{\infty\}$, and $M > m$.[8] Then, the prize is awarded to the player that has made the highest effort (ties are discussed below). Each player, depending on his effort and on whether he has achieved the prize or not, gets a certain payoff. Before formally defining an effort-prize game, we define a *effort-prize form* as a triple $(\{b_i\}_{i \in N}, \{p_i\}_{i \in N}, \{T_i\}_{i \in N})$ as follows:

- *Base payoff functions.* For each $i \in N$, there is a weakly decreasing and continuous function $b_i : [m, M] \to \mathbb{R}$. For each level of effort $e \in [m, M]$, $b_i(e)$ denotes the payoff of player i when his effort is e but he does not get the prize (*i.e.*, his effort is not among the highest ones).

- *Prize payoff functions.* For each $i \in N$, there is a weakly decreasing and continuous function $p_i : [m, M] \to \mathbb{R}$ with $p_i(m) > 0$. For each level of effort $e \in [m, M]$, $p_i(e)$ denotes the extra payoff of player i when his effort is e and it is the highest one (no ties) and, hence, he gets the prize.

- *Tie payoff functions.* For each $i \in N$, $T_i : [m, M] \times 2^N \backslash \{\emptyset\} \to \mathbb{R}$, determines $i's$ "share of the prize" when there is a tie. The element in $[m, M]$ denotes the effort made by player i and the subset of N is the set of players with the highest effort. These functions T_i have the following properties:

  T1) $T_i(e, \{i\}) = p_i(e)$, *i.e.*, if i is the only winner he gets the prize.

  T2) For each S such that $i \notin S$, $T_i(e, S) = 0$, *i.e.*, if i is not a winner he gets no extra payoff.

  T3) For each $e \in [m, M]$, each $S \in 2^N \backslash \{\emptyset\}$, with $|S| > 1$, we have

  i) for each $i \in S$, if $p_i(e) \geq 0$, then $T_i(e, S) \leq p_i(e)$. Winners get at most their prize payoff at e, provided that it is non negative,

  ii) for each $i \in S$, if $p_i(e) < 0$, then $T_i(e, S) < 0$. If i's valuation of the prize at e is negative, he must get something negative,[9]

  iii) if $\sum_{i \in S} p_i(e) > 0$, then there is $i \in S$ such that $T_i(e, S) < p_i(e)$. If sum of the prize payoffs of the tied players at e is positive, then at least one of them would be better off if he were the only winner.

Note that both base and prize payoff functions can be different for the different players. Hence, situations where the valuation of the prize may be different across players are included. Note as well that we allow for constant prize functions, *i.e.*, the effort does not necessarily affect the extra payoff when getting the prize. We have defined tie functions in a very general way, indeed, unnatural shares of the prize can be defined within our family

[8] We are making the following abuse of notation. If $M = +\infty$, then $[m, M] := [m, \infty)$.

[9] Here we even allow for situations in which players "lose more" when they are tied than they would lose being alone. Note that this is the case, for instance, in first price auctions when players are tied at bids that exceed their valuations.

of tie functions. Nonetheless, we show below that most of the results do not depend on the chosen tie functions as far as they satisfy properties T1-T3.

Let $f := (\{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$ be an effort-prize form. Now, the associated *effort-prize game with pure strategies* for the players in N, denoted by EP^f_{pure}, is defined as $EP^f_{\text{pure}} := (\{E_i\}_{i\in N}, \{u_i\}_{i\in N})$, where, for each $i \in N$, $E_i := [m, M]$ and the payoff functions are defined as follows. For each effort profile $\sigma = (e_1, \ldots, e_n) \in [m, M]^n$, let w^σ denote the set of winners, *i.e.*, $w^\sigma := \text{argmax}_{i\in N}\, e_i$. Now, for each $i \in N$ and each $\sigma = (e_1, \ldots, e_n) \in [m, M]^n$, $u_i(\sigma) := b_i(e_i) + T_i(e_i, w^\sigma)$.

The effort-prize form f is fixed throughout the paper. Moreover, we assume, without loss of generality, that $m = 0$, that is, players have to make non-negative efforts.[10] Now, according to the definition of game EP^f_{pure}, for each $i \in N$, $b_i(0)$ can be interpreted as the minimum right of player i, *i.e.*, regardless of what the other players do, player i can always ensure himself $b_i(0)$ with the strategy 0.

We show below what the effort-prize form is for each of the models discussed in Section 2. Let $i \in N$ and $S \subseteq N$ such that $i \in S$. Then,

Model 1. FPA: $b_i(e) = 0$, $p_i(e) = v_i - e$, and $T_i(e, S) = p_i(e)|S|^{-1}$.

Model 2. APA: $b_i(e) = -e$, $p_i(e) = v_i$, and $T_i(e, S) = p_i(e)|S|^{-1}$.

Model 3. PCT: $b_i(e) = -L_i - e$, $p_i(e) = L_i + W_i$, and $T_i(e, S) = L_i + W_i|S|^{-1}$.

Model 4. BM: $b_i(e) = 0$, $p_i(e) = (\bar{p} - e)D(\bar{p} - e) - c_i(D(\bar{p} - e))$, and the tie function now is $T_i(e, S) = (\bar{p} - e)D(\bar{p} - e)|S|^{-1} - c_i(D(\bar{p} - e)|S|^{-1})$.

Model 5. MS: $b_i(e) = (\bar{p} - e)n_i - c_i(n_i)$, $p_i(e) = (\bar{p} - e)m - c_i(n_i + m) + c_i(n_i)$, and the tie function now is $T_i(e, S) = (\bar{p} - e)m|S|^{-1} - c_i(n_i + m|S|^{-1}) + c_i(n_i)$.

Model 6. FEG: $b_i(e) = 0$, $p_i(e) = \omega(\rho(e))$, and $T_i(e, S) = \omega(\rho(e))|S|^{-1}$.

Model 7. MM: $b_i(e) = -\frac{\mu}{2}e$, $p_i(e) = \frac{1-\mu}{2}(1 - e)$, and $T_i(e, S) = \frac{1-\mu}{2}(1 - e)|S|^{-1}$.

Model 8. TG: $b_i(e) = \alpha_i\delta^e$, $p_i(e) = P\delta^e$, and $T_i(e, S) = P\delta^e|S|^{-1}$.

Now it is immediate to check that, for each model, the functions above lead to a well defined effort-prize form. Indeed, the only thing that is not entirely straightforward is to check that the tie payoff functions in models BM and MS satisfy T3; but they do because we have assumed that the cost functions exhibit decreasing average costs.

Remark 1. *Note that our effort-prize games do not extend other well known models of contests such as the war of attrition, the second price auction, or the second price all-pay auction. In an effort-prize game, if a player is the only winner, his payoff only depends on the effort he has made, whereas in the latter models it only depends on the losers' efforts. Indeed, this is also the reason why subgame perfection plays no role in effort-prize games. In our model, the players have to decide their effort once and for all. This is not the case, for instance, in the war of attrition, where at each time, players have to choose whether they stay in the game (*i.e.*, they put some more effort) or they quit.*[11]

Note that in the general definition of the model, both the base and prize payoff functions are weakly decreasing. Nonetheless, it is quite natural to assume that either the b_i functions

[10] Let $f = (\{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$ be an effort-prize form such that, in EP^f_{pure}, $E_i := [m, M]$ and $m \neq 0$. Then, we just need to consider the effort-prize form $f' = (\{b'_i\}_{i\in N}, \{p'_i\}_{i\in N}, \{T'_i\}_{i\in N})$, where each $b'_i : [0, M - m] \to \mathbb{R}$ is defined by $b'_i(e) := b_i(e + m)$, and similarly for the p'_i and T'_i functions. The games EP^f_{pure} and $EP^{f'}_{\text{pure}}$ are equivalent from the strategic point of view.

[11] [13] contains a general model for war of attrition that extends, for instance, second-price all-pay auctions. Moreover, regarding the discussion we made in the Introduction about silent and non-silent timing games, refer to [12] for a paper on general *non-silent* timing games and to [15] for a paper in which silent timing games are applied to R&D models (patent races).

or the p_i functions are strictly decreasing so that we ensure that the payoff functions are sensitive to the effort implemented by the players.

Assumption A1. *For each $i \in N$, the function $b_i(\cdot)$ is strictly decreasing.*

Assumption A1′. *For each $i \in N$, the function $p_i(\cdot)$ is strictly decreasing.*

Depending on whether we assume A1 or A1′ a relevant part of the contest at hand is changed. Under A1, the payoff of the players is strictly decreasing in e, regardless of whether they get the prize or not; models APA, PCT, MS, MM, and TG fall within this category. On the other hand, under A1′, only the payoff achieved when getting the prize has to be strictly decreasing in e; this situation corresponds with the FPA and FEG models. Moreover, as far as $(\bar{p}-e)D(\bar{p}-e)-c_i(D(\bar{p}-e))$ is strictly decreasing in e, the BM model also satisfies A1′. Since $-c_i(D(\bar{p}-e))$ is strictly decreasing in e, a sufficient condition for the former requirement is that $eD(\bar{p}-e)$ is weakly increasing in e; that is, given an increase Δe in the discount, the corresponding increase $\Delta D(\bar{p}-e)$ in the demand is such that $\frac{\Delta D(\bar{p}-e)}{\Delta e} \leq 1$. In words, provided that a given firm takes the whole market for sure, then it has no incentives to make further discounts, *i.e.*, if the price decreases, the increase in demand does not compensate the losses per unit. Refer to Section 6 for a discussion on the necessity of this requirement.

Now, for each $i \in N$, let $\bar{e}_i := \max_{e\in[0,M]}\{b_i(0) \leq b_i(\bar{e}_i)+p_i(\bar{e}_i)\}$. The interpretation of the $\bar{e}_i$ is as follows. If $\bar{e}_i < M$, then $b_i(0) = b_i(\bar{e}_i)+p_i(\bar{e}_i)$. That is, $\bar{e}_i$ is an upper bound for the effort that i is willing to make because higher efforts are weakly dominated by 0 and, under A1, they are strictly dominated. Henceforth, we assume, without loss of generality, that players are ordered such that $i < j$ implies that $\bar{e}_i \geq \bar{e}_j$. That is, the players with lower indices are the ones willing to make higher efforts.

Assumption A2. *For each $i \in N$, $\bar{e}_i < M$.*

Note that A2 is satisfied by all the models presented in Section 2 with the exception of FEG. Moreover, this requirement becomes very natural when translated into the different cases. Next result shows that, under A1 and A2, effort-prize games do not have Nash equilibria (in pure strategies).

Proposition 1. *If the effort-prize game EP^f_{pure} satisfies* A1 *and* A2*, then it does not have any Nash equilibrium.*

Proof. Suppose $\sigma = (e_1,\ldots,e_n) \in [m,M]^n$ is a Nash equilibrium of EP^f_{pure}. By A1 and A2, strategies above $\bar{e}_i$ are strictly dominated for player i. Hence, for each $i \in N$, $e_i \leq \bar{e}_i < M$. By A1, for each $i \in N$, the function $b_i(\cdot)+p_i(\cdot)$ is strictly decreasing. Hence, if $|w^\sigma| = 1$, the winner would be better off by doing less effort, but still ensuring himself to be the only winner. Hence, $|w^\sigma| > 1$. So there is a tie at $e \in [m,M)$. We distinguish two cases:

$\sum_{i\in w^\sigma} \boldsymbol{p_i(e)} > \mathbf{0}$**:** By T3, there is $i \in w^\sigma$ such that $T_i(e,w^\sigma) < p_i(e)$. Now, since functions $b_i(\cdot)$ and $p_i(\cdot)$ are continuous, there is $\varepsilon > 0$ such that $u_i(\sigma) = b_i(e)+T_i(e,w^\sigma) < b_i(e+\varepsilon)+p_i(e+\varepsilon) = u_i(\sigma_{-i},e+\varepsilon)$. Hence, player i would deviate.

$\sum_{i\in w^\sigma} \boldsymbol{p_i(e)} \leq \mathbf{0}$**:** Now, there is $i \in w^\sigma$ such that $p_i(e) \leq 0$. Hence, by T3, $T_i(e,w^\sigma) \leq 0$. Hence, $u_i(\sigma) = b_i(e)+T_i(e,w^\sigma) \leq b_i(e)$. Hence, since $b_i(\cdot)$ is strictly decreasing, player i would be better off by playing 0 and getting $b_i(0) > b_i(e)$. □

Remark 2. *In general, when two players are tied, one of them can get a higher payoff by putting ε effort more and getting the whole prize. The problem in many similar models is* *that, since the sets of strategies are continuous, there is no optimal reply. Hence, one could argue that the non-existence result presented above relies on the fact that players can have a continuum of strategies, but this is not the case here. The reader can verify that we cannot get rid of the latter non-existence result by discretizing the sets of strategies.*

So, for an appropriate analysis of effort-prize games we need to consider mixed strategies. Before stating formally what a mixed strategy is, we introduce some notation concerning distribution functions.

Let $F : \mathbb{R} \to [0,1]$ be a function satisfying i)F is non decreasing, ii) F is right-continuous, and iii) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$. Such a function is always a *distribution function*, corresponding to some probability measure P in $\mathbb{R}$, such that, for each $x \in \mathbb{R}$, $F(x) = P(\{y \in \mathbb{R} : x \geq y\})$. On the other hand, each probability measure P in $\mathbb{R}$ defines, by the previous formula, a distribution function F.[12] We introduce now some other notations related to a distribution function F. For each $x \in \mathbb{R}$ we denote $\lim_{y\uparrow x} F(y) = P(\{y \in \mathbb{R} : x > y\})$ by $F(x^-)$. If there is $x > 0$ such that for each $a, b \in \mathbb{R}$ with $a < x < b$ we have $F(b^-) > F(a)$, *i.e.*, the probability of choosing an element in (a, b) is positive, then x is an element of the *support* of F. The support of the distribution function F is denoted by $S(F)$. One easily verifies that $S(F)$ is a closed set. The set of *jumps* (discontinuities) of F is $J(F) := \{x \in \mathbb{R} : F(x) > F(x^-)\}$, *i.e.*, the set real numbers which are chosen with positive probability.

Now, we go back to the effort-prize game. Formally, we define a *mixed strategy* as a distribution function G such that $S(G) \subseteq [0, M]$, *i.e.*, the support of a mixed strategy must be a subset of the set of pure strategies. Let $\mathcal{G}$ denote the set of all mixed strategies. Note that for, a mixed strategy $G \in \mathcal{G}$, the set $J(G)$ denotes those pure strategies which are chosen with positive probability.

If player i chooses pure strategy e and all other players choose mixed strategies $\{G_j\}_{j\neq i}$, then the expected payoff for player i is

$$u_i(G_1, \ldots, G_{i-1}, e, G_{i+1}, \ldots, G_n) = A + B + C,$$

where

$$\begin{aligned} A &= \prod_{j\neq i} G_j(e^-)(b_i(e) + p_i(e)), \quad B = \big(1 - \prod_{j\neq i} G_j(e)\big)b_i(e), \text{ and} \\ C &= \sum_{i\in S, S\subseteq N} \Big(\prod_{j\notin S} G_j(e^-) \prod_{j\in S\setminus\{i\}} (G_j(e) - G_j(e^-))\Big)T_i(e, S). \end{aligned}$$

That is, A is the probability that i wins alone multiplied by the corresponding payoff, B is the probability that i does not win multiplied by the corresponding payoff, and C is the payoff originated in the different ties in which i can be involved.

If player i also chooses a mixed strategy G_i, whereas all other players stick to mixed strategies $\{G_j\}_{j\neq i}$, then the expected payoff for player i can be computed by the use of the Lebesgue-Stieltjes integral:

$$u_i(G_1, \ldots, G_n) = \int u_i(G_1, \ldots, G_{i-1}, e, G_{i+1}, \ldots, G_n) dG_i(e). \qquad (1)$$

Note that, with a slight abuse of notation, the functions u_i do not only denote payoffs to players when pure strategies are played, but also when mixed strategies are used.[13]

[12] See [16] for more details.

[13] We have to mention a subtle technical detail. To rigorously define the Lebesgue-Stieltjes integrals

Hence, given the effort-prize form $f := (\{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$, the associated effort-prize game for the players in N, denoted by EP^f, is defined as $EP^f := (\{X_i\}_{i\in N}, \{u_i\}_{i\in N})$, where for each $i \in N$, i's set of mixed strategies is defined by $X_i := \mathcal{G}$ and i's (expected) payoff function is defined by Eq. (1).

Finally, for notational convenience, for each strategy profile $G = (G_1, \ldots, G_n) \in \mathcal{G}^n$ and each $e \in [0, M]$, we denote the payoff $u_i(G_1, \ldots, G_{i-1}, e, G_{i+1}, \ldots, G_n)$ by $u_i^G(e)$. That is, $u_i^G(e)$ is the expected payoff of player i when he chooses the pure strategy e and all the other players act in accordance with G.

4 Characterization of the Nash Equilibria under A1

We assume A1 throughout this Section.

In this Section we present the main results we get for the models in which A1 holds. Hence, regarding the models presented in Section 2, the results in this Section apply to APA, PCT, MS, MM, and TG. Our results extend, to our unifying model, the stronger equilibrium results that had already been proved for any of the specific models. Because of the extra generality of our model, some technicalities have to be addressed. Nonetheless, the arguments underlying the proofs remain the same. Thus, since our proofs do not bring new insights, we relegate them to the Appendix.

Note that, if both A1 and A2 are assumed and $G \in \mathcal{G}^n$ is a Nash equilibrium of EP^f, then we have that, for each $i \in N$, $S(G_i) \subseteq [0, \bar{e}_i]$. Moreover, once player 1 knows that none of the other players puts positive probability above $\bar{e}_2$, the efforts in $(\bar{e}_2, \bar{e}_1]$ are strictly dominated for him. Hence, $S(G_1) \subseteq [0, \bar{e}_2]$.

Next Lemma shows that in a Nash equilibrium of EP^f no one assigns positive probability to any effort different from 0. Moreover, this leads to a remarkable simplification in the expression of $u_i^G(e)$.

Lemma 1. *Assume* A1. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Then, for each $i \in N$,*

i) $J(G_i) \cap (0, M) = \emptyset$,

ii) for each $e \in (0, M)$, $u_i^G(e) = b_i(e) + \prod_{j\neq i} G_j(e) p_i(e)$.

iii) moreover, under A2, $M \notin J(G_i)$ *and* $u_i^G(M) = b_i(M) + \prod_{j\neq i} G_j(M) p_i(M)$.

The interpretation of $u_i^G(e)$ becomes very simple now. In a Nash equilibrium, the expected payoff of each effort in $(0, M)$ is at least its base payoff, but when the player is the only winner he also gets the prize payoff (no tie-payoffs since, as far as $e \in (0, M)$, they have probability 0). Moreover, under A1 and A2, the pure strategy M is strictly dominated for every player and, hence, the previous result would be also true at M.

Now, we introduce a last element into the model. For each $i \in N$ and each $e \in [0, M]$, $b_i(0) - b_i(e)$ is a way of measuring the impact of effort e in i's base payoff; similarly, $p_i(e)$ measures the impact of e in i's prize payoff (if this is finally achieved). Now, for each $i \in N$, let $I_i : [0, M] \to \mathbb{R}$ be the *impact function* of player i, defined as $I_i(e) := (b_i(0) - b_i(e))/p_i(e)$. The functions I_i provide a way of measuring the aggregate impact of an effort e in i's potential payoff; being this impact measured with respect to i's minimum right $b_i(0)$. Note

with respect to the distribution functions we should integrate over $\mathbb{R}$. Hence, we should also define payoff functions over $\mathbb{R}$. But note that, no matter the extension of the u_i functions we consider, the integrals over $[0, M]$ remain the same (just because the support of the mixed strategies is restricted to $[0, M]$). If we do not consider integrals defined over $\mathbb{R}$, then we might have problems when calculating expected payoffs of mixed strategies that put positive probability at 0. Hence, we use $\int u_i(G_1, \ldots, G_{i-1}, e, G_{i+1}, \ldots, G_n) dG_i(e)$ to mean $\int_{\mathbb{R}} u_i^*(G_1, \ldots, G_{i-1}, e, G_{i+1}, \ldots, G_n) dG_i(e)$, where u_i^* can be any arbitrarily chosen extension of u_i to $\mathbb{R}$.

that, under A1, for each $i \in N$, $I_i(0) = 0$ and I_i is a strictly increasing function. Note that $I_i(e) = 1$ if and only if $e = \bar{e}_i$. Recall that, under A1, efforts above $\bar{e}_i$ are strictly dominated for player i.

For the characterization result we present below we need to assume A2. Moreover, we need to introduce two more assumptions: A3 and A4. These assumptions are complements to A2. More specifically, we only assume A3 as a complement to A2 and A4 as a complement to A3. The relevance of these assumptions is discussed in Subsection 4.1.

Assumption A3. Fixed ordering of the impact functions. *For each pair $i, j \in N$, if there is $e^* \in (0, M]$ such that $I_i(e^*) < I_j(e^*)$ then, for each $e \in (0, M]$, $I_i(e) < I_j(e)$.*

Informally, this Assumption says that, if the impact of a certain effort is higher for i than for j, then, for every effort the same relation holds. Note that it also implies that, if there is $e^* \in (0, M]$ such that $I_i(e^*) = I_j(e^*)$ then, for each $e \in [0, M]$, $I_i(e) = I_j(e)$.

A3 is eventually satisfied by all the models at hand in this Section and its expression becomes very natural when translated into the different models.[14] Recall that, under A2, we have that $i < j$ implies i) $\bar{e}_i \geq \bar{e}_j$ and now, because of A3, ii) for each $e \in [0, M]$, $I_i(e) \leq I_j(e)$.

Now we introduce the last assumption in the model. Suppose that $\bar{e}_1 > \bar{e}_2$. Since player 1 knows that no one but him is willing to put any effort above $\bar{e}_2$ (they are strictly dominated strategies), he can ensure himself a payoff as close to $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ as desired. So, somehow, he can ensure himself a minimum right of $b_1(\bar{e}_2) + p_1(\bar{e}_2)$. Now, using the idea of the definition of the impact functions we define $I_1^* : [0, M] \to \mathbb{R}$ by $I_1^*(e) := (b_1(\bar{e}_2) + p_1(\bar{e}_2) - b_1(e))/p_1(e)$. That is, the impact of an effort e in player 1's potential payoff but with respect to $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ instead of $b_1(0)$. Since we are assuming A1, $I_1^*(\cdot)$ is strictly increasing. Note that $I_1^*(0) > 0 = I_2(0)$ and $I_1^*(\bar{e}_2) = 1 = I_2(\bar{e}_2)$. Next assumption just says that functions $I_1^*(\cdot)$ and $I_2(\cdot)$ cannot cross below $\bar{e}_2$.

Assumption A4. *If $\bar{e}_1 > \bar{e}_2$ and $\bar{e}_2 = \bar{e}_3$, then, for each $e \in (0, \bar{e}_2)$, $I_2(e) < I_1^*(e)$.*

We only use the previous assumption combined with A3. It is a weak requirement and, moreover, as a complement to A3, it turns to be even weaker. Indeed, it follows the same motivation but only applies in very specific cases; that is, if it is not the case that $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$, then A4 is not needed. Again, the reader can check that the models discussed above satisfy this property.[14]

Let $G^* \in \mathcal{G}^n$ be the strategy profile defined for players 1, 2, and each $i \in N \backslash \{1, 2\}$ as follows:

$$G_1^*(e) = \begin{cases} 0 & e < 0 \\ I_2(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_2^*(e) = \begin{cases} 0 & e < 0 \\ I_1^*(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_i^*(e) = \begin{cases} 0 & e < 0 \\ 1 & e \geq 0 \end{cases}.$$

The payoffs associated with G^* are $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ for player 1 and, for each $i \neq 1$, $b_i(0)$. If $\bar{e}_1 > \bar{e}_2$, then $b_1(\bar{e}_2) + p_1(\bar{e}_2) > b_1(0)$. On the other hand, if $\bar{e}_1 = \bar{e}_2$, then $b_1(\bar{e}_2) + p_1(\bar{e}_2) = b_1(0)$ and $I_1^*(\cdot) = I_1(\cdot) = I_2(\cdot)$. Hence, in the latter case players 1 and 2 play the same mixed strategy (and get the same payoffs). Now, we are ready to present the characterization result. It shows that, under Assumption A1, the use of mixed strategies allows to recover the existence of the Nash equilibrium with few extra requirements.

Theorem 1. *Assume* A1, A2, A3, *and* A4. *Then,*

[14]Only model MS might not met the assumption if we allow simultaneously for different numbers of loyal consumers and different cost functions for the different firms.

i) If either $n = 2$ or $\bar{e}_1 > \bar{e}_2 > \bar{e}_3$, then G^* is the unique Nash equibrium of EP^f.

ii) Otherwise, EP^f has a continuum of Nash equibria and G^* is one of them.

All the Nash equilibria of EP^f give raise to the same payoffs: $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and, for each $i \neq 1$, $\eta_i = b_i(0)$.

We refer the reader to the Appendix for the expressions of the Nash equilibria referred to in the previous result.[15]

4.1 Implications of the Characterization Result under A1

The main implication of Theorem 1 is the following: the characterization results presented in [4] can be translated to any other of the models satisfying A1-A4 and they remain valid. Indeed, every existing equilibrium result in any of the models satisfying A1-A4 can be translated to any other model satisfying A1-A4. Note that, with our general model, we have provided the appropriate language to do such translations and, moreover, to do them in a completely rigorous way.[16]

More specifically, the implications of our characterization result are noteworthy in the MS model. It extends the results included in [14] to an arbitrary number of firms. That is, even if the number of loyal consumers is different for the different firms, Theorem 1 can be applied regardless of the number of firms. Moreover, we can go even further and allow as well for different cost functions across firms and still apply Theorem 1; in this case we just have to ensure that A3 and A4 and are still met. On the other hand, the existing results for models PCT and TG only refer to very specific configurations of the parameters. Theorem 1 extends them to any chosen configuration of the primitives of the two models.

Discussion of the Assumptions

Setting aside A1, which is the assumption that characterizes the type of effort-prize games at hand in this Section, it is natural to wonder what happens if some of the assumptions A2-A4 are not met; A2 is the most relevant of the three. If A2 is not satisfied it is possible that, in equilibrium, ties happen with positive probability at M (by Lemma 1 this was not possible under A2). Hence, the role of tie functions becomes more important now. Below, we briefly discuss some of the different possibilities for the Nash equilibria when A2 is violated. In particular, the existence of pure Nash equilibria is sometimes recovered:[17]

i) If $M = \bar{e}_1 > \bar{e}_2$, we are almost in the same situation as before. If A3 and A4 are met, then the same result as in Theorem 1 still holds (with $\bar{e}_1 > \bar{e}_2$).

ii) If there is $S \subseteq N$, $|S| > 1$, such that $e_i = M$ if and only if $i \in S$. Then, for each $S' \subseteq S$ such that

   a) for each $i \in S'$, $T_i(M, S') + b_i(M) \geq b_i(0)$ and

   b) for each $j \in S \backslash S'$, $T_j(M, S' \cup \{j\}) + b_j(M) \leq b_j(0)$,

   there is a Nash equilibrium in which players in S' put probability 1 at M and players in $N \backslash S'$ put probability 1 at 0.

[15] The reader can also check that the existing results for the models discussed in this Section are particular cases of Theorem 1 and of the equilibrium expressions presented in the Appendix.

[16] In [4], apart from giving a complete characterization of the Nash equilibria for the all-pay auction, it is also discussed whether they are revenue equivalent or not depending on the configurations of the valuations.

[17] Now we do not try to give a characterization result as Theorem 1. This is because the set of Nash equilibria would depend on the specific configurations of the $\bar{e}_i$ parameters, on the tie payoff functions, and on whether A3 and A4 are met or not. Hence, a clean result as Theorem 1 is not possible here (at least we have not been able to find it).

iii) Moreover, depending on the tie functions and on the extent up to which A3 and A4 hold for the different configurations of the $\bar{e}_i$ parameters, there can exist mixed Nash equilibria similar to those in Theorem 1.

Finally, the assumptions A1-A4 are sufficient conditions to get Theorem 1. For this characterization result A2 is crucial, but it would also be interesting to explore the tightness of the result and study whether some of the assumptions A3 and A4 can be weakened.

5 Characterization of the Nash Equilibria under A1′

We assume A1′ throughout this Section. Then, we present two characterization results depending on whether A2 is met or not. Hence, the result under A2 applies to FPA and BM (again, $(\bar{p}-e)D(\bar{p}-e)-c_i(D(\bar{p}-e))$ has to be strictly decreasing in e) and the result when A2 is not satisfied applies to FEG.

First of all, we introduce a preliminary result.

Lemma 2. *Assume* A1′. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Let $e \in [0, M]$ and let $i \in N$ be such that $\bar{e}_i > e$. Then, the probability that i participates in a tie at e is* 0.

Proof. Suppose that there is positive probability that i participates in a tie at e. Let $S \subseteq N$ be a coalition containing i for which a tie at e has positive probability. If $\sum_{j\in S} p_j(e) > 0$, by T3, there is $j \in S$ such that $T_j(e,S) < p_j(e)$. Now, since functions $b_j(\cdot)$ and $p_j(\cdot)$ are continuous, there is $\varepsilon > 0$ such that $b_j(e)+T_j(e,S) < b_j(e+\varepsilon)+p_j(e+\varepsilon)$. Now, it is easy to check that the same holds for $u_j^G(e)$. That is, there is $\varepsilon' > 0$ such that $u_j^G(e) < u_j^G(e+\varepsilon')$.[18] Hence, player j would be better off by moving his probability in e to somewhere in $(e, e+\varepsilon']$. Suppose now that $\sum_{j\in S} p_j(e) \leq 0$. Since $e < \bar{e}_i$, $p_i(e) > 0$. Hence, there is $j \in S$ such that $p_j(e) < 0$. Hence, by T3, $T_j(e,S) < 0$ and $u_j^G(e) = b_j(e) + T_j(e,S) < b_j(e)$. Hence, player j would be better off by moving his probability at e to 0. □

Under A1′, the prize payoff functions are strictly decreasing and the base payoff functions are weakly decreasing. Within this framework, once A1 is not assumed, the more natural case is the one in which the base payoff functions, apart from not being strictly decreasing, are constant; indeed, this is the case in FPA, BM, and FEG. Henceforth, we assume that, for each $i \in N$, there is $b_i \in \mathbb{R}$ such that, for each $e \in [0, M]$, $b_i(e) = b_i$. Recall that, for each player $i \in N$, $\bar{e}_i$ is such that efforts above $\bar{e}_i$ are weakly dominated strategies (under A1, they were strictly dominated). Since we are assuming that the base payoffs are constant, if $\bar{e}_i < M$, then $b_i = b_i + p_i(\bar{e}_i)$ and, hence, $p_i(\bar{e}_i) = 0$.

Now, given a mixed strategy profile $G \in \mathcal{G}^n$, we define $e_G := \inf\{e \in \bigcup_{i\in N} S(G_i) : \prod_{i\in N} G_i(e) > 0\}$. If $e_G < M$, then e_G is the smallest effort such that $e > e_G$ implies that, for each $i \in N$, $G_i(e) > 0$. Similarly, if $e_G < M$, given $G \in \mathcal{G}^n$ and an effort $e \in [0, M]$, the probability of getting the prize at e is: i) zero if $e < e_G$ and ii) strictly positive if $e > e_G$ (at e_G both things might happen). The effort level e_G is essential for the characterization result we present in Theorem 2. Before we introduce a Lemma that almost pins down the value e_G must take in a Nash equilibrium (if any). Moreover, it shows that no one puts positive probability above e_G. Since, by definition of e_G, there is $i \in N$ such that $G_i(e_G^-) = 0$, we have that player i puts probability 1 at e_G. Furthermore, the latter implies that, with probability 1, the prize will be awarded at effort e_G.

[18] If there are no discontinuities in $(e, e+\varepsilon]$, then a continuity argument for that interval does the job. If there are discontinuities of $u_j^G(\cdot)$ in $(e, e+\varepsilon]$, then they make the function take even higher values.

Lemma 3. *Assume* A1′ *and that, for each $i \in N$, $b_i(\cdot)$ equals constant $b_i \in \mathbb{R}$. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Then,*

i) for each $i \in N$, $G_i(e_G) = 1$,

ii) $e_G \in [\bar{e}_2, \bar{e}_1]$, and

iii) for each $i \in N$, if $\bar{e}_i < e_G$, then $G_i(e_G) - G_i(e_G^-) = 0$.

Proof. **i)** If $e_G = M$, then there is nothing to prove. Hence, we assume that $e_G < M$. We divide the proof in two steps:

There is no $i \in N$ such that $G_i(e_G) < 1$ and, for each $j \neq i$, $G_j(e_G) = 1$: Suppose there is such i. Let $e > e_G$ be such that $e \in S(G_i)$. Now, since for each $j \neq i$, $G_j(e_G) = 1$, there is $\varepsilon > 0$ such that $e_G + \varepsilon < e$ and $u_i^G(e_G + \varepsilon) = b_i + p_i(e_G + \varepsilon) > b_i + p_i(e) = u_i^G(e)$. Contradiction with $e \in S(G_i)$ (e is not a best reply for i against G_{-i}).

There is no $S \subseteq N$, $|S| > 1$, such that, for each $i \in S$, $G_i(e_G) < 1$: Suppose there is such S and assume, without loss of generality, that S is maximal, *i.e.*, if $j \notin S$, then $G_j(e_G) = 1$. For each $i \in S$, $\bar{e}_i > e_G$, since, otherwise, by definition of e_G, strategies above e_G would be strictly dominated for player i. Now, if, for each $i \in S$, $G_i(e_G) - G_i(e_G^-) > 0$, then, by definition of e_G, the probability that players in S participate in a tie at e_G is positive and we get a contradiction with Lemma 2. Hence, there is $j \in S$ such that $G_j(e_G) - G_j(e_G^-) = 0$ and the probability of a tie at e_G is zero. Now, combining the latter with the definition of e_G, the probability of winning with a strategy $e \leq e_G$, is zero. Let $i \in S$. There is $\varepsilon > 0$ such that, if $e \leq e_G$, e is strictly dominated by $e_G + \varepsilon < \bar{e}_i$. Hence, $G_i(e_G) = 0$. Now, $u_i^G(e_G) = b_i$ and, since $|S| > 1$, the function $u_i^G(\cdot) = b_i + \prod_{j \neq i} G_j(\cdot) p_i(\cdot)$ is continuous at e_G. Hence, since $e_G \in S(G_i)$ and $G_i(e_G) = 0$, there is $\varepsilon > 0$ such that $[e_G, e_G + \varepsilon] \in S(G_i)$. Now, $u_i^G(e_G + \varepsilon) > b_i$ and $\lim_{\delta \to 0^+} u_i^G(e_G + \delta) = b_i$. But, since all the strategies in the support of a Nash equilibrium must lead to the same payoff, the latter is not possible.

ii) By i), for each $i \in N$, $G_i(e_G) = 1$. Hence, by definition of e_G, there is $i \in N$ such that $G_i(e_G) = 1$ and $G_i(e_G^-) = 0$, *i.e.*, player i puts probability 1 at e_G. Fix player i. Also by i), the probability of winning at e_G is 1. Hence, we immediately have that $e_G \leq \bar{e}_1$, since, otherwise, player i would be better off by moving his probability at e_G to 0. Now, suppose that $e_G < \bar{e}_2$. Then, there is $j \neq i$ such that $e_G < \bar{e}_j$. By Lemma 2, the probability that j participates in a tie at e_G is 0. Now, since $\bar{e}_j > e_G$ and, for each $e \in [0, e_G)$, $u_j^G(e) = b_j$, there is again $\varepsilon > 0$ such that player j can improve by playing pure strategy $e_G + \varepsilon < \bar{e}_j$, winning the prize for sure and getting a payoff $b_j + p_j(e_G + \varepsilon) > b_j + p_j(\bar{e}_j) = b_j$.

iii) Let $j \in N$ be such that $\bar{e}_j < e_G$ and $G_j(e_G) - G_j(e_G^-) > 0$. Then, by i), the probability that j wins at e_G is positive. Now, for each $S \subseteq N$, $j \in S$, by T3, $T_j(e_G, S) \leq p_j(e_G) < 0$. Hence, $u_j^G(e_G) < b_j$ and player j would be better off moving his probability at e_G to 0. □

Next Theorem fully characterizes the structure of the Nash equilibria under A2.

Theorem 2. *Assume* A1′, A2, *and that, for each $i \in N$, $b_i(\cdot)$ equals constant $b_i \in \mathbb{R}$.*

i) Let $\bar{e}_1 > \bar{e}_2$. Then, EP^f has no Nash equilibrium in pure strategies but it has a continuum of mixed Nash equilibria. Moreover, the equilibrium payoffs are such that $\eta_1 \in (b_1, b_1 + p_1(\bar{e}_2)]$ and, for each $i \neq 1$, $\eta_i = b_i$.

ii) Let $\bar{e}_1 = \bar{e}_2$. Then, the set of Nash equilibria of EP^f is nonempty if and only if there is $S \subseteq N$, $|S| > 1$, such that, for each $i \in S$, $T_i(\bar{e}_2, S) = 0$.

Indeed, whenever players in S play pure strategy $\bar{e}_2$ and players in $N \backslash S$ put probability 0 at $\bar{e}_2$ we have a Nash equilibrium of EP^f. Moreover, the equilibrium payoffs are constant across equilibria: for each $i \in N$, $\eta_i = b_i$. Finally, if $n = 2$ and $T_1(\bar{e}_2, \{1,2\}) = T_2(\bar{e}_2, \{1,2\}) = 0$, then the strategy profile in which both players play the pure strategy $\bar{e}_2$ is the unique Nash equilibrium.

Proof. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. By Lemma 3, $e_G \in [\bar{e}_2, \bar{e}_1]$ and there is $i \in N$ such that $G_i(e_G^-) = 0$ and $G_i(e_G) = 1$. That is, player i plays pure strategy e_G.

i) $\boldsymbol{\bar{e}_1 > \bar{e}_2}$**.** Now, $i = 1$, since, otherwise, either player 1 would like to deviate to $e_G + \varepsilon < \bar{e}_1$ or player i would like to deviate to $e = 0$. Now, we claim that player 1 wins for sure at e_G with no ties. Ties at e_G can only have positive probability when $e_G = \bar{e}_2$. But, since player 1 would participate in such ties and $\bar{e}_1 > \bar{e}_2$, the latter is ruled out by Lemma 2. Hence, when G is played, player 1 gets $b_1 + p_1(e_G)$ and each $i \neq 1$ gets b_i. Now, for each $i \neq 1$ and each $e \in [0, M]$, $u_i^G(e) \leq b_i$. Hence, for each $i \neq 1$, i) $S(G_i) \subseteq [0, e_G]$, ii) since ties at e_G have probability 0, $G_i(e_G) - G_i(e_G^-) = 0$, and iii) i has no incentive to deviate from G.

Now, we show that $e_G < \bar{e}_1$. Suppose $e_G = \bar{e}_1$. Then, player 1 gets payoff $b_1 + p_1(\bar{e}_1) = b_1$. Then, since there is $e < \bar{e}_G$ such that $\prod_{j \neq 1} G_j(e) > 0$, we have $u_1^G(e) = b_1 + \prod_{j \neq 1} G_j(e) p_1(e) > b_1$. Hence, player 1 would move his probability in e_G to e. Hence, in a Nash equilibrium, $e_G \in [\bar{e}_2, \bar{e}_1)$. To ensure that player 1 has no incentive to deviate we need that, for each $e \in [0, M]$, $u_1^G(e_G) - u_1^G(e) \geq 0$. If $e > e_G$, since $p(\cdot)$ is strictly decreasing, $u_1^G(e) = b_1 - p_1(e) < b_1 - p_1(e_G) = u_1^G(e_G)$ and we are done. Hence, we need that, for each $e \in [0, \bar{e}_G)$, $u_1^G(e_G) - u_1^G(e) = b_1 + \prod_{j \neq 1} G_j(e_G) p_1(e_G) - b_1 - \prod_{j \neq 1} G_j(e) p_1(e) = p_1(e_G) - \prod_{j \neq 1} G_j(e) p_1(e) \geq 0$. Hence, for each $e \in [0, \bar{e}_G)$, the following equation must hold,[19]

$$\prod_{j \neq 1} G_j(e) \leq \frac{p_1(e_G)}{p_1(e)}. \tag{2}$$

Now, there is a continuum of mixed strategies that can be defined so that Eq. (2) is satisfied. Note that, once Eq. (2) is met, player 1 has no incentives to put efforts below $\bar{e}_2$. Finally, when e_G takes the different values in $[\bar{e}_2, \bar{e}_1]$ we get the different payoffs for player 1.

ii) $\boldsymbol{\bar{e}_1 = \bar{e}_2}$**.** Now, $e_G = \bar{e}_2$ and player i is such that $\bar{e}_i = \bar{e}_2 = e_G$. Hence, $p_i(\bar{e}_2) = 0$ and $u_i^G(\bar{e}_2) = b_i + p_i(\bar{e}_2) = b_i$. Now, we show that if there is no coalition S, $|S| > 1$, such that, for each $i \in S$, $T_i(\bar{e}_2, S) = 0$, then G cannot be a Nash equilibrium. Suppose there is $j \in N$, $j \neq i$, such that $G_j(\bar{e}_2) - G_j(\bar{e}_2^-) > 0$. Now, by statement i) of Lemma 3, the probability of a tie at $\bar{e}_2$ is positive. But, by assumption, some of the the tied players has a negative tie payoff at $\bar{e}_2$. Hence, such player would be better off moving his probability at $\bar{e}_2$ to 0. Suppose, on the other hand, that for each $j \in N$, $j \neq i$, we have $G_j(\bar{e}_2) - G_j(\bar{e}_2^-) = 0$ and $G_j(\bar{e}_2) = 1$. Now, there is $e < \bar{e}_2$ such that $\prod_{j \neq i} G_j(e) > 0$ and, hence, $u_i^G(e) = b_i + \prod_{j \neq i} G_j(e) p_i(e) > b_i$. So player i would move his probability in $\bar{e}_2$ to e.

Let $\bar{S} := \{i \in N : \bar{e}_i = \bar{e}_2\}$. Now, G satisfies the following properties:

1) For each $j \in N$, $S(G_j) \subseteq [0, \bar{e}_2]$ (Lemma 3)

2) For each $j \notin \bar{S}$, j puts probability 0 at $\bar{e}_2$, *i.e.*, $G_j(\bar{e}_2) - G_j(\bar{e}_2^-) = 0$ (Lemma 3).

3) There are at least two players that play pure strategy $\bar{e}_2$. Suppose, on the contrary, that only player i plays pure strategy $\bar{e}_2$. Then, there is $e < \bar{e}_2$ such that $\prod_{j \neq i} G_j(e) > 0$. Then, $u_i^G(e) = b_i + \prod_{j \neq i} G_j(e) p_i(e) > b_i$. So player i would move his probability in $\bar{e}_2$ to e.

[19]Note that Eq. (2) is consistent with the requirement $\prod_{j \neq 1} G_j(e_G) = 1$ imposed by statement i) of Lemma 3.

4) If $S' \subseteq \bar{S}$ is such that the probability of a tie at $\bar{e}_2$ among the players in S' is positive according to G, then, for each $i \in S'$, $T_i(\bar{e}_2, S') = 0$. Suppose there is $j \in S'$ such that $T_j(\bar{e}_2, S') \neq 0$. Since $T_j(\bar{e}_2, S') \leq p_j(\bar{e}_2) = 0$, we have $T_j(\bar{e}_2, S') < 0$. So player j would move his probability in $\bar{e}_2$ to 0.

Now it is straightforward to check that every strategy profile satisfying properties 1)-4) is a Nash equilibrium of EP^f. The existence of one such profile is ensured just by defining G as the strategy profile in which players in S play pure strategy $\bar{e}_2$ and all the others play, for instance, a pure strategy $e \in [0, \bar{e}_2)$. Finally, the uniqueness when $n = 2$ follows from property 3) above. □

Remark 3. *Note that the proof of Theorem 2 fully characterizes the set of Nash equilibria in the different cases. In particular, it shows that, if $\bar{e}_1 = \bar{e}_2$, then the Nash equilibrium may fail to exist. Theorem 2 also shows the non-existence of Nash equilibrium in pure strategies when $\bar{e}_1 > \bar{e}_2$. The source of the latter non-existence is the fact that, when player 2 is playing $\bar{e}_2$, there is no optimal reply for player 1. Hence, as in standard FPA, we can recover existence by discretizing the sets of strategies of the players (indeed, doing it for player 1 would suffice).*

After the result above it is natural to wonder why, if $\bar{e}_1 > \bar{e}_2$, we do loss the uniqueness of the equilibrium payoffs that we had under A1. The answer is as follows. Under A1, pure strategies above $\bar{e}_2$ were strictly dominated for players different from player 1. Now, since we are assuming the $b_i(\cdot)$ functions are constant, such strategies are only weakly dominated. Hence, players different from 1 can use them to threaten player 1 and, hence, new equilibrium payoffs can be supported. Nonetheless, it is clear that many refinements of Nash equilibrium concept would allow us to get rid of the equilibria in which player 1 gets less that $b_1 + p_1(\bar{e}_2)$. On the other hand, being the $b_i(\cdot)$ functions constant, it is not possible to coordinate players' mixed strategies over a common support, as we could do under A1; this is explicitly used in the proof of Lemma 3 when we show that "There is no $S \subseteq N$, $|S| > 1$, such that, for each $i \in S$, $G_i(e_G) < 1$".

Now, we turn to the characterization result when A2 is not met. Let $S^M := \{i \in N : \bar{e}_i = M\}$.

Theorem 3. *Assume* A1′, *that* A2 *is not satisfied, and that, for each $i \in N$, $b_i(\cdot)$ equals constant $b_i \in \mathbb{R}$.*

i) Let $|S^M| = 1$, i.e., *$M = \bar{e}_1 > \bar{e}_2$. Then, EP^f has no Nash equilibrium in pure strategies but it has a continuum of mixed Nash equilibria. Moreover, the equilibrium payoffs are such that $\eta_1 \in [b_1 + p_1(M), b_1 + p_1(\bar{e}_2)] \backslash \{b_1\}$ and, for each $i \neq 1$, $\eta_i = b_i$.*

ii) Let $|S^M| > 1$, i.e., *$M = \bar{e}_1 = \bar{e}_2$. Then, the set of Nash equilibria of EP^f is nonempty if there is $S \subseteq N$ such that, for each $i \in S$, $T_i(M, S) \geq 0$; if $S = \{i\}$, then $T_i(M, S) > 0$; and, for each $j \notin S$, $T_j(M, S \cup \{j\}) \leq 0$.*

Indeed, whenever players in S play pure strategy $\bar{e}_2$ and players in $N \backslash S$ put probability 0 at $\bar{e}_2$ we have a Nash equilibrium of EP^f. Finally, $\eta_i = b_i$ if either $\bar{e}_i < M$ or $p_i(M) = 0$.

iii) Let $|S^M| > 1$, i.e., *$M = \bar{e}_1 = \bar{e}_2$. If for each $S \subseteq N$ and each $i \in S$, $T_i(M, S) = \frac{p_i(M)}{|S|}$, then $G \in \mathcal{G}^n$ is a Nash equilibrium of EP^f if and only if*

- *for each $i \notin S^M$, i puts probability 0 at M and*
- *for each $i \in S^M$, if $p_i(M) > 0$, then i puts probability 1 at M.*

Proof. i) follows very similar lines to the proof of i) in Theorem 2 and ii) and iii) follow immediately from Lemma 3. □

Remark 4. *There are two important differences between the statement ii) in Theorem 2 and that in Theorem 3. First, although the conditions for existence of Nash equilibria are very similar, the one in Theorem 2 is a necessary and sufficient condition and the one in Theorem 3 is only sufficient. The reason for this is the following. Since* A2 *is not met, players can get positive payoffs at e_G (under* A2*, if $\bar{e}_1 = \bar{e}_2$, then the highest payoff at e_G was* 0*). Now, using the freedom we have in the tie payoff functions we can construct games with Nash equilibria that do not satisfy the sufficient condition. Second, if* A2 *is not met, then the equilibrium payoffs need not be unique. Just think of a situation in which at M all players get 1 if they are the only winners and* 0 *otherwise. Then, whenever a player puts probability 1 at M and all the other put probability* 0 *at M we have a Nash equilibrium and the payoffs are not always the same. Finally, iii) illustrates that for "normal" tie payoff functions a full characterization is easy to achieve.*

5.1 Implications of the Characterization Result under A1′

According to the discussion in Remark 3, some of the technical difficulties that arise when studying Nash equilibrium in effort-prize games can be overcome by discretizing the sets of strategies of the players. Nonetheless, Theorem 2 shows that mixed strategies can also be used to recover existence of equilibrium in situations where pure equilibria do not exist. Moreover, some of the mixed equibria involve strategic behaviors that had not been noticed before in any of the specific models at hand. The discussion below illustrates this point.

We begin with the FPA model. Theorem 2 says that, when the two highest valuations are different, *i.e.,* $v_1 > v_2$, there is no equilibrium in pure strategies but the existence of equilibrium is always recovered with the use of mixed strategies. In this case, instead of mixing we could also use the discretizing technique to recover existence. Moreover, player 1 does not get $v_1 - v_2$ in all the equilibria. Indeed, all the payoffs in the interval $(0, v_1 - v_2]$ can be achieved as equilibrium payoffs for player 1; the latter leading to different revenues for the auctioneer. Nonetheless, these equilibria in which player 1 gets less that $v_1 - v_2$ are based on "incredible" threats of the other players. Hence, many equilibrium refinements can be used to pin down the equilibria with payoff $v_1 - v_2$ for player 1. On the other hand, when $v_1 = v_2$, since, for each $S \subseteq N$ and each $i \in S$ such that $v_i = v_2$, we have $T_i(v_2, S) = 0 = T_i(v_2, \{i\})$, Theorem 2 ensures the existence of at least one Nash equilibrium. In all these equilibria there are two players that bid v_2 with probability one. Moreover, all the players get a expected payoff of 0; indeed, in this case we might even omit the word expected, *i.e.*, all the players get payoff 0 for sure.

Now, we discuss BM. We begin by identifying Bertrand Paradox as a special implication of Theorem 2. When the cost functions are constant and equal across firms, we have that the marginal cost is common for all firms, say $c = \bar{p} - e$ for some discount e. Marginal cost is chosen with probability one in equilibrium. This is an equilibrium because, when cost functions are constant, we have that, for each $S \subseteq N$ and each $i \in S$, $T_i(v_2, S) = 0 = T_i(v_2, \{i\})$. However, even if the cost functions are equal across firms, if they exhibit strictly decreasing average costs, we have that $T_i(v_2, S) < 0$ whenever $|S| > 1$ and $i \in S$. Hence, Theorem 2 says that in this case there is no Nash equilibrium, even if the firms can use mixed strategies. Moreover, for the latter non-existence result, discretization would not help either. Next Corollary formally states the result sketched above.

Corollary 1. *Take a general Bertrand competition model (BM) with n firms. If the cost function is common for all firms and exhibits strictly decreasing average costs, then the associated effort-prize game does not have any Nash equilibrium (neither pure, nor mixed).*

Proof. Immediate from Theorem 2. □

Now, let firms 1 and 2 be the ones with the "cheapest" production costs and let firm 1 have the cheapest of the two. Then, Theorem 2 says that there is an equilibrium in which firm 1 gets the whole market at the price at which firm 2 would get 0 profit if getting all the market. Hence, player 1 would make a positive profit. Nonetheless, as in the FPA model, we can also support in equilibrium all the profits for firm 1 between 0 and the previous one; firms different from 1 always get 0 in equilibrium.

Finally, the FEG model. Theorem 3 extends the results presented for [9]. This model is special because it is the only one for which A2 is not met. This leads to a kind of effort-prize games for which the payoffs may be different across equilibria depending on the specific tie functions at hand.

6 New Applications and Further Extensions

So far we have discussed the implications of our main results, Theorems 1 and 2, in already existing models. We want to emphasize that the scope of our general effort-prize game goes beyond that. We think that the main objective of future research would be to look for economic situations that can be modeled as effort-prize games and study the implications of our results within that situations. Indeed, most of the models we have discussed in this paper present very simple effort-prize forms, whereas our model allows for much more general ones. Moreover, it might be interesting to elaborate more on the tightness of the assumptions we have taken in the different results. Below, as a matter of example, we present two different situations that can be modeled using effort-prize games.

Combination of FPA and APA: Consider the following auction. First, players submit bids. The player with the higher bid pays his bid and gets the object. Now, for each loser, there is a positive probability that he has also to pay his bid (similarly, we could assume that each loser has to pay a given percentage of his bid). Now, it is easy to check that this new auction can be modeled as an effort-prize game which, moreover, satisfies A1. Hence, the results in Section 4 immediately apply for this type of auctions.

Combination of BM and MS: In Varian's model of sales it is assumed that the number of strategic consumers does not vary as a function of the price. Hence, this model could be extended to a situation in which, as we had in BM, the demand of the strategic consumers decreases as price increases (in BM all the players are strategic). Doing this, we have a common model for BM and MS, being the former the one in which all firms have 0 loyal consumers. Now, this leads to a new effort-prize game with the following primitives: i) The demand function for the strategic consumers, ii) the cost functions, iii) the number of loyal consumers of each firm, and i) the upper bound for the price to be set. Hence, it might be worth to study the extent up to which our results can be applied for the different configurations of the latter primitives.

6.1 A Possible Extension of our Unifying Model

Finally, we also want to discuss on a further generalization of our model. Since the payoff of a player should not increase with his effort unless he gets the prize, it is natural to assume that the base payoff functions are weakly decreasing. On the other hand, there are natural situations in which the prize payoff might increase as a function of the effort; maybe for some intervals or maybe for the whole set of efforts. Indeed, this is what might happen in general Bertrand competition models where, for high prices, the demand function might be very responsive to small changes; although, at the end, for very low prices, they responsiveness would be negligible (the latter might lead to changes in the growth of $(\bar{p}-e)D(\bar{p}-e) - c_i(D(\bar{p}-e))$ as a function of e). Hence, below we present an extension of our model that would account for these situations. First, we remove the requirement of weakly decreasing prize payoff functions (WDP). Second, consider the following assumptions:

Assumption B1. For each $i \in N$, there is a unique $\bar{e}_i \leq M$ such that $b_1(0) = b_i(\bar{e}_i) + p_1(\bar{e}_i)$. Moreover, for each $e > \bar{e}_i$, $b_i(0) < b_i(\bar{e}_i) + p_1(\bar{e}_i)$.

This assumption follows the same idea of A2, *i.e.*, player i does not want to get the prize with efforts above $\bar{e}_i$. That is, even if the payoff of the prize is increasing in e, at some point the decrease in the base payoff should dominate the increase in in the prize payoff.

Assumption B2. If $\bar{e}_1 > \bar{e}_2$, then $b_1(\cdot) + p_1(\cdot)$ is strictly decreasing in $[\bar{e}_2, M]$.

B2 says that, for player 1, the impact of the effort in the base payoff should dominate that in the prize payoff no later than $\bar{e}_2$.

Note that the combination of A1, B1, and B2 is weaker than the combination of A1, A2, and WDP. Similarly, the combination of B1 and B2 is weaker than the combination of A1′, A2, and WDP. Nonetheless, we claim that, if WDP is not assumed, then B1 and B2 are basically what we need for the essence of the results we have presented in this paper to carry out, *i.e.*, A2 has to replaced by B1 and B2, and A1′ is removed. The word essence is because, in the situations where there are a continuum of Nash equibria, new equilibria might appear, but no new equilibrium payoffs. The above generalization, would allow in particular, to account for Bertrand competition models without assuming that $(\bar{p}-e)D(\bar{p}-e) - c_i(D(\bar{p}-e))$ is strictly decreasing in e.

A Appendix

Some of the proofs in this Appendix are straightforward adaptations of some others already published for some of the models we generalized in this paper. On the other hand, many others require to be more careful. Moreover, this is the first time that the complete series of results we present below are stated and proved all together under the same notation.

Proof of Lemma 1. **i)** Suppose that there is $i \in N$ such that $J(G_i) \cap (0, M) \neq \emptyset$. Without loss of generality assume that $i = 1$. Let $a \in J(G_1) \cap (0, M)$. Now, if there is $i \neq 1$ such that $G_i(a) = 0$, then for each $e \in [0, a]$, $u_1^G(e) = b_1(e)$. Since function $u_1^G(\cdot)$ is strictly decreasing on $[0, a]$, player 1 would be better off moving the probability in a to 0. Hence, for each $i \in N$, $G_i(a) > 0$. Now, for each $i \neq 1$, the function $u_i^G(\cdot)$ is discontinuous at a, *i.e.*, since $G_1(a) > G_1(a^-)$, there are $a_1 < a$, $a_2 > a$, and $\varepsilon > 0$ such that for each $i \neq 1$ and each $e \in [a_1, a)$, $u_i^G(a_2) - u_i^G(e) \geq \varepsilon$. Hence, if player $i \neq 1$ puts positive probability on $[a_1, a)$, then he can increase his payoff by at least $\varepsilon(G_i(a^-) - G_i(a_1^-))$ by moving all this probability to a_2. Hence, for each $i \neq 1$ and each $e \in [a_1, a)$, $G_i(e) = G_i(a)$, *i.e.*, the distribution functions of the players different from 1 are constant in $[a_1, a)$. Now, if no player $j \neq i$ puts positive probability at a, we have that $u_1^G(\cdot)$ is strictly decreasing in $[a_1, a]$. Hence, player 1 can improve his payoff by moving some probability from a to a_1. So there is $j \neq i$ such that $a \in J(G_j)$. Let $S := \{j \in N : a \in J(G_j)\}$. Hence, the probability of a tie at a is positive. As we did in the proof of Proposition 1, we distinguish two cases:

$\sum_{i \in S} p_i(a) > 0$**:** By T3, there is $j \in S$ such that $T_j(a, S) < p_j(a)$. Now, since functions $b_j(\cdot)$ and $p_j(\cdot)$ are continuous, there is $\varepsilon > 0$ such that $b_j(a) + T_j(a, S) < b_j(a + \varepsilon) +$

$p_j(a+\varepsilon)$. Now, it is easy to check that the same holds for $u_j^G(a)$. That is, there is $\varepsilon' > 0$ such that $u_j^G(a) < u_j^G(a+\varepsilon')$.[20] Hence, player j would be better off by moving his probability in a to somewhere in $(a, a+\varepsilon']$.

$\sum_{i\in S} p_i(a) \leq 0$**:** Now, there is $j \in S$ such that $p_j(a) \leq 0$. Hence, by T3, $T_j(a,S) \leq 0$ and $u_j^G(a) = b_j(a) + T_j(a,S) \leq b_j(a)$. Now, since $b_j(\cdot)$ is strictly decreasing, player j would be better off by moving his probability at a to 0.

ii) Let $e \in (0,M)$ and $i \in N$. Because of i), for each $j \in N$, $G_j(e) = G_j(e^-)$. Hence, $u_i^G(e) = \prod_{j\neq i} G_j(e)(b_i(e)+p_i(e)) + \big(1-\prod_{j\neq i} G_j(e)\big)b_i(e) = b_i(e) + \prod_{j\neq i} G_j(e)p_i(e)$.

iii) Under A2, the strategy M is strictly dominated for every player and the result is straightforward. □

Recall that by Lemma 1 we now that, in a Nash equilibrium $G \in \mathcal{G}^n$, for each $i \in N$ and each $e \in (0,M)$, $G_i(e) = G_i(e^-)$. The latter implies that the payoff functions are continuous on $(0,M)$. Moreover, recall that, under A2, the previous consideration is true also at M. Next we present a series of results that are needed to prove Theorems 4, 5, and 6.

Lemma 4. *Assume* A1. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Then, the probability of a tie under G in $[0,M)$ is* 0 *and there is $i \in N$ such that $G_i(0) = 0$. Moreover, under* A2 *the probability of a tie at M is also* 0.

Proof. Because of Lemma 1, the only tie that can happen with positive probability in $[0,M)$ is at 0. This can only happen if, for each $i \in N$, $G_i(0) > 0$. Since for each $i \in N$, $G_i(0^-) = 0$, then for each $i \in N$, $u_i^G(0) = (1-\prod_{j\neq i} G_j(0))b_i(0) + \prod_{j\in N} G_j(0)T_i(0,N)$. Now, since for each $i \in N$, $p_i(0) > 0$, by T3, there are $i \in N$ and $\varepsilon > 0$ such that $u_i^G(\varepsilon) > u_i^G(0)$. Hence, player i would deviate from G. The observation for ties at M under A2 is implied by Lemma 1. □

Next result shows that, in equilibrium, an effort cannot belong to the support of the strategy of exactly one player.

Lemma 5. *Assume* A1 *and* A2. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Let $i \in N$ and $e \in S(G_i)$. Then, there is $j \neq i$ such that $e \in S(G_j)$.*

Proof. Let $e \in S(G_i)$ such that $e \notin \bigcup_{j\neq i} S(G_j)$. By A2, $e < M$. We distinguish two cases:

Case 1: $e > 0$**.** Since for each $j \in N$, $S(G_j)$ is a closed set, also $\cup_{j\neq i} S(G_j)$ is a closed set. Hence, its complement is an open set. Hence, there are $e_1 < e < e_2$ such that $[e_1,e_2] \subseteq [0,M] \setminus \cup_{j\neq i} S(G_j)$. Moreover, since the functions G_j are constant outside the support, for each $j \neq i$, $G_j(e_2) = G_j(e_1)$. Hence, for each $a \in [e_1,e_2]$ and each $j \neq i$, $G_j(a) = G_j(e_2)$. So the function $u_i^G(\cdot) = b_i(\cdot) + \prod_{j\neq i} G_j(\cdot)p_i(\cdot)$ is strictly decreasing on $[e_1,e_2]$. Since $e \in S(G_i)$, $G_i(e_2^-) > G_i(e_1)$, *i.e.*, player i puts positive probability on (e_1,e_2). Hence, i can strictly improve his payoff by moving all this probability to e_1.

Case 2: $e = 0$**.** Let $b > 0$ be the smallest element in $\cup_{j\neq i} S(G_j)$ (recall that all the $S(G_j)$ are closed). Now, for each $j \neq i$, $G_j(b) = 0$. Again, if $G_i(b^-) > G_i(0)$, *i.e.*, if player i puts positive probability on $(0,b)$, then similar arguments as in Case 1 can be used to show that player i can strictly improve his payoff by moving this probability to 0. So $G_i(b^-) = G_i(0)$ and hence, since $0 \in S(G_i)$, we have $G_i(0) > 0$. Moreover, for each $a \in (0,b]$, $G_i(a) = G_i(b)$ (this is relevant only for the case $n = 2$). Hence, for each

[20] If there are no discontinuities in $(a, a+\varepsilon]$, then a continuity argument for that interval does the job. If there are discontinuities of $u_j^G(\cdot)$ in $(a, a+\varepsilon]$, then they make the function take even higher values.

$j \neq i$, the function $u_j^G(\cdot) = b_j(\cdot) + \prod_{k \neq j} G_k(\cdot)p_j(\cdot)$ is strictly decreasing on $(0, b]$. Let $a \in (0, b)$ and $j \neq i$ be such that $b \in S(G_j)$. Now, $\varepsilon := u_j^G(a) - u_j^G(b) > 0$. Since $u_j^G(\cdot)$ is continuous on $(0, M]$, there is $\delta > 0$ such that for each $a' \in [b, b+\delta]$, $u_j^G(a) - u_j^G(a') > \frac{1}{2}\varepsilon$. Since $b \in S(G_j)$, $G_j(b + \delta) > 0 = G_j(b)$. Hence, player j can improve his payoff by moving the probability he assigns to $[b, b + \delta)$ to a. Contradiction. □

Next Lemma shows that, if some effort e does not belong the the support of any of the equilibrium strategies, then no higher effort does.

Lemma 6. *Assume* A1. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Let $e \in [0, M]$ be such that $e \notin \bigcup_{i \in N} S(G_i)$. Then, $(e, M] \cap \bigcup_{i \in N} S(G_i) = \emptyset$.*

Proof. Let $K := \bigcup_{i \in N} S(G_i)$. Then, K is closed and $e \notin K$. Suppose that $(e, M] \cap K \neq \emptyset$. Let $\hat{e} := \min\{a \in K : a > e\}$. Let $i \in N$ be such that $\hat{e} \in S(G_i)$. Since for each $j \in N$, $[e, \hat{e}) \cap S(G_j) = \emptyset$, we have that, for each $j \in N$, $G_j(e) = G_j(\hat{e})$. Since for each $a \in [e, M]$, $u_i^G(a) = b_i(a) + \prod_{j \neq i} G_j(a)p_i(a)$, the function $u_i^G(\cdot)$ is strictly decreasing on $[e, \hat{e}]$. By the continuity of $u_i^G(\cdot)$ at $\hat{e}$, there is $\varepsilon > 0$ such that for each $a \in [\hat{e}, \hat{e} + \varepsilon]$, $u_i^G(e) > u_i^G(a)$. Hence, efforts in $[\hat{e}, \hat{e} + \varepsilon]$ are strictly dominated for player i. Hence, G_i is constant on $[\hat{e}, \hat{e} + \varepsilon]$. Contradicting the fact that $\hat{e} \in S(G_i)$. □

Lemma 7. *Assume* A1. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Then, for each $i \in N$, $0 \in S(G_i)$.*

Proof. Let $i \in N$ be such that $0 \notin S(G_i)$. Let $a > 0$ be the smallest element in $S(G_i)$. Then, for each $e \in [0, a)$, $G_i(e) = 0$. Hence, for each $j \neq i$ and each $e \in [0, a]$, $u_j^G(e) = b_j(e) + \prod_{k \neq j} G_k(e)p_j(e) = b_j(e)$. So the function $u_j^G(\cdot)$ is strictly decreasing on $[0, a]$. Hence, for each $j \neq i$, $(0, a) \cap S(G_j) = \emptyset$. Now, we can take $a^* \in (0, a)$ such that $a^* \notin \bigcup_{j \in N} S(G_j)$. Then, by Lemma 6, $[a^*, M] \cap \bigcup_{j \in N} S(G_j) = \emptyset$. Contradiction with $a \in S(G_i)$. □

Recall now that whenever we assume A2 and A3 we also assume, without loss of generality, that players are ordered so that $\bar{e}_1 \geq \bar{e}_2 \geq \ldots \geq \bar{e}_n$.

Lemma 8. *Assume* A1, A2, *and* A3. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. If $\bar{e}_1 > \bar{e}_2$, then $G_1(0) = 0$ and, for each $i \neq 1$, $G_i(0) > 0$.*

Proof. Suppose there is $i \neq 1$ such that $G_i(0) = 0$. Then, the function $u_1^G(\cdot) = b_1(\cdot) + \prod_{j \neq 1} G_j(\cdot)p_i(\cdot)$ is continuous at 0. We already now that for each $j \in N$, $G_j(\bar{e}_2) = 1$. Hence,

$$u_1^G(\bar{e}_2) = b_1(\bar{e}_2) + p_1(\bar{e}_2) > b_1(0) = u_1^G(0).$$

Now, due to the continuity of $u_1^G(\cdot)$ at 0, we have that there is $\varepsilon > 0$ such that for each $e \in [0, \varepsilon]$, $u_1^G(\bar{e}_2) > u_1^G(e)$. Hence, $[0, \varepsilon] \cap S(G_1) = \emptyset$, contradicting the fact that $0 \in S(G_1)$ (Lemma 7).

Now, since, by Lemma 4, there is $i \in N$ such that $G_i(0) = 0$, we have $G_1(0) = 0$. □

In the next results we study how the equilibrium payoffs must be. Since each player can ensure himself a payoff $b_i(0)$ by playing pure strategy 0, we know that for each $i \in N$, his equilibrium payoff must be at least $b_i(0)$. Moreover, recall that, by a straight forward best reply argument, we know that in a Nash equilibrium $G \in \mathcal{G}^n$, for each $i \in N$, the function $u_i^G(\cdot)$ is constant in $S(G_i)$.

Lemma 9. *Assume* A1, A2, *and* A3. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. For each $i \in N$, let η_i denote i's equilibrium payoff. Let $i, j \in N$ be such that $\eta_i = b_i(0)$, $\eta_j \geq b_j(0)$, and $\bar{e}_i > \bar{e}_j$. Let $e > 0$, $e \in S(G_j)$. Then, $G_i(e) > G_j(e)$.*

Proof. Since $e \in S(G_j)$, $u_j^G(e) = \eta_j$. On the other hand, $u_i^G(e) \le \eta_i$ since, otherwise, player i would deviate to pure strategy e. Now

$$b_i(0) = \eta_i \ge u_i^G(e) = b_i(e) + \prod_{k \ne i} G_k(e) p_i(e) \quad \Rightarrow \quad \prod_{k \ne i} G_k(e) \le \frac{b_i(0) - b_i(e)}{p_i(e)} = I_i(e), \text{ and}$$

$$b_j(0) \le \eta_j = u_j^G(e) = b_j(e) + \prod_{k \ne j} G_k(e) p_j(e) \quad \Rightarrow \quad \prod_{k \ne j} G_k(e) \ge \frac{b_j(0) - b_j(e)}{p_j(e)} = I_j(e).$$

Since $e > 0$, by Lemma 7, for each $k \in N$, $G_k(e) > 0$. Hence, dividing the two expressions above and using A3, we get $\frac{G_j(e)}{G_i(e)} \le \frac{I_i(e)}{I_j(e)} < 1$. Hence, $G_i(e) > G_j(e)$. □

Lemma 10. *Assume* A1, A2, *and* A3. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. For each $i \in N$, let η_i denote i's equilibrium payoff. If there is $i \in N$ such that $\eta_i > b_i(0)$, then*

i) $G_i(0) = 0$,

ii) for each $j \ne i$, $G_j(0) > 0$ and $\eta_j = b_j(0)$, and

iii) if $j \ne i$ is such that $\bar{e}_j < \max_{k \ne i} \bar{e}_k$, then $G_j(0) = 1$.

Proof. **i)** Since $u_i^G(0) = b_i(0) < \eta_i$, it has to be the case that $G_i(0) = 0$.

ii) By Lemma 7, 0 belongs to the support of player i's strategy. Hence, there is $\varepsilon > 0$, such that G_i is strictly increasing in $(0, \varepsilon]$. Moreover, for each $e \in (0, \varepsilon]$, $u_i^G(e) = \eta_i$. Hence, the function $u_i^G(\cdot) = b_i(\cdot) + \prod_{j \ne i} G_j(\cdot) p_i(\cdot)$ has a discontinuity at 0. Hence, it has to be the case that for each $j \ne i$, $G_j(0) > 0$. Moreover, this implies that for each $j \ne i$, $\eta_j = u_j^G(0) = b_j(0)$.

iii) Let $j \ne i$ be such that $\bar{e}_j < \max_{k \ne i} \bar{e}_k$. Suppose that $G_j(0) < 1$. Let $\hat{e}$ be the maximum effort in $S(G_j)$. Note that, $G_j(\hat{e}) = 1$. Now, let $k \ne i$ be such that $\bar{e}_k \ge \bar{e}_j$. We have $\eta_j = b_j(0)$, $\eta_k = b_k(0)$, $\bar{e}_k > \bar{e}_j$, and $\hat{e} \in S(G_j)$. Hence, by Lemma 9, $G_k(\hat{e}) > G_j(\hat{e}) = 1$, but this is not possible. □

Next Proposition pins down what the equilibrium payoffs must be (if any).

Proposition 2. *Assume* A1, A2, A3, *and* A4. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. For each $i \in N$, let η_i denote i's equilibrium payoff. Then, $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and, for each $i \ne 1$, $\eta_i = b_i(0)$.*

Proof. We distinguish several cases.

Case 1: $\bar{e}_1 > \bar{e}_2$. According to Lemma 8, $G_1(0) = 0$ and, for each $i \ne 1$, $G_i(0) > 0$. Hence, for each $i \ne 1$, $\eta_i = u_i^G(0) = b_i(0)$. Moreover, since for each $i \in N$, $G_i(\bar{e}_2) = 1$, we have $\eta_1 \ge u_1^G(\bar{e}_2) = b_1(\bar{e}_2) + p_1(\bar{e}_2)$. Now, suppose that $\eta_1 > u_1^G(\bar{e}_2)$. Then, we distinguish two cases:

Case 1.1: $\bar{e}_2 > \bar{e}_3$. By the continuity of $u_1^G(\cdot)$ at $\bar{e}_2$, there is $\varepsilon > 0$ such that for each $e \in [\bar{e}_2 - \varepsilon, \bar{e}_2]$, $\eta_1 > u_1^G(e)$. Hence, $S(G_1) \subseteq [0, \bar{e}_2 - \varepsilon]$. Now there is $\delta > 0$ such that player 2 can get more than $b_2(0)$ by putting all his probability at $\bar{e}_2 - \varepsilon + \delta$.

Case 1.2: $\bar{e}_2 = \bar{e}_3$. Let $\hat{e}$ be the maximum effort in $S(G_1)$. Since $\eta_1 > u_1^G(\bar{e}_2)$, $\hat{e} < \bar{e}_2$. Now, by Lemma 5 there is $i \ne 1$ such that $\hat{e} \in S(G_i)$. Now, since $\eta_1 > u_1^G(\bar{e}_2) = b_1(\bar{e}_2) + p_1(\bar{e}_2)$, we have

$$\eta_1 = u_1^G(\hat{e}) = b_1(\hat{e}) + \prod_{j \ne 1} G_j(\hat{e}) p_1(\hat{e}) \quad \Rightarrow \quad \prod_{j \ne 1} G_j(\hat{e}) = \frac{\eta_1 - b_1(\hat{e})}{p_1(\hat{e})} > I_1^*(\hat{e}), \text{ and}$$

$$b_i(0) = \eta_i = u_i^G(\hat{e}) = b_i(\hat{e}) + \prod_{j \ne i} G_j(\hat{e}) p_i(\hat{e}) \quad \Rightarrow \quad \prod_{j \ne i} G_j(\hat{e}) = \frac{b_i(0) - b_i(\hat{e})}{p_i(\hat{e})} = I_i(\hat{e}).$$

Hence, dividing these two expressions we get $\frac{G_i(\hat{e})}{G_1(\hat{e})} > \frac{I_1^*(\hat{e})}{I_i(\hat{e})} \geq \frac{I_1^*(\hat{e})}{I_2(\hat{e})}$. Now, by A4, since $G_1(\hat{e}) = 1$ and $\hat{e} < \bar{e}_2$, we have $G_i(\hat{e}) > 1$, but this is not possible.

Case 2: $\bar{e}_1 = \bar{e}_2$. Now we have to prove that for each $i \in N$, $\eta_i = b_i(0)$.

Step 1: If $\bar{e}_1 > \bar{e}_i$, then $\eta_i = b_i(0)$. Suppose, on the contrary, that there is $i \in N$ such that $\bar{e}_1 > \bar{e}_i$ and $\eta_i > b_i(0)$. By Lemma 10 we have i) $G_i(0) = 0$, ii) for each $j \neq i$, $G_j(0) > 0$ and $\eta_j = b_j(0)$, and iii) if $j \neq i$ is such that $\bar{e}_j < \bar{e}_1$, then $G_j(0) = 1$. Let $\hat{e}$ be the maximum effort in $S(G_i)$. Note that, by continuity, $G_i(\hat{e}) = 1$. By Lemma 5 there is $j \neq i$ such that $\hat{e} \in S(G_j)$. Indeed, by iii) we must have $\bar{e}_j = \bar{e}_1 > \bar{e}_i$. Now, by Lemma 9, $G_j(\hat{e}) > G_i(\hat{e}) = 1$. Hence, we have a contradiction.

Step 2: There is $i \in N$ such that $\bar{e}_1 = \bar{e}_i$ and $\eta_i = b_i(0)$. Let $j \in N$ be such that $\bar{e}_1 = \bar{e}_j$. If j puts positive probability at 0, then $\eta_j = u_j^G(0) = b_j(0)$ and we are done. If $G_j(0) = 0$, let $i \neq j$ be such that $\bar{e}_1 = \bar{e}_i$ (it exists because we are assuming $\bar{e}_1 = \bar{e}_2$). Now, since i) i's payoff function is continuous at 0, ii) by Lemma 7, $0 \in S(G_i)$, and iii) $u_i^G(0) = b_i(0)$, we have that $\eta_i = b_i(0)$.

Step 3: If $\bar{e}_1 = \bar{e}_i$, then $\eta_i = b_i(0)$. Suppose there is $i \in N$ such that $\bar{e}_1 = \bar{e}_i$ and $\eta_i > b_i(0)$. Let $\hat{e}$ be the maximum effort in $S(G_i)$. Note that $\hat{e} < \bar{e}_1$. Now, let $j \in N$ be the one found in Step 2. That is, $\bar{e}_1 = \bar{e}_i$ and $\eta_j = b_j(0)$. Now,

$$b_i(0) = \eta_i \geq u_i^G(\hat{e}) = b_i(\hat{e}) + \prod_{k \neq i} G_k(\hat{e}) p_i(\hat{e}) \quad \Rightarrow \quad \prod_{k \neq i} G_k(\hat{e}) \leq \frac{b_i(0) - b_i(\hat{e})}{p_i(\hat{e})} = I_i(\hat{e}), \text{ and}$$

$$b_j(0) < \eta_j = u_j^G(\hat{e}) = b_j(\hat{e}) + \prod_{k \neq j} G_k(\hat{e}) p_j(\hat{e}) \quad \Rightarrow \quad \prod_{k \neq j} G_k(\hat{e}) > \frac{b_j(0) - b_j(\hat{e})}{p_j(\hat{e})} = I_j(\hat{e}).$$

But now, by A3 and using that $G_i(a) = 1 \geq G_j(a)$, $\prod_{k \neq j} G_k(\hat{e}) > I_j(\hat{e}) = I_i(\hat{e}) = \prod_{k \neq i} G_k(\hat{e}) \geq \prod_{k \neq j} G_k(\hat{e})$. Hence, we have a contradiction. □

Note that, according to the previous result, if $\bar{e}_1 = \bar{e}_2$, then $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2) = b_1(\bar{e}_1) + p_1(\bar{e}_1) = b_1(0)$.

Next Corollary says that, if the players are ordered according to their maximum admissible efforts, then the players whose higher admissible effort is smaller that the one of player 2 put no effort in equilibrium. That is, they play pure strategy 0 and get their minimum right $b_i(0)$.

Corollary 2. *Assume* A1*,* A2*,* A3*, and* A4*. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f and let $i \in N$. Then,*

i) $G_i(\bar{e}_2) = 1$,

ii) if $\bar{e}_2 > \bar{e}_i$*, then* $G_i(0) = 1$,

iii) if $\bar{e}_1 > \bar{e}_i$*, then* $G_i(0) > 0$*, and*

iv) if $\bar{e}_1 > \bar{e}_2$*, then* $G_1(0) = 0$.

Proof. **i)** Straightforward from the equilibrium payoffs.

ii) Suppose $G_i(0) < 1$. Let $\hat{e}$ be the maximum effort in $S(G_i)$. Now, $\bar{e}_2 > \bar{e}_i$ and by Proposition 2, $\eta_2 = b_2(0)$ and $\eta_i = b_i(0)$. Hence, by Lemma 9, $G_2(\hat{e}) \geq G_i(\hat{e}) = 1$ and we have a contraditcion.

iii) If $\bar{e}_1 > \bar{e}_2$, then, by Proposition 2, $\eta_1 = b_1(\bar{e}_2)+p_1(\bar{e}_2) > b_1(0)$. Hence, by Lemma 10, for each $i \neq 1$, $G_i(0) > 0$. If $\bar{e}_1 = \bar{e}_2$, then $\bar{e}_2 > \bar{e}_i$ and hence, by i), $G_i(0) = 1 > 0$.

iv) By Proposition 2, if $\bar{e}_1 > \bar{e}_2$, then $\eta_1 = b_1(\bar{e}_2)+p_1(\bar{e}_2) > b_1(0)$. Hence, by Lemma 10, $G_1(0) = 0$. □

Some of the Lemmas below can be proved without using all the assumptions, but we have tried to find a compromise between the tightness of the partial results and the complexity of the proofs. Next Lemma says that, in equilibrium, every effort in $[0,\bar{e}_2]$ has to belong to the support of the strategy of at least two players.

Lemma 11. *Assume* A1*,* A2*,* A3*, and* A4*. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f and let $i \in N$. Then, each $e \in [0,\bar{e}_2]$ belongs to the support of at least two players.*

Proof. First, we show that $\bar{e}_2$ belongs to the support of at least two players. If there is $i \in N$ such that $\bar{e}_2 \in S(G_i)$, then, by Lemma 5, we are done. Hence, suppose that $\bar{e}_2 \notin \bigcup_{i\in N} S(G_i)$. Now, let $\hat{e} = \max\{e \in \bigcup_{i\in N} S(G_i)\}$. Note that, by Corollary 2, for each $i \in N$, $G_i(\bar{e}_2) = 1$. Hence, $\hat{e} < \bar{e}_2$. Now, there is $\varepsilon > 0$ such that player 2 can get more than $b_2(0) = b_2(\bar{e}_2) + p_2(\bar{e}_2)$ by playing the pure strategy $\bar{e}_2 - \varepsilon$.

Now, we already now that the statement is true for $e = \bar{e}_2$. Let $e \in (0,\bar{e}_2)$. By Lemma 6, since both 0 and $\bar{e}_2$ belong to the support of some player, there is $i \in N$ such that $e \in S(G_i)$. Now, by Lemma 5, there is $j \neq i$ such that $e \in S(G_j)$. □

Lemma 12. *Assume* A1*,* A2*,* A3*, and* A4*. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Let $i,j \in N$ be such that $\bar{e}_i = \bar{e}_j$. Let $e > 0$, $e \in S(G_i) \cap S(G_j)$. Then, $G_i(e) = G_j(e)$.*

Proof. By Proposition 2, $\eta_i = b_i(0)$ and $\eta_j = b_j(0)$. Now,

$$b_i(0) = \eta_i = u_i^G(e) = b_i(e) + \prod_{k\neq i} G_k(e)p_i(e) \quad\Rightarrow\quad \prod_{k\neq i} G_k(e) = \frac{b_i(0)-b_i(e)}{p_i(e)} = I_i(e), \text{ and}$$

$$b_j(0) = \eta_j = u_j^G(e) = b_j(e) + \prod_{k\neq j} G_k(e)p_j(e) \quad\Rightarrow\quad \prod_{k\neq j} G_k(e) = \frac{b_j(0)-b_j(e)}{p_j(e)} = I_j(e).$$

Since $e > 0$, by Lemma 7, for each $k \in N$, $G_k(e) > 0$. Hence, dividing the two expressions above and using A3, we get $\frac{G_j(e)}{G_i(e)} = \frac{I_i(e)}{I_j(e)} = 1$. Hence, $G_i(e) = G_j(e)$. □

In words, previous Lemma says that, if the mixed strategies of two "equal" players are increasing at some e, then they must coincide at e.

Lemma 13. *Assume* A1*,* A2*,* A3*, and* A4*. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Let $i \in N$ be such that $\bar{e}_i = \bar{e}_2$ and let $a > 0$ belong to $S(G_i)$. Then, $[a,\bar{e}_2] \subsetneq S(G_i)$.*

Proof. Since $a \in S(G_i)$ and $a > 0$, there are a_1 and a_2 such that $a_1 < a < a_2$ and G_i is increasing in (a_1,a_2). Suppose now that the statement in this Lemma is not true. That is, there is $b > a_2$ such that $b < \bar{e}_2$ and $(a_2,b) \notin S(G_i)$. Then, by the continuity of G_i away from 0, $G_i(a_2) = G_i(b)$. Now, by Lemma 11, there are $\varepsilon > 0$ and players $j,j' \neq i$ such that $[a_2,a_2+\varepsilon] \subsetneq S(G_j) \cap S(G_{j'})$. Moreover, by Corollary 2, either $\bar{e}_j = \bar{e}_2$ or $\bar{e}_{j'} = \bar{e}_2$ (or both). Assume, without loss of generality, that $\bar{e}_j = \bar{e}_2$. Hence, by Lemma 12, $G_j(a_2) = G_i(a_2) > 0$. Now, by Proposition 2, for each $e \in [a_2,a_2+\varepsilon]$, $b_j(0) = \eta_j = u_j^G(e) = u_i^G(a_2) = \eta_i = b_i(0) \geq u_i^G(e)$. Hence, for each $e \in [a_2,a_2+\varepsilon]$,

$$b_i(0) = \eta_i \geq u_i^G(e) = b_i(e) + \prod_{k\neq i} G_k(e)p_i(e) \quad\Rightarrow\quad \prod_{k\neq i} G_k(e) \leq \frac{b_i(0)-b_i(e)}{p_i(e)} = I_i(e), \text{ and}$$

$$b_j(0) = \eta_j = u_j^G(e) = b_j(e) + \prod_{k\neq j} G_k(e)p_j(e) \quad\Rightarrow\quad \prod_{k\neq j} G_k(e) = \frac{b_j(0)-b_j(e)}{p_j(e)} = I_j(e).$$

Since $e > 0$, by Lemma 7, for each $k \in N$, $G_k(e) > 0$. Hence, dividing the two expressions above and using A3, we get $\frac{G_j(e)}{G_i(e)} \leq \frac{I_i(e)}{I_j(e)} = 1$. Hence, $G_j(e) \leq G_i(e)$. Contradicting that $G_i(a_2) = G_j(a_2)$, $G_j(\cdot)$ increasing on $(a_2, a_2 + \varepsilon)$, and $G_i(\cdot)$ constant on $(a_2, a_2 + \varepsilon)$. □

Corollary 3. *Assume* A1, A2, A3, *and* A4. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Let $i \in N$ and $a > 0$ be such that $a \notin S(G_i)$. Then, for each $e \in (0, a)$, $a \notin S(G_i)$. Moreover, $G_i(0) > 0$.*

Proof. The first part follows from Lemma 13. The second one follows from Lemma 7. □

Next Lemma says, among other things, that the support of at least two players must coincide with $[0, \bar{e}_2]$.

Lemma 14. *Assume* A1, A2, A3, *and* A4. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. Then,*

i) there are $i, j \in N$, $i \neq j$, such that $S(G_i) = S(G_j) = [0, \bar{e}_2]$ and $G_i(0) = 0$, i.e., *$G_i(\cdot)$ is continuous.*

ii) if $\bar{e}_1 = \bar{e}_2$, then $\bar{e}_i = \bar{e}_j = \bar{e}_1$ and $G_i(0) = G_j(0) = 0$, i.e., *$G_j(\cdot)$ is also continuous,*

iii) if $\bar{e}_1 > \bar{e}_2$, then $i = 1$.

iv) if $\bar{e}_2 > \bar{e}_3$, then $i = 1$ and $j = 2$.

Proof. **i)** By Lemma 4, there is $i \in N$ such that $G_i(0) = 0$. Hence, there is $\varepsilon > 0$ such that $[0, \varepsilon] \in S(G_i)$. By Lemma 5 there is $j \neq i$ and $\delta > 0$ such that $[0, \delta] \in S(G_j)$. Now, by Lemma 13, $S(G_i) = S(G_j) = [0, \bar{e}_2]$.

ii) Suppose now that $\bar{e}_1 = \bar{e}_2$. By Corollary 2, if $\bar{e}_2 = \bar{e}_k$, then $G_k(0) = 1$. Hence, for the i and j found above it must hold that $\bar{e}_i = \bar{e}_j = \bar{e}_1$. Now, using Lemma 12, we easily conclude that also $G_j(0) = 0$.

iii) If $\bar{e}_1 > \bar{e}_2$, by Lemma 8, $G_1(0) = 0$ and we are done.

iv) if $\bar{e}_2 > \bar{e}_3$, then, by Corollary 2, for each $k > 2$, $G_k(0) = 1$. Hence, we have $i = 1$ and $j = 2$. □

Lemma 15. *Assume* A1, A2, A3, *and* A4. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. If $\bar{e}_1 > \bar{e}_2$, then $\prod_{i \neq i} G_i(0) = I_1^*(0)$.*

Proof. By Proposition 2, $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$. By Lemma 8, $G_1(0) = 0$. And, by Lemma 7, $0 \in S(G_1)$. Hence, using the continuity of $G_1(\cdot)$ away from 0 (Lemma 1), there is $\varepsilon > 0$ such that $[0, \varepsilon] \in S(G_1)$. Now, for each $e \in (0, \varepsilon]$, $u_1^G(e) = \eta_1 = b_1(e) + \prod_{j \neq 1} G_j(e) p_1(e)$, and hence, $\prod_{j \neq 1} G_j(e) = I_1^*(e)$. Now, recall that the distribution functions are right-continuous and $I_1^*(\cdot)$ is continuous in $[0, M]$. Hence, $\prod_{j \neq 1} G_j(e) = \lim_{e \to 0} \prod_{j \neq 1} G_j(e) = \lim_{e \to 0} I_1^*(e) = I_1^*(0)$. □

Now we are ready to present the main theorems. For simplicity, we present them assuming that $n \geq 3$. The case $n = 2$ and $\bar{e}_1 > \bar{e}_2$ is covered by Theorem 4. The case $n = 2$ and $\bar{e}_1 = \bar{e}_2$ is covered by Theorem 6. These results show that, under A1, A2, A3, and A4, the Nash equilibrium always exists. Moreover, they provide a characterization of the whole set of Nash equilibrium depending on the values of the $\bar{e}_i$ parameters. Nonetheless, note that because of Proposition 2 we already know that all the equilibria lead to the same payoffs.

In Theorem 4, we show the existence and uniqueness result of the Nash equilibrium when $\bar{e}_1 > \bar{e}_2 > \bar{e}_3$.[21]

[21]This result is an adaptation of [7, Theorem 2] to our framework.

Theorem 4. *Assume* A1, A2, *and* A3. *Let* $G^* = (G_i^*)_{i \in N}$ *be the strategy profile defined for players* 1, 2, *and* $i \notin N \backslash \{1, 2\}$ *as follows:*

$$G_1^*(e) = \begin{cases} 0 & e < 0 \\ I_2(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_2^*(e) = \begin{cases} 0 & e < 0 \\ I_1^*(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_i^*(e) = \begin{cases} 0 & e < 0 \\ 1 & e \geq 0 \end{cases}.$$

If $\bar{e}_1 > \bar{e}_2 > \bar{e}_3$, *then* G^* *is the unique Nash equilibrium of* EP^f. *Moreover, the equilibrium payoffs are* $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ *and, for each* $i \neq 1$, $\eta_i = b_i(0)$.

Proof. "$\Rightarrow$" Note that in this Theorem we do not need A4 because the case $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$ is ruled out by assumption. Suppose $G \in \mathcal{G}^n$ is a Nash equilibrium of EP^f. By Corollary 2, for each $i \in N$, $G_i = G_i^*$. By Lemma 14, $S(G_1) = S(G_2) = [0, \bar{e}_2]$. By Proposition 2, $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and $\eta_2 = b_2(0)$. Now, for each $e \in (0, \bar{e}_2]$, $\eta_1 = u_1^G(e) = b_1(e) + G_2(e)p_1(e)$. Hence, $G_2(e) = I_1^*(e) = G_1^*(e)$. Similarly, using $u_2^G(e)$ we get $G_2(e) = G_2^*(e)$. Hence, G^* is the unique possible Nash equilibrium.

"$\Leftarrow$" Straightforward computations show that for each $i \in N$, if $e \in S(G_i^*)$, then $u_i^{G^*}(e) = \eta_i$. Now we check that no player has incentives to deviate from G^*. Let $i \in N$.

Case 1: $\boldsymbol{\bar{e}_i = \bar{e}_1}$**.** We have $i = 1$. Player 1 cannot get more than $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ outside $S(G_1^*) = [0, \bar{e}_2]$.

Case 2: $\boldsymbol{\bar{e}_i = \bar{e}_2}$**.** We have $i = 2$. Player 2 cannot get more than $b_2(0)$ outside $S(G_2^*) = [0, \bar{e}_2]$.

Case 3: $\boldsymbol{\bar{e}_i < \bar{e}_2}$**.** We have $i \geq 3$. Since $\bar{e}_i < \bar{e}_2$, player i cannot get more that $b_i(0)$ with strategies in $[e_2, M]$. Let $e \in (0, \bar{e}_2)$. Then,

$$u_i^G(e) = b_i(e) + \prod_{j \neq i} G_j^*(e)p_i(e) = b_i(e) + G_1^*(e)G_2^*(e)p_i(e) = b_i(e) + I_2(e)I_1^*(e)p_i(e).$$

Suppose now that $u_i^G(e) > b_i(0)$. Then, $b_i(e) + I_2(e)I_1^*(e)p_i(e) > b_i(0)$. Hence, $I_i(e) < I_2(e)I_1^*(e)$. Recall that for each $e \in (0, \bar{e}_2)$, $J(e) < 1$. Hence, $I_i(e) < I_2(e)I_1^*(e) < I_2(e)$, contradiction with $\bar{e}_i < \bar{e}_2$. □

Now, we formally define the set of strategy profiles NE^1. All of them lead to the equilibrium payoffs and, indeed, Theorem 5 shows that all them are Nash equilibria and, moreover, they are the only ones when $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$.[22]

Henceforth, let $m \in \{2, \ldots, n\}$ denote the player such that $\bar{e}_m = \bar{e}_2 > \bar{e}_{m+1}$. Assume that $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$ ($m \geq 3$). By Lemma 6, in a Nash equilibrium player 1 randomizes continuously in $[0, \bar{e}_2]$. On the other hand, by Corollary 2, for each $i \in N$ such that $\bar{e}_i < \bar{e}_2$, i plays the pure strategy 0. Hence, we only need to worry about the players in $\{2, \ldots, m\}$. A strategy profile in NE^1 is characterized by a vector $b = (b_2, \ldots, b_m) \in [0, \bar{e}_2]^{m-1}$ such that there is $i \in \{2, \ldots, m\}$ such that $b_i = 0$. Let B denote the set of all such vectors. Now, let $b \in B$. The interpretation of the vector b is the following: i) if $b_i < \bar{e}_2$, then $S(G_i) = \{0\} \cup [b_i, \bar{e}_2]$ and ii) if $b_i = \bar{e}_2$, $S(G_i) = \{0\}$. For each $j \in \{2, \ldots, m\}$, let $H(b_j)$ and $L(b_j)$ be the sets of players $i \in \{2, \ldots, m\}$ such that $b_i > b_j$ and $b_i \leq b_j$, respectively. For simplicity, assume that $0 = b_2 \leq \ldots \leq b_m$. We define $G^b = (G_i^b)_{i \in N}$ as follows. If $i > m$, player i plays the pure strategy 0. Now,

[22]The forthcoming descriptions of the equilibrium strategy profiles and the statements of Theorems 5 and 6 are adaptations to our general framework of the ones in [1, Theorem 1] and [4, Theorem 2], respectively. On the other hand, our proofs address a pair of inaccuracies in the proofs included in the papers above.

$\boldsymbol{e \in [b_m, \bar{e}_2]}$**:**

$$\text{For each } i \in \{2,\dots,m\},\quad G_i^b(e) = I_1^*(e)^{\frac{1}{m-1}} = I_1^*(e)^{\frac{1}{|L(b_j)|}}.$$
$$\text{For player 1,}\quad G_1^b(e) = I_2(e) I_1^*(e)^{-\frac{m-2}{m-1}} = I_2(e) I_1^*(e)^{-\frac{|L(b_j)|-1}{|L(b_j)|}}.$$

$\boldsymbol{e \in [b_j, b_{j+1})}$**,** $j \in \{m-1,\dots,2\}$**:**

$$\text{For each } i \in H(b_j),\quad G_i^b(e) = G_i^b(b_i).$$
$$\text{For each } i \in L(b_j),\quad G_i^b(e) = I_1^*(e)^{\frac{1}{|L(b_j)|}} \Big(\prod_{k\in H(b_j)} G_k^b(b_k)\Big)^{-\frac{1}{|L(b_j)|}}.$$
$$\text{For player 1,}\quad G_1^b(e) = I_2(e) I_1^*(e)^{-\frac{|L(b_j)|-1}{|L(b_j)|}} \Big(\prod_{k\in H(b_j)} G_k^b(b_k)\Big)^{-\frac{1}{|L(b_j)|}}.$$

Straightforward computations show that for each $b \in B$, $\prod_{i\neq 1} G_i^b = \prod_{i\in\{2,\dots,n\}} G_i^b = I_1^*(0)$.

Now we formally define the set NE^1. Let $G \in \mathcal{G}^n$. Then, $G \in NE^1$ if and only if there is $b \in B$ such that $G = G^b$. In words, the strategy profiles in NE^2 can be summarized as follows. There is a continuum of Nash equilibria. In each of them the following statements hold: i) players $i \in N$ such that $\bar{e}_2 > \bar{e}_i$ play pure strategy 0, ii) Player 1 randomizes continuously in $[0, \bar{e}_2]$, iii) each other player $i \neq 1$ puts positive probability at 0 and randomizes continuously on some interval $(b_i, \bar{e}_2]$ with $b_i = 0$ for at least one $i \neq 1$ ($b_i = 1$ implies that i plays pure strategy 0), iv) whenever an effort $e > 0$ belongs to the support of 2 players different from player 1, their distribution functions coincide at e, and v) the product of the probabilities put at 0 by players different from player 1 is $I_1^*(0)$.

Theorem 5. *Assume* A1, A2, A3, *and* A4. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. If $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$, then G is a Nash equilibrium of EP^f if and only if $G \in NE^1$. Again, the equilibrium payoffs are $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and, for each $i \neq 1$, $\eta_i = b_i(0)$.*

Proof. "⇒" First we show that, if $G \in \mathcal{G}^n$ is Nash equilibrium, then it belongs to NE^1. We show that the five points presented above must be satisfied. i) is implied by Corollary 2, ii) is is implied by Lemma 14, iii) is implied by the combination of Corollary 2, Lemma 13, and Lemma 14; being the latter for the part $b_i = 0$ for at least one $i \neq 1$), iv) is Lemma 12, and, finally, v) is Lemma 15.

"⇐" Let $b \in B$. Straightforward computations show that for each $i \in N$, if $e \in S(G_i^b)$, then $u_i^{G^b}(e) = \eta_i$. We check that no player has incentives to deviate from G^b. Let $i \in N$.

Case 1: $\boldsymbol{\bar{e}_i = \bar{e}_1}$**.** We have $i = 1$. Player 1 cannot get more than $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ outside $S(G_1^*) = [0, \bar{e}_2]$.

Case 2: $\boldsymbol{\bar{e}_i = \bar{e}_2}$**.** We have $i \in \{2,\dots,m\}$. Player i cannot get more than $b_i(0)$ with strategies in $[e_2, M]$. Suppose that there is $a \in [0, \bar{e}_2]$ such that $u_i^{G^b}(a) > b_i(0)$. We already know that for each $e \in S(G_i^b)$, $u_i^{G^b}(e) = b_i(0)$ and also that $u_i^{G^b}(0) = b_i(0)$. Hence, $a \in (0, b_i)$. Now, by iii) above, there is $j \in \{2,\dots,m\}$, $j \neq i$ be such that $[a, \bar{e}_2] \in S(G_j)$. Now, $u_i^{G^b}(b_i) = u_j^{G^b}(b_i)$ and G_j is strictly increasing in (a, b_i). Hence, $G_j(a) < G_j(b_i) = G_i(b_i) = G_i(a)$. Now, $u_i^{G^b}(a) = b_i(a) + \prod_{k\neq i} G_k(a) p_i(a)$ and, since $\bar{e}_i = \bar{e}_j$ and $G_j(a) < G_i(a)$, we have $u_i^{G^b}(a) < b_j(a) + \prod_{k\neq j} G_k(a) p_j(a) = \eta_j = b_j(0) = b_i(0)$, and we have a contradiction.

Case 3: $\boldsymbol{\bar{e}_i < \bar{e}_2}$**.** Analogous to Case 3 in the proof of Theorem 4. □

Whenever $n > 2$, the set B has a continuous of elements. Hence, the previous result says that there is a continuous of Nash equilibria, all of them leading to the same profile of payoffs.

Now, we turn to the case $\bar{e}_1 = \bar{e}_2$ and define the set of strategy profiles NE^2. By Corollary 2, for each $i \in N$ such that $\bar{e}_i < \bar{e}_2$, i plays the pure strategy 0. We need to worry about the players in $\{1, \ldots, m\}$. Again, a strategy profile in NE^2 is characterized by a vector $b = (b_1, \ldots, b_m) \in [0, \bar{e}_2]^{m-1}$ such that there are $i, j \in \{1, \ldots, m\}$, $i \neq j$, such that $b_i = b_j = 0$. Let $\bar{B}$ denote the set of all such vectors. Now, let $b \in \bar{B}$. The interpretation of the vector b is the same as above. Now, for each $j \in \{1, \ldots, m\}$, let $\bar{H}(b_j)$ and $\bar{L}(b_j)$ be the sets of players $i \in \{1, \ldots, m\}$ such that $b_i > b_j$ and $b_i \leq b_j$, respectively. For simplicity, assume that $0 = b_1 \leq \ldots \leq b_m$. We define $\bar{G}^b = (\bar{G}^b_i)_{i \in N}$ as follows. If $i > m$, player i plays the pure strategy 0. Now,[23]

$\boldsymbol{e \in [b_m, \bar{e}_2]}$**:**

$$\text{For each } i \in \{1, \ldots, m\}, \quad \bar{G}^b_i(e) = I_1(e)^{\frac{1}{m}} = I_1(e)^{\frac{1}{|L(b_j)|}}.$$

$\boldsymbol{e \in [b_j, b_{j+1})}$**,** $j \in \{m-1, \ldots, 1\}$**:**

$$\begin{aligned} &\text{For each } i \in \bar{H}(b_j), \quad \bar{G}^b_i(e) = \bar{G}^b_i(b_i). \\ &\text{For each } i \in \bar{L}(b_j), \quad \bar{G}^b_i(e) = I^*_1(e)^{\frac{1}{|\bar{L}(b_j)|}} \left(\textstyle\prod_{k \in \bar{H}(b_j)} \bar{G}^b_k(b_k) \right)^{-\frac{1}{|\bar{L}(b_j)|}}. \end{aligned}$$

Now we formally define the set NE^2. Let $G \in \mathcal{G}^n$. Then, $G \in NE^2$ if and only if there is $b \in \bar{B}$ such that $G = \bar{G}^b$. In words, the strategy profiles in NE^2 can be summarized as follows. If $n = 2$ or $\bar{e}_1 = \bar{e}_2 > \bar{e}_3$, the Nash equilibrium is unique and players 1 and 2 play the same strategy. If $n > 2$ and $\bar{e}_1 = \bar{e}_2 = \bar{e}_3$, then there is a continuum of Nash equilibria. In each of them the following statements hold: i) players $i \in N$ such that $\bar{e}_2 > \bar{e}_i$ play pure strategy 0, ii) at least two players randomize continuously in $[0, \bar{e}_2]$, iii) each other player i randomizes continuously on some interval $[b_i, \bar{e}_2]$ and puts positive probability at 0 whenever $b_i > 0$ ($b_i = 1$ implies that i plays pure strategy 0), and iv) whenever an effort $e > 0$ belongs to the support of 2 or more players, their distribution functions coincide at e.

Theorem 6. *Assume* A1, A2*, and* A3*. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of EP^f. If $\bar{e}_1 = \bar{e}_2$, then G is a Nash equilibrium of EP^f if and only if $G \in NE^2$. Moreover, the equilibrium payoffs are, for each $i \in N$, $\eta_i = b_i(0)$.*

Proof. "$\Rightarrow$" First we show that, if $G \in \mathcal{G}^n$ is Nash equilibrium, then it belongs to NE^2. We show that the five points presented above must be satisfied. i) is implied by Corollary 2, ii) is is implied by Lemma 14, iii) is implied by the combination of Corollary 2 and Lemma 13, and, finally, iv) is Lemma 12.

"$\Leftarrow$" Let $b \in B$. Straightforward computations show that for each $i \in N$, if $e \in S(G^b_i)$, then $u^{G^b}_i(e) = \eta_i$. We check that no player has incentives to deviate from G^b. Let $i \in N$.

Case 1: $\boldsymbol{\bar{e}_i = \bar{e}_2}$**.** Analogous to Case 2 in the proof of Theorem 5.

Case 2: $\boldsymbol{\bar{e}_i < \bar{e}_2}$**.** Analogous to Case 3 in the proof of Theorem 4. □

Again, whenever $n > 2$, the set B has a continuous of elements (otherwise it has only one $b = (0, 0)$). Hence, the previous result says that there is a continuous of Nash equilibria, all of them leading to the same profile of payoffs.

[23] In the corresponding expressions for the equilibrium strategies when $\bar{e}_1 = \bar{e}_2$ included in [1, 4] there is a minor typo. They wrote $|\bar{L}(b_j)| - 1$ instead of $|\bar{L}(b_j)|$ in the expressions of the G_i functions.

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