Departamento de Estatística e Investigación Operativa
UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

SOLUTION CONCEPTS AND AXIOMATIZATIONS IN COOPERATIVE GAME THEORY

SILVIA MARÍA LORENZO FREIRE

November 2006

Realizado el acto público de defensa y mantenimiento de esta tesis doctoral el día 17 de noviembre de 2006, en la Facultad de Matemáticas de la Universidad de Santiago de Compostela, ante el tribunal formado por:

Presidente:	Dr. D. Ignacio García Jurado
Vocales:	Drª. Dª. Gloria Fiestras Janeiro
	Dr. D. Manfred J. Holler
	Dr. D. Ruud Hendrickx
Secretario:	Dr. D. Gustavo Bergantiños Cid

siendo directores el Dr. D. José M. Alonso Meijide y la Drª. Dª. Balbina V. Casas Méndez obtuvo la máxima calificación de SOBRESALIENTE CUM LAUDE. Además, esta tesis ha cumplido los requisitos necesarios para la obtención del DOCTORADO EUROPEO.

A mis padres

Acknowledgements

First of all, I would like to express my gratitude to my supervisors, José María Alonso Meijide and Balbina Casas Méndez, for all the support they have offered me during these years. I have been able to finish this thesis thanks to their invaluable interest and patience.

The first chapter of this thesis would probably be empty if it not were for Gustavo Bergantiños and Leticia Lorenzo. As far as the second chapter is concerned, I began to work on it during the second year of the PhD with Balbina and Ignacio García Jurado and I finished it with the help of Balbina and Ruud Hendrickx while I was visiting the Department of Econometrics and Operations Research of Tilburg University. Gloria Fiestras Janeiro and Manfred J. Holler are involved in the papers on which the third paper is based. I am very grateful to all of them.

I am indebted with the people in Department of Statistics of the University of Vigo for their hospitality and the pleasant time I spent here, in special with Gustavo, Leticia, María, Juan, Alberto, Juan Carlos, and Javier.

I would also like to thank my family and friends for their moral support.

Finally, I must say that this thesis has been financed by the Spanish Ministry for Science and Technology, FEDER, and Xunta de Galicia through projects BEC2002-04102-C02-01, SEJ2005-07637-C02-01, and PGIDIT03PXIC20701PN.

Contents

Introduction

Game Theory is a branch of Mathematics which studies the decision-making in multi-person situations, where the outcome depends on everyone's choice. The relevance of this discipline comes from its application to many other academic fields, such as Economics, Political Science, Sociology, Philosophy, Computer Science, and Biology.

Although there were some earlier works related to it, Game Theory did not really exist as a unique field until John von Neumann published the article "Zur Theorie der Gesellschaftsspiele" in the year 1928, where he proved the Minimax Theorem for two-person zero-sum games. John von Neumann's work culminated in the year 1944 with the publication of the book "Theory of Games and Economic Behavior ", in collaboration with Oskar Morgenstern. In 1950, John Nash defined the Nash equilibrium, which is considered as one of the most important concepts in Game Theory. From that moment, the contributions to Game Theory experienced a substantial increase. In 1994, the game theorists John Harsanyi, John Nash, and Reinhard Selten won the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel. Later on, in 2005, the contribution of two other game theorists, Robert Aumann and Thomas Schelling, in the field of Economics was again rewarded with the Nobel Prize.

Game Theory is divided in two important subjects: the non-cooperative and the cooperative Game Theory. In the non-cooperative Game Theory, a game is a detailed model of all the moves available to the players. By contrast, in the cooperative approach, it is assumed that binding agreements are possible, and it abstracts away from the detailed bargaining procedures, describing only the outcomes which result when the players come together in different combinations.

In this thesis we focus on the cooperative branch. Moreover, we divide it in three different and independent parts. Chapter 1 is devoted to analyze the cost spanning tree problems. Chapter 2 shows new results on bankruptcy problems and multi-issue allocation situations. Finally, Chapter 3 deals with power indices. Whereas the cost spanning tree problems and the bankruptcy problems are closer to the field of Economics, the power indices are useful tools in Politics.

Chapter 1

Rules in minimum cost spanning tree problems

1.1 Introduction

Consider the following situation: a group of agents want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. They do not care whether they are connected directly or indirectly to the source. This kind of problems are studied in minimum cost spanning tree problems. Many real situations can be modeled in this way. For instance, Bergantiños and Lorenzo (2004) studied a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supply. Other examples are communication networks, such as telephone, Internet, wireless telecommunication, or cable television.

A relevant issue of this literature is to define algorithms for constructing minimal cost spanning trees. Kruskal (1956) and Prim (1957) provided two algorithms for finding minimal cost spanning trees. Another important issue is how to allocate the cost associated with the minimal cost spanning trees among agents. Bird (1976), Kar (2002), and Dutta and Kar (2004) introduced several rules. Moreover, Bird (1976) associated with each minimal cost spanning tree problem a cooperative game with transferable utility. According to this game, each coalition pays the cost of connecting agents in the coalition to the source, assuming that the agents outside the coalition are not present. Kar (2002) studied the Shapley value of this game whereas Granot and Huberman (1981 and 1984) studied the core and the nucleolus. Feltkamp et al. (1994) introduced the equal remaining obligation rule, which was studied by Bergantiños and Vidal-Puga (2004, 2005a, 2005b, and 2005c).

All the rules mentioned before allocate the cost among the agents taking into account only the cost matrix. In some situations, it could make sense to use further information. For instance, in the case of Bergantiños and Lorenzo (2004), we can also take into account the income of each villager, which can be represented by a weight system. One of the main objectives of this chapter is to study nice rules allocating the cost among agents using both, the cost matrix and the weight system. We will do it by considering several families of weighted Shapley values.

Other rules which do not only depend on the cost matrix were defined by Tijs et al. (2005). These rules are the obligation rules and are associated with obligation functions. Tijs et al. (2005) proved that obligation rules satisfy two appealing properties: population monotonicity (if a new agent joins the society, nobody will be worse off) and strong cost monotonicity (if the connection cost between any pair of agents increases, nobody will be better off). Obligation rules were also studied in Moretti et al. (2005). In Bergantiños and Lorenzo-Freire (2006) and Lorenzo-Freire and Lorenzo (2006), we prove that some families of weighted Shapley rules are obligation rules. This is a quite surprising result because they are defined in a completely different way. As a consequence of it, these families also satisfy population monotonicity and strong cost monotonicity.

Bergantiños and Vidal-Puga (2005a) proved that the equal remaining obligation rule, which is an obligation rule, is the only rule satisfying population monotonicity, strong cost monotonicity, and equal share of extra cost. In Bergantiños and Lorenzo-Freire (2006), we modify the property of equal share of extra cost, considering a weight system and defining the property of weighted share of extra cost with respect to the weight system. Moreover, we prove that there is a unique rule in minimum cost spanning tree problems satisfying population monotonicity, strong cost monotonicity, and weighted share of extra cost with respect to the weight system. This rule is the weighted Shapley value of a game for this weight system and we define it as the optimistic weighted Shapley rule. Notice that the first two properties are related to the cost matrix whereas the last one is also related to the weight system.

In Lorenzo-Freire and Lorenzo (2006), we give the first characterization of the obligation rules by means of two appealing properties: population monotonicity and a property of additivity suitable for the minimum cost spanning tree problems, called restricted additivity. This result is not only relevant for the characterization itself, but also provides us with an easy way to calculate the obligation functions associated with the rules.

Chapter 1 is organized as follows. In Section 2 we introduce minimum cost spanning tree problems. In Section 3 we introduce some families of weighted Shapley rules and in Section 4 we introduce the obligation rules. In Section 5 we study the relationship of the optimistic weighted Shapley rules with the obligation rules. In Section 6 we present the axiomatic characterization of the optimistic weighted Shapley rules. Finally, Section 7 is devoted to the characterization of the family of obligation rules.

1.2 Minimum cost spanning tree problems

In this section we introduce minimum cost spanning tree problems and the notation used in the chapter.

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite subset $N \subset \mathcal{N}$, an order π on N is a bijection $\pi : N \longrightarrow \{1, \ldots, |N|\}$ where, for all $i \in N$, $\pi(i)$ is the position of agent i. Let $\Pi(N)$ denote the set of all orders in N. Given $\pi \in \Pi(N)$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π, i.e.,

$$Pre(i, \pi) = \{j \in N \mid \pi(j) < \pi(i)\}.$$

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where N is the set of agents and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ represents the cost of a direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for all $i, j \in N_0$ and that $c_{ii} = 0$ for all $i \in N_0$. Since $c_{ij} = c_{ji}$, we will work with undirected arcs, i.e., $(i, j) = (j, i)$. We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say that $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, more briefly referred to as an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an mcstp (N_0, C), we denote the *mcstp induced by* C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i,j) \mid i,j \in N_0, i \neq j\}$. The elements of g are called *arcs*.

Given a network g and a pair of different nodes i and j, a *path* from i to j (in g) is a sequence of different arcs $\{(i_{s-1}, i_s)\}_{s=1}^p$ that satisfy $(i_{s-1}, i_s) \in g$ for all $s \in \{1, 2, ..., p\}$, with $i = i_0$ and $j = i_p$.

We say that $i, j \in N_0$ are *connected* (in g) if there exists a path from i to j. A *cycle* is a path from i to i.

We say that $i,\ j \in S \subset N_0$, $i \neq j$ are (C,S)*-connected* if there exists a path g_{ij} from i to j satisfying that for all $(k,l) \in g_{ij}$, $k, l \in S_0$ and $c_{kl} = 0$. We say that $S \subset N_0$ is a C*-component* if two conditions hold. Firstly, for all $i, j \in S$, i and j are (C,S)-connected. Secondly, S is maximal, i.e., if $S \subsetneq T$ there exist $i, j \in T$, $i \neq j$ such that i and j are not (C,T)-connected. Norde et al. (2004) proved that the set of C-components is a partition of N_0.

A *tree* is a network where, for each $i \in N$, there is a unique path from i to the source.

We denote the set of all networks over N_0 as $\mathcal{G}^N$ and the set of networks over N_0 in such a way that every agent in N is connected to the source as $\mathcal{G}_0^N$.

Given an mcstp (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c\left(N_0, C, g\right) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimal tree* for (N_0, C), more briefly referred to as an *mt*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well established in the literature on mcstp that an mt exists, even though it does not necessarily have to be unique. Given an mcstp (N_0, C), we denote the cost associated with any mt t in (N_0, C) as $m(N_0, C)$.

One of the most important issues addressed in the literature on mcstp is how to divide $m(N_0, C)$ among the agents. To do it, different cost allocation rules can be considered.

A *cost allocation rule* is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ for each mcstp (N_0, C) and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

1.3 The weighted Shapley rules

In this section we introduce several families of cost allocation rules, which consist of weighted Shapley values for different TU games.

A cooperative game with transferable utility, *TU game*, is a pair (N, v) where $N \subset \mathcal{N}$ and $v : 2^N \to \mathbb{R}$ is the *characteristic function* that assigns to each coalition $S \in 2^N$ the value the agents in the coalition obtain when they cooperate, given by $v(S)$. Moreover, it is assumed that $v(\emptyset) = 0$.

The *dual game* of a TU game (N, v) is a game (N, v^*) such that $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subset N$.

A quite standard approach for defining rules in some problems is based on TU games. We first associate with each problem a TU game. In the case of mcstp, two games can be considered: the pessimistic game, defined by Bird (1976), and the optimistic game, defined in Bergantiños and Vidal-Puga (2005b).

The pessimistic game. The pessimistic game associated with an mcstp (N_0, C) is the TU game denoted by (N, v_C). For each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

Notice that $v_C(S)$ denotes the cost of connecting agents in S to the source, assuming that agents in $N \setminus S$ are not present.

The optimistic game. The optimistic game related to an mcstp (N_0, C) is denoted by $\left(N, v_C^+\right)$.

Given an mcstp (N_0, C) and $S, T \subset N$ such that $S \cap T = \emptyset$, $\left(S_0, C^{+T}\right)$ is the mcstp obtained from (N_0, C), assuming that agents in S have to be connected, agents in T are already connected, and agents in S can connect to the source through agents in T. Formally, $c_{ij}^{+T} = c_{ij}$ for all $i, j \in S$ and $c_{0i}^{+T} = \min_{j \in T_0} c_{ji}$ for all $i \in S$.

For each $S \subset N$,

$$v_C^+(S) = m\left(S_0, C^{+(N \setminus S)}\right).$$

Notice that $v_C^+(S)$ is the minimal cost of connecting agents in S to the source, assuming that agents in $N \setminus S$ are already connected, and agents in S can connect to the source through agents in $N \setminus S$.

Given an mcstp (N_0, C) and an mt t, Bird (1976) defined the *minimal network* (N_0, C^t) associated with t as follows: $c_{ij}^t = \max_{(k,l) \in g_{ij}} \{c_{kl}\}$, where g_{ij} denotes the unique path in t from i to j. Note that we should obtain the same cost matrix if we consider a different mt for the original mcstp. Proof of this can be found, for instance, in Aarts and Driessen (1993). Moreover, $C^t \leq C$.

We define the *irreducible form* of an mcstp (N_0, C) as the minimal network $(N_0, C^*) = (N_0, C^t)$ associated with a particular mt t. If (N_0, C^*) is an irreducible form, we say that C^* is an *irreducible matrix*. Note that a matrix is irreducible if reducing the cost of any arc, the cost of connecting agents to the source is also reduced.

Given an mcstp (N_0, C), we can associate two new TU games using the irreducible form instead of the original mcstp: (N, v_{C^*}) and $\left(N, v_{C^*}^+\right)$.

Once the associated TU game has been chosen, we compute a solution for TU games in the associated TU game. Thus, the rule in the original problem is defined as the solution applied to the TU game associated with the original problem.

Given a family of TU games H, a *solution* on H is a function f which assigns to each TU game $(N, v) \in H$ the vector $(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^N$, where the real number $f_i(N, v)$ is the payoff of $i \in N$ in the game (N, v) according to f. There are several solutions for TU games. One of the most common solutions is the Shapley value.

The *Shapley value* (Shapley, 1953b) is a solution which assigns to each TU game (N, v) the vector $Sh(N, v)$ where

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \left[v\left(Pre\left(i, \pi\right) \cup \{i\}\right) - v\left(Pre\left(i, \pi\right)\right)\right] \text{ for all } i \in N.$$

In the literature on mcstp there are several rules defined using solutions for an associated TU game. For instance, Kar (2002) studied the Shapley value of

(N, v_C) whereas Bergantiños and Vidal-Puga (2005a) studied the Shapley value of (N, v_{C^*}).

Bergantiños and Vidal-Puga (2005b) studied the relationship between the TU games associated with mcstp and their Shapley values. These results can be summarized as follows.

Lemma 1.1 *For each mcstp* (N_0, C),

(*a*) *If* C *is an irreducible matrix,* $v_C(S) + v_C^+(N \setminus S) = m(N_0, C)$ *for all* $S \subset N$, *i.e.,* (N, v_C) *and* (N, v_C^+) *are dual games.*

(*b*) $v_C^+ = v_{C^*}^+$.

(*c*) $Sh\left(N, v_C^+\right) = Sh\left(N, v_{C^*}^+\right) = Sh\left(N, v_{C^*}\right)$.

Thus, we can define two different Shapley rules for mcstp: $Sh(N, v_C)$ and $Sh\left(N, v_C^+\right)$. $Sh(N, v_C)$ is called the *pessimistic Shapley rule* and $Sh\left(N, v_C^+\right)$ is called the *optimistic Shapley rule*.

Shapley (1953a) introduced the family of weighted Shapley values for TU games. Each weighted Shapley value associates a positive weight with each player. These weights are the proportions in which the players share in unanimity games. Later on, Kalai and Samet (1987) studied this family.

Given $N \subset \mathcal{N}$ and $w = \{w_i\}_{i \in N}$, we say that w is a *weight system* for N if $w_i > 0$ for all $i \in N$.

Take $N \subset \mathcal{N}$ and a weight system $w = \{w_i\}_{i \in N}$. The *weighted Shapley value* Sh^w associates with each TU game (N, v) a vector $Sh^w(N, v) \in \mathbb{R}^N$ such that for each $i \in N$,

$$Sh_i^w(N, v) = \sum_{\pi \in \Pi(N)} p_w(\pi)\left[v(Pre(\pi, i) \cup \{i\}) - v(Pre(\pi, i))\right]$$

where $p_w(\pi) = \prod_{j=1}^{n} \frac{w_{\pi^{-1}(j)}}{\sum_{k=1}^{j} w_{\pi^{-1}(k)}}$.

It is well-known that the Shapley value is a weighted Shapley value where $w_i = w_j$ for all $i, j \in N$.

We now apply this idea to mcstp through the optimistic and pessimistic games.

From now on, we will say that $w = \{w_i\}_{i \in \mathcal{N}}$ is a *weight system* for $\mathcal{N}$ if $w_i > 0$ for all $i \in \mathcal{N}$. Given the weight system w and $N \subset \mathcal{N}$, we denote $w_N = \{w_i\}_{i \in N}$.

- We say that ψ is an *optimistic weighted Shapley rule* for mcstp if there exists a weight system $w = \{w_i\}_{i \in \mathcal{N}}$ such that for each *mcstp* (N_0, C),

$$\psi(N_0, C) = Sh^{w_N}\left(N, v_C^+\right).$$

  Lemma 1.1 (b) says that $(N, v_C^+) = (N, v_{C^*}^+)$. Then, $Sh^{w_N}(N, v_C^+) = Sh^{w_N}(N, v_{C^*}^+)$.

- We say that ψ is a *pessimistic weighted Shapley rule* for mcstp if there exists a weight system $w = \{w_i\}_{i \in \mathcal{N}}$ such that for each *mcstp* (N_0, C),

$$\psi(N_0, C) = Sh^{w_N}(N, v_C).$$

- We say that ψ is a *pessimistic weighted Shapley rule of the irreducible form* for mcstp if there exists a weight system $w = \{w_i\}_{i \in \mathcal{N}}$ such that for each *mcstp* (N_0, C),

$$\psi(N_0, C) = Sh^{w_N}(N, v_{C^*}).$$

Remark 1.1 *Kalai and Samet (1987) assume that the population of agents is fixed. Thus, they define the weight system with respect to N. Since we work with the property of population monotonicity, we can not make this assumption. Hence, we have defined the weight system with respect to the set of possible agents $\mathcal{N}$.*

Example 1.1 *In this example we compute the three families of weighted Shapley rules. For it, we consider the mcstp (N_0, C) and (N_0, C^*) given by the Figures 1.1 and 1.2, respectively.*

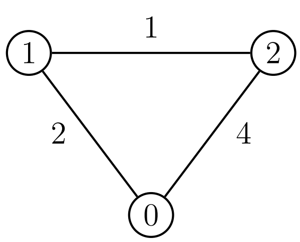

Figure 1.1: mcstp (N_0, C)

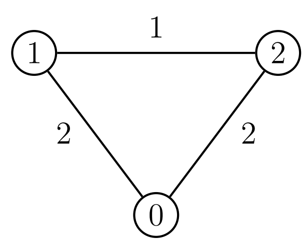

Figure 1.2: mcstp (N_0, C^*)

Firstly, we obtain the pessimistic and optimistic games. The values of the three games appear in the next table.

S	$\emptyset$	$\{1\}$	$\{2\}$	N
$v_C^+(S)$	0	1	1	3
$v_C(S)$	0	2	4	3
$v_{C^*}(S)$	0	2	2	3

Once we have obtained the values of the three games, we calculate the three families of weighted Shapley rules. So, for each weight system w, we have that

$$Sh^{w^N}(N_0, v_C^+) = \left(1 + \frac{w_1}{w_1 + w_2}, 1 + \frac{w_2}{w_1 + w_2}\right),$$

$$Sh^{w^N}(N_0, v_C) = \left(2 - \frac{3w_1}{w_1 + w_2}, 1 + \frac{3w_1}{w_1 + w_2}\right),$$

and

$$Sh^{w^N}(N_0, v_{C^*}) = \left(1 + \frac{w_2}{w_1 + w_2}, 1 + \frac{w_1}{w_1 + w_2}\right).$$

Note that we can choose a weight system w in such a way that the three weighted Shapley rules are different. For example, if $(w_1, w_2) = \left(\frac{3}{4}, \frac{1}{4}\right)$ then $Sh^{w^N}(N_0, v_C^+) = \left(\frac{7}{4}, \frac{5}{4}\right)$, $Sh^{w^N}(N_0, v_C) = \left(-\frac{1}{4}, \frac{13}{4}\right)$, and $Sh^{w^N}(N_0, v_{C^*}) = \left(\frac{5}{4}, \frac{7}{4}\right)$.

1.4 Obligation rules

There are several algorithms to compute an mt. One of them was defined by Kruskal (1956). The idea of the algorithm is to construct a tree by sequentially adding arcs with the lowest cost and without introducing cycles. Formally, Kruskal algorithm is defined as follows.

We start with $A(C) = \{(i,j) \mid i,j \in N_0, i \neq j\}$ and $g^0(C) = \emptyset$.

Stage 1: Take an arc $(i,j) \in A(C)$ such that $c_{ij} = \min_{(k,l) \in A(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. We have that
$(i^1(C), j^1(C)) = (i,j)$, $A(C) = A(C) \setminus \{(i,j)\}$, and $g^1(C) = \{(i^1(C), j^1(C))\}$.

Stage p+1. We have defined the sets $A(C)$ and $g^p(C)$. Take an arc $(i,j) \in A(C)$ such that $c_{ij} = \min_{(k,l) \in A(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Two cases are possible:

1. $g^p(C) \cup \{(i,j)\}$ has a cycle. *Go to the beginning of Stage p+1* with $A(C) = A(C) \setminus \{(i,j)\}$ and $g^p(C)$ the same.

2. $g^p(C) \cup \{(i,j)\}$ has no cycles. Take $(i^{p+1}(C), j^{p+1}(C)) = (i,j)$, $A(C) = A(C) \setminus \{(i,j)\}$, and $g^{p+1}(C) = g^p(C) \cup \{(i^{p+1}(C), j^{p+1}(C))\}$. *Go to Stage p+2.*

This process is completed in n stages. We say that $g^n(C)$ is a tree obtained following Kruskal algorithm. Notice that this algorithm leads to a tree, but that this is not always unique.

When there is not ambiguity, we write A, g^p, and (i^p, j^p) instead of $A(C)$, $g^p(C)$, and $(i^p(C), j^p(C))$, respectively.

Given a network g, let $P(g) = \{T_k(g)\}_{k=1}^{n(g)}$ denote the partition of N_0 in connected components induced by g. Formally, $P(g)$ is the only partition of N_0 satisfying the following two properties:

- If $i, j \in T_k(g)$, i and j are connected in g.

- If $i \in T_k(g)$, $j \in T_l(g)$ and $k \neq l$, i and j are not connected in g.

Given a network g and $i \in N_0$, let $S(P(g), i)$ denote the element of $P(g)$ to which i belongs.

Tijs et al. (2005) defined obligation rules for mcstp. We present this definition in a little bit different way from the original in order to adapt it to our objectives.

For each $S \in 2^N \setminus \{\emptyset\}$, let $\Delta(S) = \left\{x \in \mathbb{R}_+^S \mid \sum_{i \in S} x_i = 1\right\}$ be the simplex in $\mathbb{R}^S$. Given $N \subset \mathcal{N}$, an *obligation function* for N is a map o assigning to each $S \in 2^N \setminus \{\emptyset\}$ a vector $o(S) \in \Delta(S)$ satisfying that for each $S, T \in 2^N \setminus \{\emptyset\}$, $S \subset T$ and $i \in S$, $o_i(S) \geq o_i(T)$.

In the same way we defined a rule for each weight system, in the case of the obligation rules, for each obligation function o we have an *obligation rule* ϕ^o. The idea is as follows. At each stage of Kruskal algorithm, an arc is added to the network. The cost of this arc will be paid by the agents who benefit from adding this arc. Each of these agents pays the difference between his obligation before the arc is added to the network and after it is added. See Tijs et al. (2005) for a more detailed discussion.

We now define obligation rules formally. Given an mcstp (N_0, C), let g^n be a tree obtained applying Kruskal algorithm to (N_0, C). For all $i \in N$, the obligation rule is given by

$$\phi_i^o (N_0, C) = \sum_{p=1}^{n} c_{i^p j^p} \left(o_i \left(S \left(P \left(g^{p-1} \right), i \right) \right) - o_i \left(S \left(P \left(g^p \right), i \right) \right) \right),$$

where, by convention, $o_i(T) = 0$ if the source is in T.

Remark 1.2 *From the definition of obligation rules, it is not clear that ϕ^o is an allocation rule for mcstp. For instance, ϕ^o could depend on the tree obtained through Kruskal algorithm. Tijs et al. (2005) proved that ϕ^o is an allocation rule in mcstp.*

Example 1.2 *Given the mcstp (N_0, C) described in the Figure 1.3, we calculate the associated family of obligation rules.*

We consider, as it is shown in the Figure 1.3, the mt $\{(1,2),(2,3),(0,1)\}$ obtained by Kruskal algorithm.

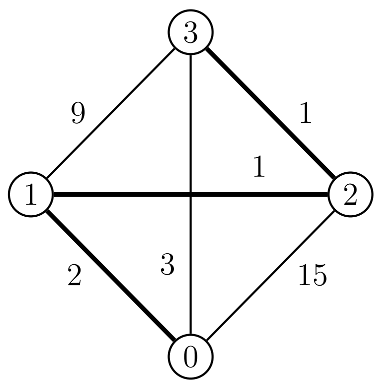

Figure 1.3: mcstp (N_0, C)

In the next table we describe the quantity assigned to each agent in the different stages.

Arc	*Agent* 1	*Agent* 2	*Agent* 3
$(1,2)$	$1(o_1(\{1\}) - o_1(\{1,2\}))$	$1(o_2(\{2\}) - o_2(\{1,2\}))$	0
$(2,3)$	$1(o_1(\{1,2\}) - o_1(N))$	$1(o_2(\{1,2\}) - o_2(N))$	$1(o_3(\{3\}) - o_3(N))$
$(0,1)$	$2(o_1(N) - o_1(N_0))$	$2(o_2(N) - o_2(N_0))$	$2(o_3(N) - o_3(N_0))$

Finally, we obtain that

$$\phi^o(N_0, C) = (1 + o_1(N), 1 + o_2(N), 1 + o_3(N)),$$

with $o_1(N)$, $o_2(N)$, $o_3(N) \geq 0$ *and* $o_1(N) + o_2(N) + o_3(N) = 1$.

1.5 On the connection between obligation and optimistic weighted Shapley rules

The main result of this section says that optimistic weighted Shapley rules are obligation rules. Moreover, the obligation function associated with the obligation rule is obtained in a very intuitive way. The obligation of each agent i in a coalition S (of course, $i \in S$) is proportional to his weight.

Theorem 1.1 *Let* φ^w *be the optimistic weighted Shapley rule associated with the weight system* w. *Thus, for all mcstp* (N_0, C),

$$\varphi^w(N_0, C) = \phi^{o^{N,w}}(N_0, C),$$

where the obligation function $o^{N,w}$ *is given by*

$$o_i^{N,w}(S) = \frac{w_i}{\sum_{j \in S} w_j} \text{ for all } S \in 2^N \setminus \{\emptyset\} \text{ and } i \in S.$$

Remark 1.3 *The family of obligation rules corresponding to the obligation functions* $o^{N,w}$ *appears in Example 3 of Tijs et al. (2005).*

Before proving the Theorem 1.1, we need some results about optimistic weighted Shapley rules and obligation rules.

Next proposition says that the optimistic weighted Shapley rule (obligation rule) of a cost matrix C coincides with the optimistic weighted Shapley rule (obligation rule) of its irreducible matrix C^*.

Proposition 1.1 *Let* (N_0, C) *be an mcstp and let* C^* *be the irreducible matrix associated with* C.

(*a*) *If* w *is a weight system for* $\mathcal{N}$, $\varphi^w(N_0, C^*) = \varphi^w(N_0, C)$.

(*b*) *If* o *is an obligation function for* N, $\phi^o(N_0, C^*) = \phi^o(N_0, C)$.

Proof.

(a) By definition, we have that $\varphi^w(N_0, C^*) = Sh^{w_N}\left(N, v^+_{C^*}\right)$ and $\varphi^w(N_0, C) = Sh^{w_N}\left(N, v^+_C\right)$. By Lemma 1.1 (b), $\varphi^w(N_0, C^*) = \varphi^w(N_0, C)$.

(b) Let $t = \{(i^p(C), j^p(C))\}_{p=1}^n$ be an mt in (N_0, C) obtained through Kruskal algorithm.

We know that $c^*_{ij} = \max_{(k,l)\in g_{ij}} \{c_{kl}\}$, where g_{ij} denotes the unique path in t from i to j. It is well known that if t is an mt in (N_0, C), t is also an mt in (N_0, C^*).

If $C^* = C$, the result holds. Assume that $C^* \neq C$. Thus, there exists an arc (i', j') such that $c_{i'j'} > c^*_{i'j'} = \max_{(k,l)\in g_{i'j'}} \{c_{kl}\}$. Let $\left(i^{p'}(C), j^{p'}(C)\right)$ be such that $c_{i^{p'}(C)j^{p'}(C)} = \max_{(k,l)\in g_{i'j'}} \{c_{kl}\}$ and $g_{i'j'} \subset g^{p'}(C)$. Let C' be such that $c'_{i'j'} = c^*_{i'j'}$ and $c'_{kl} = c_{kl}$ if $(k,l) \neq (i', j')$.

We can apply Kruskal algorithm to C' in such a way that Stages 1, 2,..., and p' coincide with Stages 1, 2, ..., and p' applied to C, i.e., $g^{p'}(C) = g^{p'}(C')$ and $A(C) = A(C')$ at the beginning of Stage $p'+1$.

At the beginning of Stage $p'+1$ of Kruskal algorithm applied to C', we can take the arc (i', j'). Since $g_{i'j'} \subset g^{p'}(C) = g^{p'}(C')$, there is a cycle in $g^{p'}(C') \cup \{(i', j')\}$. Thus, (i', j') is not selected, i.e., $(i', j') \neq \left(i^{p'+1}(C'), j^{p'+1}(C')\right)$.

Now, it is easy to conclude that we can proceed in such a way that for all $q = p'+1, ..., n$, the arc selected in Stage q of Kruskal algorithm applied to C' coincides with $(i^q(C), j^q(C))$, i.e., $(i^q(C'), j^q(C')) = (i^q(C), j^q(C))$. Thus, $g^n(C') = g^n(C) = t$ and, for all $p = 1, ..., n$, $c_{i^p(C)j^p(C)} = c'_{i^p(C)j^p(C)}$.

Because of the definition of the obligation rules, we can conclude that

$$\phi^o(N_0, C) = \phi^o(N_0, C').$$

If $C^* = C'$, the result holds. Assume that $C^* \neq C'$. Thus, there exists an arc $(i'', j''), (i'', j'') \neq (i', j')$ such that $c_{i''j''} > c^*_{i''j''} = \max_{(k,l)\in g_{i''j''}} \{c_{kl}\}$. Let $\left(i^{p''}(C), j^{p''}(C)\right)$ be such that $c_{i^{p''}(C)j^{p''}(C)} = \max_{(k,l)\in g_{i''j''}} \{c_{kl}\}$ and $g_{i''j''} \subset g^{p''}(C)$. Let C'' be such that $c''_{i''j''} = c^*_{i''j''}$ and $c''_{kl} = c'_{kl}$ if $(k,l) \neq (i'', j'')$. Using similar arguments to those used for C and C', we can prove that $\phi^o(N_0, C') = \phi^o(N_0, C'')$.

Repeating this procedure a finite number of steps, say m, we obtain that

$$\phi^o(N_0, C) = ... = \phi^o\left(N_0, C^{(m-1)'}\right) = \phi^o\left(N_0, C^{(m)'}\right),$$

where $C^{(m)\prime} = C^{*}$. ■

Norde et al. (2004) showed that every mcstp can be written as a non-negative combination of mcstp where the cost of the arcs are 0 or 1. We now present this result in a little bit different way in order to adapt it to our objectives.

Lemma 1.2 *For each mcstp (N_0, C), there exists a family $\{C^q\}_{q=1}^{m(C)}$ of cost matrices and a family $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying three conditions:*

(1) $C = \sum\limits_{q=1}^{m(C)} x^q C^q$.

(2) *For each $q \in \{1, ..., m(C)\}$, there exists a network g^q such that $c_{ij}^q = 1$ if $(i,j) \in g^q$ and $c_{ij}^q = 0$ otherwise.*

(3) *Take $q \in \{1, ..., m(C)\}$ and $\{i, j, k, l\} \subset N_0$. If $c_{ij} \leq c_{kl}$, then $c_{ij}^q \leq c_{kl}^q$.*

We now present expressions of optimistic weighted Shapley rules and obligation rules in terms of the family $\{C^q\}_{q=1}^{m(C)}$.

Proposition 1.2 *Let (N_0, C) be an mcstp.*

(*a*) *If w is a weight system for $\mathcal{N}$,*

$$\varphi^w (N_0, C) = \sum_{q=1}^{m(C)} x^q Sh^{w_N} \left(N, v_{C^q}^{+}\right).$$

(*b*) *If o is an obligation function for N,*

$$\phi^o (N_0, C) = \sum_{q=1}^{m(C)} x^q \phi^o (N_0, C^q).$$

Proof.

(*a*) Given $S \subset N$, we know that $v_C^{+}(S) = m\left(S_0, C^{+(N \setminus S)}\right)$. Moreover, $c_{ij}^{+(N \setminus S)} = c_{ij}$ for all $i, j \in S$ and $c_{0i}^{+(N \setminus S)} = \min\limits_{j \in (N \setminus S)_0} c_{ji}$ for all $i \in S$.

By condition (3) of Lemma 1.2, if $c_{0i}^{+(N \setminus S)} = c_{i'i}$ with $i' \in N \setminus S$, then $(c^q)_{0i}^{+(N \setminus S)} = c_{i'i}^q$ for all $q = 1, ..., m(C)$. This means that for all $i, j \in S_0$, $c_{ij}^{+(N \setminus S)} = \sum\limits_{q=1}^{m(C)} x^q \left(c^q\right)_{ij}^{+(N \setminus S)}$.

Let t_S be an mt in $\left(S_0, C^{+(N\backslash S)}\right)$. We know that t_S can be obtained through Kruskal algorithm. By condition (3) of Lemma 1.2, for all $q = 1, ..., m(C)$, t_S can also be obtained applying Kruskal algorithm to $\left(S_0, (C^q)^{+(N\backslash S)}\right)$. Thus,

$$\begin{aligned} v_C^+(S) &= m\left(S_0, C^{+(N\backslash S)}\right) = c\left(S_0, C^{+(N\backslash S)}, t_S\right) = \sum_{(i,j)\in t_S} c_{ij}^{+(N\backslash S)} \\ &= \sum_{(i,j)\in t_S} \sum_{q=1}^{m(C)} x^q \left(c^q\right)_{ij}^{+(N\backslash S)} = \sum_{q=1}^{m(C)} x^q \sum_{(i,j)\in t_S} \left(c^q\right)_{ij}^{+(N\backslash S)} \\ &= \sum_{q=1}^{m(C)} x^q m\left(S_0, (C^q)^{+(N\backslash S)}\right) = \sum_{q=1}^{m(C)} x^q v_{C^q}^+(S). \end{aligned}$$

Kalai and Samet (1987) proved that Sh^{w_N} is additive on the characteristic function, i.e., $Sh^{w_N}(N, v_1 + v_2) = Sh^{w_N}(N, v_1) + Sh^{w_N}(N, v_2)$ for all $w_N,$ v_1, and v_2. Moreover, for each TU game (N, v) and each $\alpha \in \mathbb{R}$, $Sh^{w_N}(N, \alpha v) = \alpha Sh^{w_N}(N, v)$. Thus,

$$\varphi^w(N_0, C) = \sum_{q=1}^{m(C)} x^q Sh^{w_N}\left(N, v_{C^q}^+\right).$$

(b) Let $g^n(C) = \{(i^p(C), j^p(C))\}_{p=1}^n$ be an mt obtained applying Kruskal algorithm to C.

Let $q \in \{1, ..., m(C)\}$. By condition (3) of Lemma 1.2, if we apply Kruskal algorithm to C^q, we can obtain $g^n(C^q) = \{(i^p(C^q), j^p(C^q))\}_{p=1}^n$ such that

$$(i^p(C^q), j^p(C^q)) = (i^p(C), j^p(C)) \text{ for all } p = 1, ..., n.$$

Thus, for all $i \in N$,

$$\phi_i^o(N_0, C) = \sum_{p=1}^n c_{i^p(C)j^p(C)} \left(o_i\left(S\left(P\left(g^{p-1}(C)\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C)\right), i\right)\right)\right)$$

$$= \sum_{p=1}^n \sum_{q=1}^{m(C)} x^q c_{i^p(C)j^p(C)}^q \left(o_i\left(S\left(P\left(g^{p-1}(C)\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C)\right), i\right)\right)\right)$$

$$= \sum_{q=1}^{m(C)} x^q \sum_{p=1}^{n} c^q_{i^p(C)j^p(C)} \left(o_i\left(S\left(P\left(g^{p-1}(C)\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C)\right), i\right)\right)\right)$$

$$= \sum_{q=1}^{m(C)} x^q \sum_{p=1}^{n} c^q_{i^p(C^q)j^p(C^q)} \left(o_i\left(S\left(P\left(g^{p-1}(C^q)\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C^q)\right), i\right)\right)\right)$$

$$= \sum_{q=1}^{m(C)} x^q \phi_i^o(N_0, C^q).$$

■

We now prove Theorem 1.1 using Propositions 1.1 and 1.2.

Proof of Theorem 1.1. Let (N_0, C) be an mcstp.

It is trivial to see that if w is a weight system for $\mathcal{N}$, $o^{N,w}$ is an obligation function for N.

By Proposition 1.2, it is enough to prove that $Sh^{w_N}\left(N, v_C^+\right) = \phi^{o^{N,w}}(N_0, C)$ when C satisfies condition (2) of Lemma 1.2, i.e., there exists a network g such that $c_{ij} = 1$ if $(i,j) \in g$ and $c_{ij} = 0$ otherwise.

Let $\{T_i\}_{i=0}^m$ be the partition of N_0 in C-components. We assume, without loss of generality, that the source is in T_0.

It is easy to see that $c^*_{ij} = 0$ if i and j are in the same C-component whereas $c^*_{ij} = 1$ otherwise. Thus, C^*-components coincide with C-components.

By Proposition 1.1, it is enough to prove that $Sh^{w_N}\left(N, v_{C^*}^+\right) = \phi^{o^{N,w}}(N_0, C^*)$.

We first compute $Sh^{w_N}\left(N, v_{C^*}^+\right)$.

Given a weight system w for $\mathcal{N}$, we define the value Sh^{*w_N} for TU games following Kalai and Samet (1987).

Let $S \in 2^N \setminus \{\emptyset\}$. We consider the TU game (N, u_S^*), where $u_S^*(T) = 1$ if $S \cap T \neq \emptyset$ and $u_S^*(T) = 0$ otherwise.

The family $\{u_S^*\}_{S \in 2^N \setminus \{\emptyset\}}$ is a basis for the set of TU games with player set N. This means that for each TU game (N, v), $v = \sum\limits_{S \in 2^N \setminus \{\emptyset\}} \alpha(S, v) u_S^*$, with $\alpha(S, v) \in \mathbb{R}$.

For each TU game (N, v), we define

$$Sh_i^{*w_N}(N, v) = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha(S, v) Sh_i^{*w_N}(N, u_S^*)$$

where $Sh_i^{*w_N}(N, u_S^*) = \frac{w_i}{\sum\limits_{j \in S} w_j}$ if $i \in S$ and $Sh_i^{*w_N}(N, u_S^*) = 0$ otherwise.

Moreover, by Theorem 5 in Kalai and Samet (1987), we know that for each TU game (N, v), $Sh^{w_N}(N, v) = Sh^{*w_N}(N, v^*)$, where (N, v^*) is the dual game of (N, v).

Since C^* is irreducible, by Lemma 1.1 (a), v_{C^*} is the dual game of $v^+_{C^*}$.

Then, we have that

$$Sh^{w_N}\left(N, v^+_{C^*}\right) = Sh^{*w_N}\left(N, v_{C^*}\right).$$

For each $S \subset N$, it is easy to see that $v_{C^*}(S)$ is the number of elements of $\{T_i\}_{i=1}^m$ which have a non-empty intersection with S, i.e.,

$$v_{C^*}(S) = \left|\{k \in \{1, ..., m\} \text{ such that } T_k \cap S \neq \emptyset\}\right|.$$

Thus, $v_{C^*} = \sum\limits_{k=1}^{m} u^*_{T_k}$.

Given $i \in T_k$,

$$\varphi^w_i(N_0, C) = Sh^{*w_N}_i(N, v_{C^*}) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{w_i}{\sum\limits_{j \in T_k} w_j} & \text{if } k \neq 0. \end{cases}$$

We now compute $\phi^{o^{N,w}}(N_0, C^*)$.

If we apply Kruskal algorithm to C^*, we realize that in the first $n-m$ stages, agents in each C-component are connected among themselves, i.e., $P\left(g^{n-m}\right) = \{T_k\}_{k=0}^m$. Since $c^*_{i^p j^p} = 0$ for all $p = 1, ..., n-m$ and $o^{N,w}_i(T) = 0$ when the source is in T, $\phi^{o^{N,w}}_i(N_0, C^*) = 0$ when $i \in T_0 \setminus \{0\}$.

At Stage $n-m+1$ of Kruskal algorithm, it is possible to select the arc (i^{n-m+1}, j^{n-m+1}) such that $i^{n-m+1} \in T_1$ and $j^{n-m+1} \in T_0$. Since $c^*_{i^p j^p} = 0$ for all $p = 1, ..., n-m$ and $o^{N,w}_i(T) = 0$ when the source is in T, if $i \in T_1$,

$$\phi^{o^{N,w}}_i(N_0, C^*) = o^{N,w}_i\left(S\left(P\left(g^{n-m}\right), i\right)\right) = o^{N,w}_i(T_1) = \frac{w_i}{\sum\limits_{j \in T_1} w_j}.$$

At Stage $n-m+2$ of Kruskal algorithm, it is possible to select the arc (i^{n-m+2}, j^{n-m+2}) such that $i^{n-m+2} \in T_2$ and $j^{n-m+2} \in T_0$. Using similar arguments to those used at Stage $n-m+1$, we can conclude that

$$\phi_i^{o^{N,w}}(N_0, C^*) = \frac{w_i}{\sum_{j \in T_2} w_j}.$$

Repeating this procedure, we can conclude that for all $k = 1, ..., m$ and $i \in T_k$,

$$\phi_i^{o^{N,w}}(N_0, C^*) = \frac{w_i}{\sum_{j \in T_k} w_j}.$$

Then, we can conclude that $\varphi_i^w(N_0, C) = \phi_i^{o^{N,w}}(N_0, C)$ for all $i \in N$. ■

1.6 Characterization of the optimistic weighted Shapley rules

In this section the family of optimistic weighted Shapley rules is characterized. This characterization is based on three properties: strong cost monotonicity, population monotonicity, and weighted share of extra cost. This result is related to the axiomatic characterization of the rule $Sh(N, v_{C^*})$ obtained in Bergantiños and Vidal-Puga (2005a).

We now introduce the properties used in the axiomatic characterization of this family of rules.

Strong cost monotonicity. We say that ψ satisfies the property of strong cost monotonicity (SCM) if for all mcstp (N_0, C) and (N_0, C') such that $C \leq C'$, we have that

$$\psi(N_0, C) \leq \psi(N_0, C').$$

This property implies that if a number of connection costs increase and the rest of connection costs (if any) remain the same, no agent can be better off.

Remark 1.4 *Dutta and Kar (2004) introduced the property of cost monotonicity in mcstp as follows. Let (N_0, C) and (N_0, C') be two mcstp satisfying that $c'_{ij} > c_{ij}$ for some $i \in N, j \in N_0$ and $c'_{kl} = c_{kl}$ if $(k, l) \neq (i, j)$. Thus, $\psi_i(N_0, C) \leq \psi_i(N_0, C')$. According to this property, if a connection cost increases for an agent i and the rest of connection costs remain the same, this agent i cannot be better off.*

In Dutta and Kar (2004), the property of strong cost monotonicity was introduced. Notice that strong cost monotonicity is a strong version of cost monotonicity. Bergantiños and Vidal-Puga (2005a) proved that $Sh(N, v_C)$ satisfies cost monotonicity but fails strong cost monotonicity.

Later, Tijs et al. (2005) used a property called cost monotonicity, which is the property called strong cost monotonicity in this chapter and introduced by Bergantiños and Vidal-Puga (2005a).

Population monotonicity. We say that ψ satisfies population monotonicity (PM) if for all mcstp (N_0, C), $S \subset N$, and $i \in S$, we have

$$\psi_i (N_0, C) \leq \psi_i (S_0, C).$$

This property implies that if new agents join a "society", no agent from the "initial society" can be worse off.

Weighted share of extra cost. Consider the weight system $w = \{w_i\}_{i \in \mathcal{N}}$. The property of weighted share of extra cost (w-WSEC) says that:

Let (N_0, C) and (N_0, C') be two mcstp. Let $c_0, c_0' \geq 0$. Assuming $c_{0i} = c_0$ and $c_{0i}' = c_0'$ for all $i \in N$, $c_0 < c_0'$, and $c_{ij} = c_{ij}' \leq c_0$ for all $i, j \in N$, we have

$$\psi_i (N_0, C') = \psi_i (N_0, C) + \frac{w_i}{\sum\limits_{j \in N} w_j} (c_0' - c_0).$$

This property is interpreted as follows: a group of agents N faces a problem (N_0, C), in which all of them have the same connection cost to the source ($c_{0i} = c_0$). Moreover, this cost is greater than the connection cost between any pair of agents ($c_{ij} \leq c_0$). Under these circumstances, an optimal network implies that any one agent connects directly to the source, and that the rest connect to the source through this agent. Agents agree that the correct solution is $\psi (N_0, C)$. Assume that an error was made and that the connection cost to the source is $c_0' > c_0$. w-WSEC states that agents should share this extra cost $c_0' - c_0$ proportionally to the weights.

This property is a generalization of the property of equal share of extra cost (ESEC) defined in Bergantiños and Vidal-Puga (2005a). Under the same conditions for (N_0, C) and (N_0, C'), a rule ψ satisfies ESEC if $\psi_i (N_0, C') = \psi_i (N_0, C) + \frac{1}{n} (c_0' - c_0)$. Notice that if $w_i = w_j$ for all $i \neq j$, then w-WSEC coincides with ESEC.

Bergantiños and Vidal-Puga (2005a) proved the following lemma.

Lemma 1.3 *Consider the mcstp* (N_0, C)*.*

(a) (N_0, C) *is irreducible if and only if there exists an mt* t *in* (N_0, C) *satisfying the two following conditions:*

(a1) $t = \{(i_{p-1}, i_p)\}_{p=1}^{n}$ *where* $i_0 = 0$*.*

(a2) Given $i_p, i_q \in N_0$*,* $p < q$*, then* $c_{i_p i_q} = \max\limits_{p<r\leq q} \{c_{i_{r-1} i_r}\}$*.*

(b) Suppose that the mcstp (N_0, C) *is irreducible, i.e., there exists an mt* t *in the conditions (a1) and (a2).*

If $S = \{j_1, \ldots, j_{|S|}\} \subset N$*, with* $j_{p-1} \leq j_p$ *for all* $p = 1, \ldots, |S|$*, and we denote* $j_0 = 0$*, then* $t' = \{(j_{p-1}, j_p)\}_{p=1}^{|S|}$ *is an mt in* (S_0, C) *and* $m(S_0, C) = \sum\limits_{p=1}^{|S|} c_{j_{p-1} j_p}$*.*

(c) Suppose that the mcstp (N_0, C) *is irreducible.*

Given an agent $i \in N$*,* $v(S \cup \{i\}) - v(S) = \min\limits_{j \in S_0} \{c_{ij}\}$ *for all* $S \subset N$*.*

Theorem 1.2 *For each weight system* w*,* φ^w*, the optimistic weighted Shapley rule for the weight system* w*, is the only rule for mcstp satisfying strong cost monotonicity, population monotonicity, and weighted share of extra cost with respect to the weight system* w*.*

Proof. Let w be a weight system.

Existence.

(a) φ^w satisfies SCM.

Let (N_0, C) and (N_0, C') be such that $C \leq C'$. Tijs et al. (2005) proved that obligation rules satisfy SCM. By Theorem 1.1,

$$\varphi^w (N_0, C) = \phi^{o^{N,w}} (N_0, C) \leq \phi^{o^{N,w}} (N_0, C') = \varphi^w (N_0, C').$$

(b) φ^w satisfies PM.

To prove it, we will take into account the concept of PMAS (population monotonic allocation scheme), introduced by Sprumont (1990). A PMAS for the game (N, v) is a table $x = \{x^T\}_{T \in 2^N \setminus \{\emptyset\}}$ where $\sum\limits_{i \in T} x_i^T = v(T)$ for all $T \in 2^N \setminus \{\emptyset\}$ and $x_i^T \leq x_i^{T'}$ for all $T, T' \in 2^N \setminus \{\emptyset\}$, $i \in T' \subset T \subset N$.

Let (N_0, C) be an mcstp, $S \subset N$, and $i \in S$. Tijs et al. (2005) proved that $\left\{\phi^{o^{T,w}}(T_0, C)\right\}_{T \in 2^N \setminus \{\emptyset\}}$ is a PMAS (population monotonic allocation scheme) for the TU game (N, v_C).

Thus, $\phi_i^{o^{N,w}}(N_0, C) \leq \phi_i^{o^{S,w}}(S_0, C)$.

By Theorem 1.1, we conclude that $\varphi_i^w(N_0, C) \leq \varphi_i^w(S_0, C)$.

(c) φ^w satisfies w-WSEC.

Let (N_0, C) and (N_0, C') be two mcstp as in the definition of w-WSEC. It is easy to check that if $S \subset N$, $S \neq N$, and $i \in S$, $c_{0i}^{\prime +(N\setminus S)} = c_{0i}^{+(N\setminus S)}$. Moreover, if $S \subset N$ and $i, j \in S$, $c_{ij}^{\prime +(N\setminus S)} = c_{ij}^{+(N\setminus S)}$.

Thus,

$$v_{C'}^+(S) = \begin{cases} v_C^+(S) + (c_0' - c_0) & \text{if } S = N \\ v_C^+(S) & \text{otherwise.} \end{cases}$$

This means that $v_{C'}^+ = v_C^+ + (c_0' - c_0)u_N$, where $u_N(S) = 1$ if $S = N$ and $u_N(S) = 0$ otherwise.

Since the weighted Shapley values are additive on the characteristic function, for all $i \in N$

$$\begin{aligned} \varphi_i^w(N_0, C') &= Sh_i^{w_N}\left(N, v_{C'}^+\right) = Sh_i^{w_N}\left(N, v_C^+\right) + (c_0' - c_0) Sh_i^{w_N}(N, u_N) \\ &= Sh_i^{w_N}(N, v_C^+) + \frac{w_i}{\sum\limits_{j \in N} w_j}(c_0' - c_0) \\ &= \varphi_i^w(N_0, C) + \frac{w_i}{\sum\limits_{j \in N} w_j}(c_0' - c_0). \end{aligned}$$

Uniqueness.

Let ψ be a rule satisfying SCM, PM, and w-WSEC. We will use the induction hypothesis over $|N|$, the number of agents in N, to show that ψ and φ^w coincide.

If $|N| = 1$, $\psi_i(N_0, C) = c_{0i} = \varphi_i^w(N_0, C)$. Assume that the result holds when $|N| < \alpha$. We will prove that the result is true when $|N| = \alpha$.

As ψ satisfies SCM and $C^* \leq C$, $\psi(N_0, C^*) \leq \psi(N_0, C)$. Since $m(N_0, C^*) = m(N_0, C)$, $\psi(N_0, C^*) = \psi(N_0, C)$. By Proposition 1.1, $\varphi^w(N_0, C^*) = \varphi^w(N_0, C)$. Therefore, it is enough to prove that $\psi(N_0, C) = \varphi^w(N_0, C)$ when C is an irreducible matrix.

Let C be an irreducible matrix. We assume, without loss of generality, that $t = \{(i-1, i)\}_{i=1}^n$ is the mt satisfying conditions $(a1)$ and $(a2)$ of Lemma 1.3. We distinguish two cases:

1. There exists $j > 1$ such that $c_{(j-1)j} = \max_{i \in N}\{c_{(i-1)i}\}$.

   Take $S = \{1, \ldots, j-1\}$. Thus, $t' = \{(i-1, i)\}_{i=1}^{j-1}$ is an mt in (S_0, C) and $t'' = \{(0, j)\} \cup \{(i-1, i)\}_{i=j+1}^{n}$ is an mt in $((N \setminus S)_0, C)$. Therefore, $m(S_0, C) = \sum_{i=1}^{j-1} c_{(i-1)i}$ and $m((N \setminus S)_0, C) = c_{0j} + \sum_{i=j+1}^{n} c_{(i-1)i}$.

   By condition $(a2)$ of Lemma 1.3 and the definition of j, $c_{0j} = \max_{i \leq j}\{c_{(i-1)i}\} = c_{(j-1)j}$. Thus,

   $$m(S_0, C) + m((N \setminus S)_0, C) = \sum_{i=1}^{n} c_{(i-1)i} = m(N_0, C).$$

   Since ψ satisfies PM,

   $$\begin{array}{rcl} \psi_i(N_0, C) & \leq & \psi_i(S_0, C) \text{ for all } i \in S \text{ and} \\ \psi_i(N_0, C) & \leq & \psi_i((N \setminus S)_0, C) \text{ for all } i \in N \setminus S. \end{array}$$

   As $\sum_{i \in S} \psi_i(S_0, C) = m(S_0, C)$, $\sum_{i \in N \setminus S} \psi_i((N \setminus S)_0, C) = m((N \setminus S)_0, C)$, and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$,

   $$\psi_i(N_0, C) = \begin{cases} \psi_i(S_0, C) & \text{if } i \in S \\ \psi_i((N \setminus S)_0, C) & \text{if } i \in N \setminus S. \end{cases}$$

   Using similar arguments to those used with ψ, we can deduce that

   $$\varphi_i^w(N_0, C) = \begin{cases} \varphi_i^w(S_0, C) & \text{if } i \in S \\ \varphi_i^w((N \setminus S)_0, C) & \text{if } i \in N \setminus S. \end{cases}$$

   Since $1 \in S$ and $j \in N \setminus S$, both S and $N \setminus S$ are in the conditions of the induction hypothesis. Thus, for all $i \in S$

   $$\psi_i(N_0, C) = \psi_i(S_0, C) = \varphi_i^w(S_0, C) = \varphi_i^w(N_0, C).$$

   Analogously, $\psi_i(N_0, C) = \varphi_i^w(N_0, C)$ for all $i \in N \setminus S$.

2. $\max_{i \in N}\{c_{(i-1)i}\} = c_{01}$ and $c_{01} > c_{(i-1)i}$ for all $i \in N \setminus \{1\}$.

   Let $k \in N \setminus \{1\}$ be such that $c_{(k-1)k} = \max_{i \in N \setminus \{1\}} \{c_{(i-1)i}\}$.

   We define the mcstp $(N_0, \widehat{C})$ where $\widehat{c}_{ij} = c_{ij}$ if $0 \notin \{i,j\}$ and $\widehat{c}_{0i} = c_{(k-1)k}$ for all $i \in N$. By property $(a2)$ in Lemma 1.3 for the irreducible matrix C, $c_{0i} = c_{01}$ for all $i \in N$. Thus, $(N_0, \widehat{C})$ and (N_0, C) are in the conditions of w-WSEC. Since ψ and φ^w satisfy w-WSEC, for all $i \in N$,

$$\psi_i(N_0, C) = \psi_i(N_0, \widehat{C}) + \frac{w_i}{\sum\limits_{j \in N} w_j}\left(c_{01} - c_{(k-1)k}\right)$$

   and

$$\varphi_i^w(N_0, C) = \varphi_i^w(N_0, \widehat{C}) + \frac{w_i}{\sum\limits_{j \in N} w_j}\left(c_{01} - c_{(k-1)k}\right).$$

   It is easy to see that t is an mt in $(N_0, \widehat{C})$ satisfying conditions $(a1)$ and $(a2)$ of Lemma 1.3. Thus, $\widehat{C}$ is an irreducible matrix satisfying that $\widehat{c}_{(k-1)k} = \max_{i \in N}\{\widehat{c}_{(i-1)i}\}$. By case 1, $\psi(N_0, \widehat{C}) = \varphi^w(N_0, \widehat{C})$. Then, we conclude that $\psi(N_0, C) = \varphi^w(N_0, C)$.

■

Remark 1.5 *Bergantiños and Vidal-Puga (2005a) characterized the rule given by $Sh\left(N, v_{C^*}\right)$ as the only rule satisfying strong cost monotonicity, population monotonicity, and equal share of extra cost. Theorem 1.2 is inspired in this result. Nevertheless, from a technical point of view, the scheme of our proof is different from the scheme of the proof of Bergantiños and Vidal-Puga (2005a). For instance, we prove that optimistic weighted Shapley rules satisfy strong cost monotonicity and population monotonicity by proving that they are obligation rules. Bergantiños and Vidal-Puga (2005a) proved that $Sh\left(N, v_{C^*}\right)$ satisfies strong cost monotonicity and population monotonicity directly.*

Remark 1.6 *The properties used in Theorem 1.2 are independent.*

- Given a weight system w, we define the rule δ^w such that for all $i \in N$,

$$\delta_i^w(N_0, C) = \frac{w_i}{\sum\limits_{j \in N} w_j} m(N_0, C).$$

It is easy to show that δ^w satisfies SCM and w-WSEC.

Nevertheless, δ^w does not satisfy PM. Let (N_0, C) be such that $N = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Thus, $\delta_1^w(\{1\}_0, C) = 0$ whereas $\delta_1^w(N_0, C) = \frac{w_1}{w_1+w_2} > 0$.

- Given $S \subset N$ and $i \in S$, we define the obligation function o as

$$o_i(S) = \begin{cases} 1 & \text{if } i = \min\limits_{j \in S} \{j\} \\ 0 & \text{otherwise.} \end{cases}$$

  This obligation function appears in Example 2 in Tijs et al. (2005). We know that the corresponding obligation rule ϕ^o satisfies SCM and PM.

  Nevertheless, ϕ^o does not satisfy w-WSEC. Let (N_0, C) and (N_0, C') be such that $N = \{1, 2\}$,

$$C = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}.$$

  Thus,

$$\begin{aligned} \phi_2^o(N_0, C') &= 1 \text{ and} \\ \phi_2^o(N_0, C) + \frac{w_2}{w_1+w_2} &= 1 + \frac{w_2}{w_1+w_2} > 1. \end{aligned}$$

- Let (N_0, C) be an mcstp.

  $\Pi'(N) = \{\pi \in \Pi(N) \mid \pi(i) < \pi(j) \text{ when } c_{0i} < c_{0j} \text{ for all } i, j \in N, i \neq j\}$.

  Moreover, let $P(N_0, C) = \{P_1(N_0, C), \ldots, P_m(N_0, C)\}$ be the partition of N satisfying the following two conditions:

  1. If $k \in \{1, \ldots, m\}$ and $i, j \in P_k(N_0, C)$, then $c_{0i} = c_{0j}$.

2. If $k, k' \in \{1, \ldots, m\}$, $k < k'$, $i \in P_k(N_0, C)$ and $j \in P_{k'}(N_0, C)$, then $c_{0i} < c_{0j}$.

By simplicity and when no confusion arises, we denote $P_k = P_k(N_0, C)$ for all $k = 1, ..., m$.

Given the weight system w, an mcstp (N_0, C), and $i \in N$, we define the rule ξ^w as

$$\xi_i^w(N_0, C) = \sum_{\pi \in \Pi'(N)} \tilde{p}_w(\pi) \left[v_C^+(Pre(i, \pi) \cup \{i\}) - v_C^+(Pre(i, \pi)) \right]$$

where $\tilde{p}_w(\pi) = \prod_{k=1}^{m} p_{w_{P_k}}(\pi_{P_k})$, $p_{w_{P_k}}(\pi_{P_k})$ is as in the definition of weighted Shapley values, and for each $S \subset N$, π_S denotes the order induced in S by π (for all $i, j \in S$, $\pi_S(i) < \pi_S(j)$ if and only if $\pi(i) < \pi(j)$).

We will prove that ξ^w satisfies w-WSEC and PM but fails SCM.

We first prove that ξ^w satisfies w-WSEC. Let (N_0, C) and (N_0, C') be as in the definition of w-WSEC. Thus, $P(N_0, C) = P(N_0, C') = \{N\}$. Hence, $\xi^w(N_0, C) = \varphi^w(N_0, C)$ and $\xi^w(N_0, C') = \varphi^w(N_0, C')$. Since φ^w satisfies w-WSEC, we deduce that ξ^w also satisfies w-WSEC.

We now prove that ξ^w satisfies PM.

Let $i, j \in N$, $i \neq j$. We must prove that $\xi_i^w(N_0, C) \leq \xi_i^w((N \setminus \{j\})_0, C)$. By simplicity, we denote $N^{-j} = N \setminus \{j\}$.

We know that $\xi_i^w(N_0, C) =$

$$\sum_{\pi^{-j} \in \Pi'(N^{-j})} \sum_{\pi \in \Pi'(N), \pi_{N^{-j}} = \pi^{-j}} \tilde{p}_w(\pi) \left[v_C^+(Pre(i, \pi) \cup \{i\}) - v_C^+(Pre(i, \pi)) \right]$$

and $\xi_i^w(N_0^{-j}, C) =$

$$\sum_{\pi^{-j} \in \Pi'(N^{-j})} \tilde{p}_w\left(\pi^{-j}\right) \left[v_C^+(Pre(i, \pi^{-j}) \cup \{i\}) - v_C^+\left(Pre\left(i, \pi^{-j}\right)\right) \right].$$

Thus, $\xi_i^w(N_0, C) \leq \xi_i^w(N_0^{-j}, C)$ is a consequence of the following two claims:

Claim 1 *For each* $\pi^{-j} \in \Pi'(N^{-j})$ *and* $\pi \in \Pi'(N)$ *with* $\pi_{N^{-j}} = \pi^{-j}$,

$$\begin{aligned} & v_C^+(Pre(i,\pi) \cup \{i\}) - v_C^+\left(Pre\left(i,\pi\right)\right) \\ \leq \; & v_C^+(Pre(i,\pi^{-j}) \cup \{i\}) - v_C^+\left(Pre\left(i,\pi^{-j}\right)\right). \end{aligned}$$

Claim 2 *For each* $\pi^{-j} \in \Pi'(N^{-j})$,

$$\sum_{\pi \in \Pi'(N), \pi_{N^{-j}} = \pi^{-j}} \tilde{p}_w(\pi) = \tilde{p}_w(\pi^{-j}).$$

Proof of Claim 1. By Lemma 1.1,

$$v_C^+(Pre(i,\pi) \cup \{i\}) - v_C^+(Pre(i,\pi)) = v_{C^*}^+(Pre(i,\pi) \cup \{i\}) - v_{C^*}^+(Pre(i,\pi))$$

$$= v_{C^*}(N \setminus Pre(i,\pi)) - v_{C^*}(N \setminus (Pre(i,\pi) \cup \{i\})).$$

By Lemma 1.3 (c), we know that if C is an irreducible matrix, $S \subset N$, and $i \in S$,

$$v_C\left(S \cup \{i\}\right) - v_C\left(S\right) = \min_{j \in S_0} \left\{c_{ij}\right\}.$$

Thus,

$$v_C^+(Pre(i,\pi) \cup \{i\}) - v_C^+(Pre(i,\pi)) = \min_{l \in N_0 \setminus (Pre(i,\pi) \cup \{i\})} \left\{c_{il}^*\right\}.$$

We distinguish two cases:

1. $\pi(j) < \pi(i)$. Since $j \in Pre(i,\pi)$,

$$v_C^+(Pre(i,\pi) \cup \{i\}) - v_C^+(Pre(i,\pi)) = \min_{l \in N_0^{-j} \setminus (Pre(i,\pi^{-j}) \cup \{i\})} c_{il}^*$$

$$= v_C^+(Pre(i,\pi^{-j}) \cup \{i\}) - v_C^+(Pre(i,\pi^{-j})).$$

2. $\pi(i) < \pi(j)$. Since $j \in N \setminus (Pre(i,\pi) \cup \{i\})$,

$$v_C^+(Pre(i,\pi) \cup \{i\}) - v_C^+(Pre(i,\pi))$$

$$
\begin{aligned}
&\leq \min_{l\in N_0\setminus(Pre(i,\pi)\cup\{i,j\})} c^*_{il} \\
&= \min_{l\in N_0^{-j}\setminus(Pre(i,\pi^{-j})\cup\{i\})} c^*_{il} \\
&= v^+_C(Pre(i,\pi^{-j})\cup\{i\}) - v^+_C(Pre(i,\pi^{-j})).
\end{aligned}
$$

This finishes the proof of Claim 1. ■

Proof of Claim 2. Let $\pi^{-j}\in\Pi'(N^{-j})$. Assume, without loss of generality, that $j\in P_m\in P(N_0,C)=\{P_1,\ldots,P_m\}$. We will prove this claim applying an induction argument over the cardinality of P_m.

Assume that $|P_m|=1$, i.e., $P_m=\{j\}$. It is trivial to see that

$$\left|\pi\in\Pi'(N),\pi_{N^{-j}}=\pi^{-j}\right|=1.$$

Let π' denote the only element of this set. Since $P_m=\{j\}$, $p_{w_{P_m}}(\pi'_{P_m})=1$. Thus,

$$
\begin{aligned}
\sum_{\pi\in\Pi'(N),\pi_{N^{-j}}=\pi^{-j}}\tilde{p}_w(\pi) &= \tilde{p}_w(\pi')=\prod_{k=1}^{m}p_{w_{P_k}}(\pi'_{P_k}) \\
&= \prod_{k=1}^{m-1}p_{w_{P_k}}(\pi'_{P_k}).
\end{aligned}
$$

It is trivial to see that $P\left(N_0^{-j},C\right)=\{P_1,\ldots,P_{m-1}\}$. Since $\pi'_{N^{-j}}=\pi^{-j}$,

$$\tilde{p}_w(\pi^{-j})=\prod_{k=1}^{m-1}p_{w_{P_k}}(\pi^{-j}_{P_k})=\prod_{k=1}^{m-1}p_{w_{P_k}}(\pi'_{P_k}).$$

Assume now that Claim 2 holds when $|P_m|=r\geq 1$. We prove it when $|P_m|=r+1$.

It is trivial to see that $P\left(N_0^{-j},C\right)=\{P_1,\ldots,P_{m-1},P_m\setminus\{j\}\}$. Thus,

$$
\begin{aligned}
\sum_{\pi\in\Pi'(N),\pi_{N^{-j}}=\pi^{-j}}\tilde{p}_w(\pi) &= \sum_{\pi\in\Pi'(N),\pi_{N^{-j}}=\pi^{-j}}\left(\prod_{k=1}^{m}p_{w_{P_k}}\left(\pi_{P_k}\right)\right) \\
&= \left(\prod_{k=1}^{m-1}p_{w_{P_k}}\left(\pi^{-j}_{P_k}\right)\right)\sum_{\pi\in\Pi'(N),\pi_{N^{-j}}=\pi^{-j}}p_{w_{P_m}}\left(\pi_{P_m}\right)
\end{aligned}
$$

and

$$\tilde{p}_w(\pi^{-j}) = \left(\prod_{k=1}^{m-1} p_{w_{P_k}}\left(\pi_{P_k}^{-j}\right)\right) p_{w_{P_m \setminus \{j\}}}\left(\pi_{P_m \setminus \{j\}}^{-j}\right).$$

Then, it is enough to prove that

$$\sum_{\pi \in \Pi'(N), \pi_{N-j} = \pi^{-j}} p_{w_{P_m}}\left(\pi_{P_m}\right) = p_{w_{P_m \setminus \{j\}}}\left(\pi_{P_m \setminus \{j\}}^{-j}\right).$$

Let us assume $P_m = \{j_1, \ldots, j_r, j\}$ and $\pi^{-j}(j_1) < \ldots < \pi^{-j}(j_r)$.

Thus, $\sum_{\pi \in \Pi'(N), \pi_{N-j} = \pi^{-j}} p_{w_{P_m}}(\pi_{P_m}) =$

$$\begin{aligned}
&= \sum_{\pi \in \Pi'(N), \pi_{N-j} = \pi^{-j}, \pi(j) < \pi(j_r)} p_{w_{P_m}}(\pi_{P_m}) \\
&+ \sum_{\pi \in \Pi'(N), \pi_{N-j} = \pi^{-j}, \pi(j) > \pi(j_r)} p_{w_{P_m}}(\pi_{P_m}) \\
&= \sum_{\pi \in \Pi'(N^{-j_r}), \pi_{N \setminus \{j_r, j\}} = \pi_{N \setminus \{j_r, j\}}^{-j}} p_{w_{P_m \setminus \{j_r\}}}(\pi_{P_m \setminus \{j_r\}}) \frac{w_{j_r}}{w_j + \sum_{p=1}^{r} w_{j_p}} \\
&\quad + p_{w_{P_m \setminus \{j\}}}\left(\pi_{P_m \setminus \{j\}}^{-j}\right) \frac{w_j}{w_j + \sum_{p=1}^{r} w_{j_p}}.
\end{aligned}$$

By induction hypothesis,

$$\sum_{\pi \in \Pi'(N^{-j_r}), \pi_{N \setminus \{j_r, j\}} = \pi_{N \setminus \{j_r, j\}}^{-j}} p_{w_{P_m \setminus \{j_r\}}}(\pi_{P_m \setminus \{j_r\}}) = p_{w_{P_m \setminus \{j, j_r\}}}\left(\pi_{P_m \setminus \{j, j_r\}}^{-j}\right).$$

Moreover,

$$p_{w_{P_m \setminus \{j\}}}\left(\pi_{P_m \setminus \{j\}}^{-j}\right) = p_{w_{P_m \setminus \{j, j_r\}}}\left(\pi_{P_m \setminus \{j, j_r\}}^{-j}\right) \frac{w_{j_r}}{\sum_{p=1}^{r} w_{j_p}}.$$

Therefore,

$$\sum_{\pi \in \Pi'(N), \pi_{N-j} = \pi^{-j}} p_{w_{P_m}}(\pi_{P_m}) =$$

$$
\begin{aligned}
&= p_{w_{P_m \setminus \{j,j_r\}}}\left(\pi^{-j}_{P_m \setminus \{j,j_r\}}\right) \frac{w_{j_r}}{w_j + \sum\limits_{p=1}^{r} w_{j_p}} + p_{w_{P_m \setminus \{j\}}}\left(\pi^{-j}_{P_m \setminus \{j\}}\right) \frac{w_j}{w_j + \sum\limits_{p=1}^{r} w_{j_p}} \\
&= p_{w_{P_m \setminus \{j\}}}\left(\pi^{-j}_{P_m \setminus \{j\}}\right) \left\{ \frac{\sum\limits_{p=1}^{r} w_{j_p}}{w_j + \sum\limits_{p=1}^{r} w_{j_p}} + \frac{w_j}{w_j + \sum\limits_{p=1}^{r} w_{j_p}} \right\} \\
&= p_{w_{P_m \setminus \{j\}}}\left(\pi^{-j}_{P_m \setminus \{j\}}\right).
\end{aligned}
$$

This finishes the proof of Claim 2. ■

Nevertheless, ξ^w does not satisfy SCM. Let (N_0, C) and (N_0, C') be such that $N = \{1, 2\}$,

$$
C = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}.
$$

Thus,

$$
\begin{aligned}
\xi^w_1 (N_0, C) &= 1 + \frac{w_1}{w_1 + w_2} \text{ and} \\
\xi^w_1 (N_0, C') &= 1.
\end{aligned}
$$

1.7 Characterization of the obligation rules

In this section we give the first characterization for the family of obligation rules. This characterization is based on a property of additivity defined in Bergantiños and Vidal-Puga (2004) and the property of population monotonicity introduced before. This characterization works for any set of possible agents $\mathcal{N}$ except for sets of possible agents composed of two agents. In this situation, it is necessary to add the property of positivity. Furthermore, we use this characterization to show that the pessimistic and the optimistic weighted Shapley rules of the irreducible form are obligation rules.

Restricted additivity. We say that a cost allocation rule ψ satisfies the property of restricted additivity (RA) if

$$\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$$

for all mcstp (N_0, C) and (N_0, C') satisfying that there exists an mt $t = \{(i^0, i)\}_{i\in N}$ in (N_0, C), (N_0, C'), and $(N_0, C + C')$ and an order $\pi = (i_1, \ldots, i_n) \in \Pi(N)$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \ldots \leq c_{i_n^0 i_n}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \ldots \leq c'_{i_n^0 i_n}$. Moreover, we say that two mcstp (N_0, C) and (N_0, C') are *similar* if they satisfy these conditions.

Positivity. A rule ψ satisfies positivity (POS) if $\psi_i(N_0, C) \geq 0$ for all $i \in N$ and for all mcstp (N_0, C).

Proposition 1.3 *Let ψ be a rule such that satisfies population monotonicity and restricted additivity. Suppose that $\mathcal{N}$ is such that $|\mathcal{N}| \geq 3$. For all mcstp (N_0, C) such that $N \subset \mathcal{N}$,*

$$\psi(N_0, C) = \phi^{o^\psi}(N_0, C),$$

where o^ψ is the obligation function defined by

$$o^\psi(S) = \psi(S_0, \widehat{C}) \text{ for all } S \in 2^N \setminus \{\emptyset\}$$

and the cost matrix $\widehat{C} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$.

Proof.

First of all, we will prove that o^ψ is an obligation function.

As ψ satisfies PM, o^ψ is monotone, i.e., $o_i^\psi(T) \leq o_i^\psi(S)$ for all $S \subset T \in 2^N \setminus \{\emptyset\}$ and $i \in S$.

Moreover, as we know that $m(S_0, \widehat{C}) = 1$ for all $S \in 2^N \setminus \{\emptyset\}$, if we prove that $o_i^\psi(S) \geq 0$ for all $S \in 2^N \setminus \{\emptyset\}$ and $i \in S$, we can conclude that the vector $o^\psi(S)$ belongs to the simplex in $\mathbb{R}^S$.

Suppose that $S = \{i\}$, with $i \in N$. We know that $o_i^\psi(S) = \psi_i(S_0, \widehat{C}) = 1$. So, we will assume that $|S| > 1$.

In this case, we obtain by PM that for each $i \in S$, $\psi_j(S_0, \widehat{C}) \leq \psi_j((S_0 \setminus \{i\}), \widehat{C})$ for all $j \in S \setminus \{i\}$. Thus, $1 - \psi_i(S_0, \widehat{C}) \leq \sum_{j\in S\setminus\{i\}} \psi_j(S_0 \setminus \{i\}, \widehat{C}) = 1$ and, hence,

$o_i^{\psi}(S) = \psi_i(S_0, \widehat{C}) \geq 0.$

Let ψ be a rule satisfying PM and RA for all (N_0, C) with $N \subset \mathcal{N}$ and $|\mathcal{N}| \geq 3$. We must show that $\psi(N_0, C) = \phi^{o^{\psi}}(N_0, C)$.

By Lemma 1.2, we can consider $C = \sum\limits_{q=1}^{m(C)} x^q C^q$ with $x^q \geq 0$ for all $q = 1, \ldots, m(C)$.

Since ψ satisfies RA and mcstp $\{(N_0, x^q C^q)\}_{q=1}^{m(C)}$ are similar, we have that $\psi(N_0, C) = \sum_{q=1}^{m(C)} \psi(N_0, x^q C^q)$.

Moreover, by Proposition 1.2 (b) we know that

$$\phi^{o^{\psi}}(N_0, C) = \sum_{q=1}^{m(C)} x^q \phi^{o^{\psi}}(N_0, C^q) = \sum_{q=1}^{m(C)} \phi^{o^{\psi}}(N_0, x^q C^q).$$

Then, to finish the proof, it is sufficient to prove that $\psi(N_0, C) = \phi^{o^{\psi}}(N_0, C)$, where C is such that there exists a network g with $c_{ij} = x \geq 0$ if $(i,j) \in g$ and $c_{ij} = 0$ otherwise. Let $\{T_r\}_{r=1}^m$ be the partition of N_0 in C-components.

Now, suppose that we have an obligation rule ϕ^o. If we apply Kruskal algorithm, we can assume that in the first $n-m$ stages the agents in each component are connected among themselves, i.e., $P(g^{n-m}) = \{T_r\}_{r=1}^m$. Since $c_{i^p j^p} = 0$ for all $p = 1, \ldots, n-m$ and $o_i(T) = 0$ when the source is in T, we distinguish two cases:

1. $0 \in T_r$. In this case we have that

$$\phi_i^o(N_0, C) = 0 \text{ for all } i \in T_r.$$

2. $0 \notin T_r$. At Stage $n-m+1$ of Kruskal algorithm, it is possible to select the arc (i^{n-m+1}, j^{n-m+1}) such that $i^{n-m+1} \in T_r$ and $j^{n-m+1} = 0$. Therefore,

$$\phi_i^o(N_0, C) = c_{i^{n-m+1} j^{n-m+1}} o_i(S(P(g^{n-m}), i)) = x o_i(T_r) \text{ for all } i \in T_r.$$

On the other hand, Bergantiños and Vidal-Puga (2004) proved that $m(N_0, C) = \sum\limits_{r=1}^{m} m((T_r)_0, C)$. Therefore, since ψ satisfies PM, we have that $\psi_i(N_0, C) = \psi_i((T_r)_0, C)$ for all $i \in T_r$ and $r = 1, \ldots, m$.

We define two cost matrices $\tilde{C}$ and $\bar{C}$ by

$$\tilde{c}_{ij} = \begin{cases} 0 & \text{if } 0 \in \{i,j\} \\ c_{ij} & \text{otherwise} \end{cases} \text{ and } \bar{c}_{ij} = \begin{cases} c_{ij} & \text{if } 0 \in \{i,j\} \\ 0 & \text{otherwise} \end{cases} \text{ for all } i,j \in N_0.$$

Notice that $((T_r)_0, \tilde{C})$ and $((T_r)_0, \bar{C})$ are similar and $C = \tilde{C} + \bar{C}$. By RA, we have that

$$\psi((T_r)_0, C) = \psi((T_r)_0, \tilde{C}) + \psi((T_r)_0, \bar{C}).$$

Taking into account that $m((T_r)_0, \tilde{C}) = 0$, $m(\{i\}_0, \tilde{C}) = 0$ for all $i \in T_r$, and the property of PM, we have that $\psi_i((T_r)_0, \tilde{C}) = \psi_i(\{i\}_0, \tilde{C}) = 0$ for all $i \in T_r$.

Therefore, $\psi((T_r)_0, C) = \psi((T_r)_0, \bar{C})$.

We will study two cases:

1. $0 \in T_r$.

   We distinguish two cases again:

   (a) $c_{i0} = 0$ for all $i \in T_r$. In this case, we have that $\psi_i(N_0, C) = \psi_i((T_r)_0, C) = 0 = \phi_i^{o^\psi}(N_0, C)$ for all $i \in T_r$.

   (b) There exist $j, k \in T_r$ such that $c_{0j} = 0$ and $c_{0k} = x$.

   Following similar arguments to Bergantiños and Vidal-Puga (2004) in the proof of Proposition 3, we consider

   ${T_r}^1 = \{i \in T_r : c_{0i} = x\} \cup \{j\}$ and ${T_r}^2 = \{i \in T_r : c_{0i} = 0\} \setminus \{j\}$.

   We know that $m((T_r)_0, \bar{C}) = m(({T_r}^1)_0, \bar{C}) + m(({T_r}^2)_0, \bar{C})$. By PM, we have that $\psi_i(N_0, C) = \psi_i((T_r)_0, \bar{C}) = 0 = \phi_i^{o^\psi}(N_0, C)$ for all $i \in {T_r}^2$ and $\psi_i((T_r)_0, \bar{C}) = \psi_i(({T_r}^1)_0, \bar{C})$ for all $i \in {T_r}^1$.

   By RA, we have that $\psi(({T_r}^1)_0, \bar{C}) = \sum_{i \in {T_r}^1 \setminus \{j\}} \psi(({T_r}^1)_0, (\bar{C})^i)$ where $(\bar{c}_{0i})^i = x$ and $(\bar{c}_{kl})^i = 0$ otherwise.

   Taking into account that

   $m(({T_r}^1)_0, (\bar{C})^i) = m(\{i,j\}_0, (\bar{C})^i) + \sum_{k \in {T_r}^1 \setminus \{j,i\}} m(\{k\}_0, (\bar{C})^i)$

   and the property of PM, $\psi_k(({T_r}^1)_0, (\bar{C})^i) = 0$ for all $k \in {T_r}^1 \setminus \{j,i\}$,

   $$\psi_j(({T_r}^1)_0, (\bar{C})^i) = \psi_j(\{i,j\}_0, (\bar{C})^i), \text{ and}$$

   $$\psi_i(({T_r}^1)_0, (\bar{C})^i) = \psi_i(\{i,j\}_0, (\bar{C})^i).$$

Then, to see that $\psi_j(({T_r}^1)_0, (\bar{C})^i) = 0$ and $\psi_i(({T_r}^1)_0, (\bar{C})^i) = 0$ and, therefore, $\psi_i(N_0, C) = \psi_i((T_r)_0, \bar{C}) = \psi_i(({T_r}^1)_0, \bar{C}) = 0 = \phi_i^{o^\psi}(N_0, C)$ for all $i \in {T_r}^1$, it only remains to prove that $\psi_i(\{i,j\}_0, C) = 0$ and $\psi_j(\{i,j\}_0, C) = 0$ for the mcstp $(\{i,j\}_0, C)$ such that $c_{0j} = c_{ij} = 0$ and $c_{0i} = x$.

Since $m(\{i,j\}_0, C) = 0$, we can suppose that

$$\psi_i(\{i,j\}_0, C) = -\psi_j(\{i,j\}_0, C).$$

We will prove that $\psi_j(\{i,j\}_0, C) = 0$.

As $|\mathcal{N}| \geq 3$, we can consider the mcstp $(\{i,j,k\}, C')$ such that $c'_{0i} = x$ and $c'_{hl} = 0$ otherwise. By PM, we have that $\psi_j(\{i,j,k\}, C') = \psi_j(\{i,j\}_0, C)$.

Applying PM again, we have that $\psi_j(\{i,j,k\}, C') = \psi_j(\{j\}_0, C) = 0$. Then, $\psi_j(\{i,j\}_0, C) = 0$.

2. $0 \notin T_r$. In this case, $c_{0i} = x$ for all $i \in T_r$.

   We know that, $((T_r)_0, \bar{C}) = ((T_r)_0, x\widehat{C})$. Hence, $\psi((T_r)_0, C) = \psi((T_r)_0, x\widehat{C})$. We can conclude that

$$\phi^{o^\psi}(N_0, C) = xo^\psi(T_r) = x\psi((T_r)_0, \widehat{C}).$$

   Then, to show that $\psi(N_0, C) = \phi^{o^\psi}(N_0, C)$, we only need to prove that

$$\psi((T_r)_0, x\widehat{C}) = x\psi((T_r)_0, \widehat{C}), \text{ where } x \geq 0.$$

   We have several possibilities:

   - $x = \frac{p}{q}$ where $p, q \in \mathbb{N}$. Since ψ satisfies RA, it is straightforward that $\psi((T_r)_0, x\widehat{C}) = x\psi((T_r)_0, \widehat{C})$.
   - $x \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. There exists $\{x^p\}_{p \in \mathbb{N}}$ such that $x^p \in \mathbb{Q}^+$ for all $p \in \mathbb{N}$, $0 < x^p < x$, and $\lim_{p \to \infty} x^p = x$. Thus, for all $p \in \mathbb{N}$ and for all $i \in T_r$,

$$\begin{aligned}\psi_i((T_r)_0, x\widehat{C}) - x^p\psi_i((T_r)_0, \widehat{C}) &= \psi_i((T_r)_0, x\widehat{C}) - \psi_i((T_r)_0, x^p\widehat{C}) \\ &= \psi_i((T_r)_0, (x - x^p)\widehat{C}).\end{aligned}$$

In addition, we have that

$$0 \le \psi_i((T_r)_0, (x - x^p)\widehat{C}) \le (x - x^p) m((T_r)_0, \widehat{C}) = x - x^p.$$

Therefore,

$$\begin{aligned} 0 &\le \lim_{p\to\infty} [\psi_i((T_r)_0, x\widehat{C}) - x^p \psi_i((T_r)_0, \widehat{C})] \\ &= \psi_i((T_r)_0, x\widehat{C}) - x\psi_i((T_r)_0, \widehat{C}) \\ &\le \lim_{p\to\infty} (x - x^p) = 0. \end{aligned}$$

Then, $\psi_i((T_r)_0, x\widehat{C}) = x\psi_i((T_r)_0, \widehat{C})$ for all $i \in T_r$.

■

Theorem 1.3 *Suppose that $|\mathcal{N}| \ge 3$. A rule ψ satisfies the properties of population monotonicity and restricted additivity if and only if is an obligation rule.*

Proof.

Existence.

Tijs et al. (2005) proved that obligation rules satisfy PM. Then, we only need to show that obligation rules satisfy RA.

Consider the mcstp (N_0, C) and (N_0, C') in the conditions of the definition of RA. Thus, in the problem $(N_0, C + C')$, we have that $t = \{(i^0, i)\}_{i\in N}$ is also an mt and $c_{i_1^0 i_1} + c'_{i_1^0 i_1} \le c_{i_2^0 i_2} + c'_{i_2^0 i_2} \le \ldots \le c_{i_n^0 i_n} + c'_{i_n^0 i_n}$. Furthermore, Tijs et al. (2005) proved that obligation rules are independent of the mt chosen according to Kruskal algorithm. Therefore, given an obligation rule ϕ^o, we have that

$$\begin{aligned} \phi_i^o(N_0, C + C') &= \sum_{p=1}^{n} (c_{i_p^0 i_p} + c'_{i_p^0 i_p})(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\ &= \sum_{p=1}^{n} c_{i_p^0 i_p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\ &+ \sum_{p=1}^{n} c'_{i_p^0 i_p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\ &= \phi_i^o(N_0, C) + \phi_i^o(N_0, C'). \end{aligned}$$

Uniqueness.

If we consider the Proposition 1.3, we have that a rule ψ satisfying PM and RA is an obligation rule. ■

According to the Theorem 1.3, as the obligation rules satisfy RA and PM, we have a characterization of the family of obligation rules when $|\mathcal{N}| \geq 3$. Moreover, in the Proposition 1.3 we have given an expression to obtain the obligation function associated with an obligation rule. For any set of possible agents $\mathcal{N}$, we have similar results if we add the property of POS.

Proposition 1.4 *If a cost allocation rule ψ satisfies population monotonicity, restricted additivity, and positivity then*

$$\psi(N_0, C) = \phi^{o^{\psi}}(N_0, C),$$

where o^{ψ} is the obligation function defined by

$$o^{\psi}(S) = \psi(S_0, \widehat{C}) \text{ for all } S \in 2^N \setminus \{\emptyset\}$$

and the cost matrix $\widehat{C} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$.

Proof.

Consider a rule ψ satisfying PM, RA, and POS.

In the case of $|\mathcal{N}| \geq 3$, we take into account the Proposition 1.3 and, in the case of $|\mathcal{N}| = 2$, we can follow the same procedure as in Proposition 1.3 except the case of the mcstp $(\{i, j\}, C)$ with $c_{0j} = c_{ij} = 0$ and $c_{0i} = x$. In this case, applying the property of POS we obtain that $\psi_i(\{i, j\}, C) = \psi_j(\{i, j\}, C) = 0$. ■

Theorem 1.4 *A cost allocation rule ψ satisfies the properties of population monotonicity, restricted additivity, and positivity if and only if is an obligation rule.*

Proof.

Existence.

By definition, the obligation rules satisfy POS. Following similar arguments to Theorem 1.3, we can prove that they also satisfy PM and RA.

Uniqueness.

By Proposition 1.4 we have that a cost allocation rule ψ which satisfies the properties of PM, RA, and POS is an obligation rule. Then, the proof is finished. ■

Remark 1.7 *The properties mentioned in Theorem 1.4 are independent.*

- The egalitarian rule, $\delta_i(N_0, C) = \frac{1}{n}m(N_0, C)$ for all $i \in N$, satisfies the properties of POS and RA. Nevertheless, δ does not satisfy PM. Consider the mcstp (N_0, C), where $N = \{1, 2\}$ and $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. In this case, $\delta_1(\{1\}_0, C) = 0$ while $\delta_1(N_0, C) = \frac{1}{2}$.

- Consider the subset of orders

  $$\Pi'(N) = \{\pi \in \Pi(N) | \pi(i) < \pi(j) \text{ when } c_{0i} \leq c_{0j} \text{ for all } i, j \in N, i \neq j\}.$$

  Let β be the cost allocation rule defined for all $i \in N$ as

  $$\beta_i(N_0, C) = \frac{1}{|\Pi'(N)|} \sum_{\pi \in \Pi'(N)} (v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))).$$

  This rule satisfies PM and POS. However, it fails to satisfy RA. Consider $N = \{1, 2\}$ and the cost matrices

  $$C = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 4 & 5 \\ 4 & 0 & 2 \\ 5 & 2 & 0 \end{pmatrix}.$$

  C and C' are similar cost matrices and $\beta_1(N_0, C + C') = 7 \neq \beta_1(N_0, C) + \beta_1(N_0, C') = 2 + 4 = 6$.

- To prove that POS is independent of RA and PM, consider the rule γ such that

  1. If $\mathcal{N}$ has at least three members, $\gamma(N_0, C) = Sh(N, v_{C^*})$ for all $N \subset \mathcal{N}$.
  2. If $|\mathcal{N}| \leq 2$, $\gamma(N_0, C) = Sh(N, v_C)$ for all $N \subset \mathcal{N}$.

Since the Shapley value is additive on the characteristic function, the rule γ satisfies RA. In the case of the property of PM, we only need to prove that it is true for the case of $|\mathcal{N}| \leq 2$ because Bergantiños and Vidal-Puga (2005a) showed that $Sh(N, v_{C^*})$ satisfies PM for all mcstp (N_0, C) .

As the remainder cases are straightforward, we can assume that $N = \{i, j\}$ and $c_{0i} \leq c_{0j}$. We must prove that $\gamma_i(N_0, C) \leq c_{0i}$ and $\gamma_j(N_0, C) \leq c_{0j}$. For it, we distinguish three cases:

1. $c_{0i} \leq c_{ij} \leq c_{0j}$. We obtain that
   $\gamma_i(N_0, C) = c_{0i} + \frac{c_{ij} - c_{0j}}{2} \leq c_{0i}$ and $\gamma_j(N_0, C) = \frac{c_{ij} + c_{0j}}{2} \leq c_{0j}$.
2. $c_{0i} \leq c_{0j} \leq c_{ij}$. In this case, $\gamma_i(N_0, C) = c_{0i}$ and $\gamma_j(N_0, C) = c_{0j}$.
3. $c_{ij} \leq c_{0i} \leq c_{0j}$. We have that
   $\gamma_i(N_0, C) = c_{0i} + \frac{c_{ij} - c_{0j}}{2} \leq c_{0i}$ and $\gamma_j(N_0, C) = \frac{c_{ij} + c_{0j}}{2} \leq c_{0j}$.

This rule fails POS. Consider (N_0, C) with $C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, $\gamma(N_0, C) = (\frac{1}{2}, -\frac{1}{2})$.

Remark 1.8 *Bergantiños and Vidal-Puga (2004) defined the restricted additivity property instead of the common additivity in the mcstp, i.e.,* $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$ *for all mcstp* (N_0, C) *and* (N_0, C')*, because there is no cost allocation rule satisfying the latter one as the example described in Figures 1.4 and 1.5 shows.*

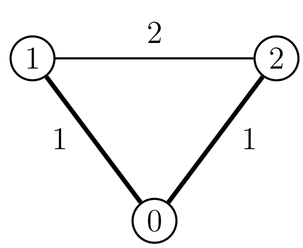

Figure 1.4: mcstp (N_0, C)

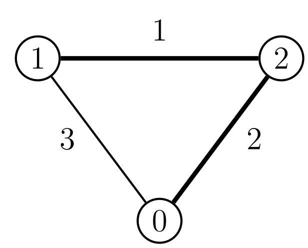

Figure 1.5: mcstp (N_0, C')

In this case $m(N_0, C) = 2$ *and* $m(N_0, C') = 3$*, while* $m(N_0, C + C') = 6$*. So, there is no cost allocation rule simultaneously satisfying efficiency and additivity.*

In accordance with Proposition 1.4, it is easy to calculate the obligation function for any obligation rule. Then, we will apply this proposition not only to show that the optimistic and pessimistic weighted Shapley rules of the irreducible form are obligation rules, but also to calculate the associated obligation functions for both weighted Shapley rules.

Corollary 1.1 *Let φ^w be the pessimistic weighted Shapley rule of the irreducible form associated with the weight system w. Thus, for all mcstp (N_0, C),*

$$\varphi^w (N_0, C) = \phi^{o^{N,w}} (N_0, C),$$

where the obligation function $o^{N,w}$ is given by

$$o_i^{N,w}(S) = \sum_{\pi \in \Pi(S \setminus \{i\})} \prod_{j=1}^{s-1} \frac{\omega_{\pi^{-1}(j)}}{\sum\limits_{k=1}^{j} \omega_{\pi^{-1}(k)} + \omega_i} \text{ for all } S \in 2^N \setminus \{\emptyset\} \text{ and } i \in S.$$

Proof.

Since we know by Lemma 1.3 (c) that

$$v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = \min_{j \in Pre(i,\pi)_0} \{c^*_{ij}\} \geq 0,$$

we have that φ^w satisfies POS.

Kalai and Samet (1987) proved that the weighted Shapley values satisfy additivity on the characteristic function. Moreover, Bergantiños and Vidal-Puga (2004) proved that $v_{(C+C')^*} = v_{C^*} + v_{C'^*}$, where (N_0, C) and (N_0, C') are two similar mcstp.

Using this results, we have that, for all weight system w,

$$\begin{aligned}
\varphi^w (N_0, C + C') &= Sh^{w_N}(N, v_{(C+C')^*}) \\
&= Sh^{w_N}(N, v_{C^*} + v_{C'^*}) \\
&= Sh^{w_N}(N, v_{C^*}) + Sh^{w_N}(N, v_{C'^*}) \\
&= \varphi^w (N_0, C) + \varphi^w (N_0, C').
\end{aligned}$$

Then, we obtain that φ^w satisfies RA.

To prove that they satisfy PM, we need to prove that

$$\begin{aligned} Sh_i^{w_N}(N, v_{C^*}) &= \sum_{\pi \in \Pi(N)} p_{w_N}(\pi) \left[v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) \right] \\ \leq & \sum_{\pi^{-j} \in \Pi(N^{-j})} p_{w_{N^{-j}}}(\pi^{-j}) \left[v_{C^*}(Pre(i,\pi^{-j}) \cup \{i\}) - v_{C^*}(Pre(i,\pi^{-j})) \right] \\ = & \; Sh_i^{w_{N^{-j}}}(N^{-j}, v_{C^*}). \end{aligned}$$

We know that $\sum_{\pi \in \Pi(N)} p_{w_N}(\pi) \left[v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) \right]$

$$\begin{aligned} = & \sum_{\pi \in \Pi(N), j \in Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in Pre(i,\pi)_0} \{c^*_{ik}\} + \sum_{\pi \in \Pi(N), j \notin Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in Pre(i,\pi)_0} \{c^*_{ik}\} \\ \leq & \sum_{\pi \in \Pi(N), j \in Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in (Pre(i,\pi) \setminus \{j\})_0} \{c^*_{ik}\} \\ + & \sum_{\pi \in \Pi(N), j \notin Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in Pre(i,\pi)_0} \{c^*_{ik}\}. \end{aligned}$$

Given a cost matrix C, we know that $C^* \leq C$. Considering the connection costs of agents in N_0^{-j}, $(C^*)^{-j} \leq C^{-j}$, where C^{-j} denotes the restriction of C to the agents in N^{-j}. Moreover, by Lemma 1.3 it is straightforward that $(C^*)^{-j}$ is an irreducible matrix. Thus, $(C^*)^{-j} \leq (C^{-j})^*$.

Then,

$$\begin{aligned} Sh_i^{w_N}(N, v_{C^*}) \leq & \sum_{\pi \in \Pi(N), j \in Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in (Pre(i,\pi) \setminus \{j\})_0} \{(c^{-j})^*_{ik}\} \\ + & \sum_{\pi \in \Pi(N), j \notin Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in Pre(i,\pi)_0} \{(c^{-j})^*_{ik}\} \\ = & \sum_{\pi^{-j} \in \Pi(N^{-j})} \left\{ \sum_{\pi \in \Pi(N), \pi_{N^{-j}} = \pi^{-j}} p_{w_N}(\pi) \min_{k \in Pre(i,\pi^{-j})_0} \{(c^{-j})^*_{ik}\} \right\}. \end{aligned}$$

Now, we will prove that

$$\sum_{\pi \in \Pi(N), \pi_{N^{-j}} = \pi^{-j}} p_{w_N}(\pi) = p_{w_{N^{-j}}}(\pi^{-j}).$$

For this purpose, we will use the induction hypothesis in the cardinality of

$N \setminus \{j\}$.

Suppose that $N = \{i, j\}$. We obtain that

$$\sum_{\pi \in \Pi(\{i,j\})} p_{w_N}(\pi) = \frac{w_i}{w_i + w_j} + \frac{w_j}{w_i + w_j} = 1 = p_{w_{N^{-j}}}(\pi^{-j}).$$

By induction hypothesis, we assume that the result is true for all the sets of agents to which agent j belongs and with cardinality below n, with $n > 2$.

Consider $N = \{i_1, \ldots, i_{n-1}, j\}$. We can assume, without loss of generality, that $\pi^{-j}(i_1) < \ldots < \pi^{-j}(i_{n-1})$.

Therefore,

$$\sum_{\pi \in \Pi(N), \pi_{N^{-j}} = \pi^{-j}} p_{w_N}(\pi)$$

$$\begin{aligned}
&= \sum_{\pi \in \Pi(N), \pi_{N^{-j}} = \pi^{-j}, \pi(j) = n} p_{w_N}(\pi) + \sum_{\pi \in \Pi(N), \pi_{N^{-j}} = \pi^{-j}, \pi(j) < n} p_{w_N}(\pi) \\
&= p_{w_{N^{-j}}}(\pi^{-j}) \frac{w_j}{w_j + \sum_{k=1}^{n-1} w_{i_k}} \\
&+ \sum_{\pi \in \Pi(N \setminus \{i_{n-1}\}), \pi_{N \setminus \{i_{n-1}, j\}} = \pi^{-j}_{N \setminus \{i_{n-1}\}}} p_{w_{N \setminus \{i_{n-1}\}}}(\pi_{N \setminus \{i_{n-1}\}}) \frac{w_{i_{n-1}}}{w_j + \sum_{k=1}^{n-1} w_{i_k}} \\
&= p_{w_{N^{-j}}}(\pi^{-j}) \frac{w_j}{w_j + \sum_{k=1}^{n-1} w_{i_k}} + p_{w_{N \setminus \{i_{n-1}, j\}}}(\pi^{-j}_{N \setminus \{i_{n-1}\}}) \frac{w_{i_{n-1}}}{w_j + \sum_{k=1}^{n-1} w_{i_k}} \\
&= p_{w_{N^{-j}}}(\pi^{-j}) \left\{ \frac{w_j}{w_j + \sum_{k=1}^{n-1} w_{i_k}} + \frac{\sum_{k=1}^{n-1} w_{i_k}}{w_j + \sum_{k=1}^{n-1} w_{i_k}} \right\} = p_{w_{N^{-j}}}(\pi^{-j}).
\end{aligned}$$

Thus,

$$\begin{aligned} Sh_i^{w_N}(N, v_{C^*}) &\leq \sum_{\pi^{-j} \in \Pi(N^{-j})} p_{w_{N^{-j}}}(\pi^{-j}) \left\{ \min_{k \in Pre(i,\pi^{-j})_0} \{(c^{-j})^*_{ik}\} \right\} \\ &= Sh_i^{w_{N^{-j}}}(N^{-j}, v_{C^*}). \end{aligned}$$

Then, by Proposition 1.4 we have that these rules are obligation rules. To obtain the corresponding obligation function, we consider an mcstp (N_0, C) and a weight system w.

The obligation function is, for all $i \in S$ and for all $S \in 2^N \setminus \{\emptyset\}$,

$$\begin{aligned} o_i^{\varphi^w}(S) &= \varphi_i^w(S_0, \widehat{C}) \\ &= \sum_{\pi \in \Pi(S)} p_{w_S}(\pi) \left[v_{\widehat{C}}(Pre(i,\pi) \cup \{i\}) - v_{\widehat{C}}(Pre(i,\pi)) \right] \\ &= \sum_{\pi \in \Pi(S)} p_{w_S}(\pi) \min_{k \in Pre(i,\pi)_0} \{\widehat{c}_{ik}\} \\ &= \sum_{\pi \in \Pi(S) : \pi(i) \neq 1} p_{w_S}(\pi) 0 + \sum_{\pi \in \Pi(S) : \pi(i) = 1} p_{w_S}(\pi) 1 \\ &= \sum_{\pi \in \Pi(S) : \pi(i) = 1} p_{w_S}(\pi) = \sum_{\pi \in \Pi(S \setminus \{i\})} \prod_{j=1}^{s-1} \frac{w_{\pi^{-1}(j)}}{\sum\limits_{k=1}^{j} w_{\pi^{-1}(k)} + w_i}. \end{aligned}$$

■

In the following corollary we prove in an alternative way, different from the one used in Theorem 1.1, that the optimistic weighted Shapley rules are obligation rules.

Corollary 1.2 *Let φ^w be the optimistic weighted Shapley rule associated with the weight system w. Thus, for all mcstp (N_0, C),*

$$\varphi^w(N_0, C) = \phi^{o^{N,w}}(N_0, C),$$

where the obligation function $o^{N,w}$ is given by

$$o_i^{N,w}(S) = \frac{w_i}{\sum\limits_{j \in S} w_j} \text{ for all } S \in 2^N \setminus \{\emptyset\} \text{ and } i \in S.$$

Proof.

Following the same procedure than in the case of the pessimistic weighted Shapley rules of the irreducible form, we may prove that the optimistic ones satisfy RA and POS.

By Kalai and Samet (1987), we know that

$$\begin{aligned} Sh_i^{w_N}(N, v_C^+) &= Sh_i^{*w_N}(N, v_{C^*}) \\ &= \sum_{\pi \in \Pi(N)} p_{w_N}^*(\pi) \left[v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) \right], \end{aligned}$$

where $p_{w_N}^*(\pi) = \prod\limits_{j=0}^{n-1} \dfrac{w_{\pi^{-1}(n-j)}}{\sum\limits_{k=0}^{j} w_{\pi^{-1}(n-k)}}$.

Using this expression, the proof of PM is similar to the one in the case of the pessimistic weighted Shapley values of the irreducible form. Therefore, we omit it.

Then, by Proposition 1.4 we have that these rules are obligation rules. To obtain the corresponding obligation function, we consider an mcstp (N_0, C) and a weight system w. The obligation function, for all $i \in S$ and for all $S \in 2^N \setminus \{\emptyset\}$, is given by

$$\begin{aligned} o_i^{\varphi^w}(S) &= \varphi_i^w(S, \widehat{C}) \\ &= \sum_{\pi \in \Pi(S)} p_{w_S}(\pi) \left[v_{\widehat{C}}^+(Pre(i,\pi) \cup \{i\}) - v_{\widehat{C}}^+(Pre(i,\pi)) \right] \\ &= \sum_{\pi \in \Pi(S)} p_{w_S}(\pi) \left[v_{\widehat{C}}(S \setminus Pre(i,\pi)) - v_{\widehat{C}}(S \setminus (Pre(i,\pi) \cup \{i\})) \right] \\ &= \sum_{\pi \in \Pi(S)} p_{w_S}(\pi) \min_{k \in (S \setminus (Pre(i,\pi) \cup \{i\}))_0} \{\widehat{c}_{ik}\} \\ &= \sum_{\pi \in \Pi(S) : \pi(i) \neq s} p_{w_S}(\pi) 0 + \sum_{\pi \in \Pi(S) : \pi(i) = s} p_{w_S}(\pi) 1 \\ &= \sum_{\pi \in \Pi(S) : \pi(i) = s} p_{w_S}(\pi) \\ &= \sum_{\pi \in \Pi(S \setminus \{i\})} p_{w_{S \setminus \{i\}}}(\pi) \frac{w_i}{\sum\limits_{j \in S} w_j} = \frac{w_i}{\sum\limits_{j \in S} w_j}. \end{aligned}$$

■

Remark 1.9 *We must point out that, in spite of the fact that the pessimistic and optimistic weighted Shapley values of the irreducible form are obligation rules, not all the pessimistic weighted Shapley rules are obligation rules.*

Consider, for instance, $Sh(N, v_C)$ for the mcstp (N_0, C), with

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}.$$

We have that $Sh_1(N, v_C) = \frac{5}{6} > \frac{1}{2} = Sh_1(\{1,2\}, v_C)$. Therefore, we know that this rule does not satisfy the property of PM. As the family of obligation rules satisfies PM, we deduce that this pessimistic weighted Shapley rule is not an obligation rule.

1.8 Concluding remarks

In this chapter we have obtained, among other things, two characterizations. We have characterized the family of obligation rules, introduced by Tijs et al. (2005), and a subfamily of obligation rules, the optimistic weighted Shapley rules, based on weighted Shapley values of a game.

There are previous characterizations of some obligation rules, such as the characterizations of $Sh\,(N, v_{C^*})$, the obligation rule where the obligations for the agents in a coalition are the same, provided by Feltkamp et al. (1994) and Bergantiños and Vidal-Puga (2004 and 2005a). But, we must say that, as far as we know, we give the first characterization of the whole family of obligation rules. However, from our point of view, the relevance of this result lies not only in the fact that there is not a previous characterization of the family, but also in the use for this purpose of a very appealing property in the context of mcstp, the property of population monotonicity.

At the view of this result, we think that interesting connected research with this characterization could consist of finding other characterizations for this family by means of properties such as the property of strong cost monotonicity, also very appealing in the context of mcstp, or looking for the family of rules which satisfies the property of population monotonicity and the family of rules which satisfies the properties of population monotonicity and strong cost monotonicity.

It could be also interesting to continue studying some obligation rules. In this context, we have studied the family of the optimistic weighted Shapley rules. In the case of these rules, it is interesting to point out that the characterization is inspired in the result by Bergantiños and Vidal-Puga (2005a), where they characterized the rule $Sh\left(N, v_{C^*}\right)$ as the only rule satisfying SCM, PM, and ESEC. Nevertheless, there are important differences between the two results. Some of them are presented below.

As we have mentioned before, from a technical point of view, the scheme of the proof for the optimistic weighted Shapley rules is different from the scheme of the proof of Bergantiños and Vidal-Puga (2005a). For instance, we prove that optimistic weighted Shapley rules satisfy SCM and PM by proving that they are obligation rules. Bergantiños and Vidal-Puga (2005a) proved that $Sh\left(N, v_{C^*}\right)$ satisfies SCM and PM directly.

On the other hand, it is not possible to generalize the result of Bergantiños and Vidal-Puga (2005a) in a trivial way. If we change Shapley value $\left(Sh\left(N, v_{C^*}\right)\right)$ by the weighted Shapley value $\left(Sh^w\left(N, v_{C^*}\right)\right)$, and equal share of extra cost by weighted share of extra cost in the result of Bergantiños and Vidal-Puga (2005a), the result does not hold because the pessimistic weighted Shapley value does not satisfy the property of weighted share of extra cost.

Finally, we must note that we have proved that the optimistic weighted Shapley rules $Sh^w\left(N, v_C^+\right)$ and the pessimistic weighted Shapley rules $Sh^w\left(N, v_{C^*}\right)$ are obligation rules. An axiomatic characterization in the case of the optimistic weighted Shapley rules has been provided. It could be interesting to make a similar study for the family $Sh^w\left(N, v_{C^*}\right)$.

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Chapter 2

Bankruptcy problems and multi-issue allocation situations

2.1 Introduction

Bankruptcy problems are situations where we have to divide an estate among a set of agents, but the estate is not enough to satisfy all the quantities the agents demand. Due to this insufficiency, bankruptcy rules appear. These rules, whose definition depends on the context, divide the estate adequately. There are many situations which can be described by means of bankruptcy models. One of the more classical examples is the case of a firm which goes bankrupt and has to divide the active (estate) among the claimants (agents) taking into account that the active is not sufficient to satisfy the claims.

In spite of being already studied in the Talmud, the first formal analysis of bankruptcy problems appears in O'Neill (1982). In this paper he not only associates a cooperative game with each bankruptcy problem but also defines the random arrival rule, which coincides with the Shapley value of this game and is characterized with a property of consistency. Other interesting papers related to the study of these problems are the ones by Aumann and Maschler (1985), Curiel et al. (1987), Moulin (1987), Young (1988), Dagan (1996), and Herrero and Villar (2001). Although we have only mentioned these frameworks, there is a wide literature about bankruptcy. Thomson (2003) is a good survey devoted to the analysis of bankruptcy problems.

In Lorenzo-Freire et al. (2005a), we study an extension of the bankruptcy problems: the multi-issue allocation situations. The multi-issue allocation situations were introduced by Calleja et al. (2005) to model bankruptcy-like problems in which the estate is divided not on the basis of a single claim for each agent, but multiple claims. These multiple claims are not the result of some exogenously difference in status or priority (see Kaminski (2000)). The multiple claims result from different issues, all with the same status. Another paper related to multi-issue allocation situations is the one by González-Alcón et al. (2003).

In Calleja et al. (2005), a multi-issue allocation solution is a function which assigns to each multi-issue allocation situation a vector whose components are the allocations the agents obtain. But, due to the fact that in multi-issue allocation situations there is a matrix of claims where each claim is the claim of an agent in an issue, we think that it could be natural to consider rules as functions where each multi-issue allocation situation is associated with a matrix representing the assignment the agents in the issues obtain, i.e., we think that it could be interesting to know what is the assignment for an agent in a specific issue. Therefore, in Lorenzo-Freire et al. (2005b), we introduce a new definition for the multi-issue allocation solutions. Taking into account this new definition, we define a new family of rules: the rules in two-stages. In these rules, we consider two stages: in the first stage, we distribute the estate using any bankruptcy rule (it will depend on the context) among the issues and, in the second one, we divide the assignment obtained for each issue among the agents, using the same bankruptcy rule. We characterize two rules of this family, the constrained equal awards rule in two stages and the constrained equal losses rule in two stages. These rules are defined taking into account the constrained equal awards and losses rules for bankruptcy problems, respectively.

As for the constrained equal awards rule for bankruptcy problems, this rule was characterized (Dagan, 1996) with the properties of equal treatment (cf. O'Neill (1982)), composition (cf. Young (1988)), and invariance under claims truncation. In Lorenzo-Freire et al. (2005b), we characterize the constrained equal awards rule in two stages with three properties, which follow the philosophy of the previous properties in the context of multi-issue allocation situations, and a quotient property. This quotient property is similar to the property introduced by Owen (1977) in the context of TU games with a priori unions.

On the other hand, the constrained equal losses in bankruptcy problems was characterized by Herrero and Villar (2001) with the properties of equal treatment, composition from minimal rights (cf. Curiel et al. (1987)), and path-independence (cf. Moulin (1987)). We characterize the constrained equal losses rule in two stages with three properties in the same line and the quotient property.

In the literature about Shapley value and other game solution concepts, a principle of reciprocity among the agents is often used. This principle was introduced by Myerson (1980). Myerson's principle of balanced contributions asserts that for any two agents the gain or loss caused to each agent when the other one leaves the game should be equal.

In their paper, Calleja et al. (2005) generalized O'Neill's random arrival rule to the class of multi-issue allocation situations, defining the proportional and queue run-to-to-the-bank rules. They also defined two games, the proportional and queue games, and showed that the rules coincide with the Shapley value in these games. Following O'Neill's characterization of the random arrival rule for bankruptcy situations by means of the property of consistency, Calleja et al. (2005) characterized the proportional and the queue rules in a similar fashion. Due to the fact that the underlying idea of an agent leaving the game is not easy to implement in multi-issue allocation situations, they have to extend their domain to the class of the so-called multi-issue allocation situations with awards. In such situations, the awarded agents are still part of the game, but any solution must give them their predetermined award.

The consideration of a set of awarded agents has perfect sense in a wide variety of situations, not only in multi-issue allocation situations. Suppose, for example, the problem of the cost allocation of a project in which some agents are invited to participate with the compromise of being allocated with some fixed quantity. In this way, with the objective of providing a general framework which contains all these situations, in Lorenzo-Freire et al. (2005a), we extend the classical model of cooperative games with transferable utility to the more general model of TU games with awards, in which any solution must allocate to the awarded agents their fixed award. For this class of games, we define a run-to-the-bank rule and characterize it in terms of a property of balanced contributions. As application of our main result, in Lorenzo-Freire et al. (2005a), we focus on the more pessimistic approach, the queue rule for multi-issue allocation situations, characterizing the queue run-to-the-bank rule with a property of balanced contributions. Similar results can be obtained for the proportional approach.

The structure of this chapter is as follows. In Section 2 we introduce bankruptcy problems. In Section 3 we define multi-issue allocation situations. Section 4 is devoted to the definition and characterization of the two-stage constrained equal awards rule and the two-stage constrained equal losses rule. Finally, in Section 5, we define the run-to-the-bank rule for TU games with awards, given a characterization of this rule.

2.2 Bankruptcy problems

A *bankruptcy problem* (O'Neill (1982)) is a triple (N, E, c) such that $N = \{1, \ldots, n\}$ is the set of agents, $E \geq 0$ represents the estate, which is the available amount to satisfy the claims of the agents, and $c \in \mathbb{R}^N_+$ is the vector of claims, where c_i denotes the claim of agent i. The main assumption of a bankruptcy problem is that the estate is not sufficient to satisfy all the claims, i.e., $0 \leq E \leq \sum\limits_{i \in N} c_i$.

A cooperative game with transferable utility, *TU game*, is a pair (N, v) where $v : 2^N \to \mathbb{R}$ is the characteristic function that assigns to each coalition $S \in 2^N$ the value the agents in the coalition obtain when they cooperate, given by $v(S)$. Moreover, it is assumed that $v(\varnothing) = 0$.

A TU game is superadditive if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$.

Given a bankruptcy problem (N, E, c), O'Neill (1982) defines the *bankruptcy game* as the TU game $(N, v^{(E,c)})$ where $v^{(E,c)}(S) = \max \left\{ E - \sum\limits_{i \in N \setminus S} c_i, 0 \right\}$ for all $S \subset N$. According to this game, each coalition receives what remains after the agents outside the coalition receive their claims.

A *bankruptcy rule* is a function ψ which assigns to every bankruptcy problem (N, E, c) a vector $\psi(N, E, c) \in \mathbb{R}^N$ such that $0 \leq \psi_i(N, E, c) \leq c_i$ for all $i \in N$ and $\sum\limits_{i \in N} \psi_i(N, E, c) = E$.

There are many bankruptcy rules in the literature. Depending on the context of the problem, a suitable bankruptcy rule is chosen.

One of the most common bankruptcy rules is the proportional rule. This rule divides the estate proportionally to the claims of the agents.

Other interesting rules are the constrained equal awards and losses rules. Whereas the constrained equal awards rule makes awards as equal as possible on condition that nobody gets more than his claim, the constrained equal losses rule makes losses as equal as possible on condition that no agent receives a negative award.

Aumann and Maschler (1985) defined the Talmud rule as the consistent extension of a solution defined for two-person bankruptcy problems: the contested garment rule. In the contested garment rule, each agent receives first the part of the estate which is left after satisfying the claim of the other agent and then the rest of the estate is equally divided between the two agents.

In the random arrival rule (O'Neill, 1982), the claims are satisfied according to the arrival order of the agents until there is no estate left. The random arrival rule is an average over all the possible arrival orders. Note that an order π on N, which has already been defined in Chapter 1, is a bijection $\pi : N \longrightarrow \{1, \ldots, |N|\}$ where, for all $i \in N$, $\pi(i)$ is the position of agent i. Let $\Pi(N)$ denote the set of all orders in N.

Next, we give the formal definition of these rules.

The proportional rule (P). Given a bankruptcy problem (N, E, c) and an agent $i \in N$,

$P_i(N, E, c) = \lambda c_i$, where $\lambda \geq 0$ is such that $\displaystyle\sum_{i \in N} \lambda c_i = E$.

The constrained equal awards rule (CEA). Given a bankruptcy problem (N, E, c) and $i \in N$,

$CEA_i(N, E, c) = \min\{c_i, \lambda\}$, where $\lambda \geq 0$ is such that $\displaystyle\sum_{i \in N} \min\{c_i, \lambda\} = E$.

The constrained equal losses rule (CEL). For each bankruptcy problem (N, E, c) and for each $i \in N$,

$CEL_i(N, E, c) = \max\{c_i - \lambda, 0\}$, where $\lambda \geq 0$ is such that $\displaystyle\sum_{i \in N} \max\{c_i - \lambda, 0\} = E$.

Note that the CEL rule is the dual of the CEA, i.e., $CEL(N, E, c) = c - CEA\left(N, \sum_{i \in N} c_i - E, c\right)$.

The Talmud rule (T). For each bankruptcy problem (N, E, c) and for each agent $i \in N$,

$$T_i(N,E,c) = \begin{cases} \min\{c_i/2, \lambda\} & \sum\limits_{i \in N} c_i/2 \geq E \\ c_i - \min\{c_i/2, \lambda\} & \sum\limits_{i \in N} c_i/2 \leq E, \end{cases}$$

where $\lambda \geq 0$ is such that $\sum\limits_{i \in N} T_i(N,E,c) = E$.

The random arrival rule (RA). For each bankruptcy problem (N,E,c) and for each $i \in N$,

$$RA_i(N,E,c) = \tfrac{1}{n!} \sum_{\pi \in \Pi(N)} \min\left\{c_i, \max\left\{E - \sum_{j \in N, \pi(j) < \pi(i)} c_j, 0\right\}\right\}.$$

2.3 Multi-issue allocation situations

A *multi-issue allocation (MIA) situation* (Calleja et al. (2005)) is a 4-tuple (R,N,E,C) where R is the finite set of issues, N is the finite set of agents, $C \in \mathbb{R}_+^{R \times N}$ is the matrix of claims, and E is the estate to be divided. Each element c_{ki} represents the amount claimed by an agent $i \in N$ in an issue $k \in R$. We assume that $0 \leq E \leq \sum\limits_{k \in R} \sum\limits_{i \in N} c_{ki}$. For all $k \in R$, $C_k = (c_{ki})_{i \in N}$ denotes the vector of claims according to issue $k \in R$.

Note that a bankruptcy problem is a MIA situation with $|R| = 1$. On the other hand, a bankruptcy situation with unions (Borm et al. (2005)) can also be expressed as a MIA situation, where the issues correspond to the unions. In the associated matrix of claims, the issues are disjoint in the sense that each agent has a claim on just one issue.

There are some examples which can be modeled as multi-issue allocation situations. The first example is related to the proposals to stabilize the greenhouse gas concentration in the atmosphere such as the Kyoto Protocol. The issues could be the six gases considered (CO_2, CH_4, NO_2, HFC, PFC, and SF_6) whereas the agents implied would be the countries that have signed the Kyoto Protocol in order to reduce their emissions of these gases. Another example may be a University with different libraries placed in several faculties. The central services of the University must decide how to allocate some money (the estate) to these libraries (issues). Each department (agent) has a number of claims on the estate.

A department can not only be implied in one faculty but also in several faculties, being possible to have claims in different libraries. A different example is related to the political decisions concerning the allocation of taxpayers' money among several services. In this case, the central government decides how to assign this money adequately. This money is not directly assigned to the services but first to the different government departments. In this way, each department (agent) has demands for all the services (issues) the department is responsible for.

A *multi-issue allocation (MIA) solution* Ψ is a function which assigns to every MIA situation (R, N, E, C) a matrix $\Psi(R, N, E, C) \in \mathbb{R}^{R\times N}$ satisfying for all $k \in R$ and $i \in N$ that $0 \leq \Psi_{ki}(R, N, E, C) \leq c_{ki}$ (reasonability) and $\sum_{k\in R}\sum_{i\in N}\Psi_{ki}(R, N, E, C) = E$ (efficiency).

2.4 The two-stages rules

In this section we introduce a natural two-stage procedure to define MIA solutions from bankruptcy rules.

Let ψ be a bankruptcy rule and let (R, N, E, C) be a MIA situation. The *two-stage rule* $\Psi^{\psi}(R, N, E, C)$ is the MIA solution obtained from the following two-stage procedure.

- *First stage:* Consider the so-called quotient bankruptcy problem (R, E, c^R), where $c^R = (c_1^R, \ldots, c_r^R) \in \mathbb{R}^R$ denotes the vector of total claims in the issues, i.e., $c_k^R = \sum_{i\in N} c_{ki}$ for all $k \in R$. Divide the amount E among the issues using bankruptcy rule ψ. In this way, we obtain $\psi(R, E, c^R) \in \mathbb{R}^R$.

- *Second stage:* For each $k \in R$, consider a new bankruptcy problem for the agents $(N, \psi_k(R, E, c^R), C_k)$ and apply the same bankruptcy rule ψ to this new bankruptcy problem. So, we obtain $\psi(N, \psi_k(R, E, c^R), C_k) \in \mathbb{R}^N$ for each $k \in R$.

The next figure shows the whole procedure:

$$
\begin{tikzcd}
(N,R,E,C) \arrow[rr, "\Psi^{\psi}"] \arrow[d] & & \Psi^{\psi}(N,R,E,C) = \begin{pmatrix} s_1^1 & \dots & s_n^1 \\ \vdots & & \vdots \\ s_1^r & \dots & s_n^r \end{pmatrix} \\
(R,E,c^R) \arrow[r, "\psi"] & \psi(R,E,c^R) \arrow[r] & \left\{ \begin{array}{ccc} (N,\psi_1(R,E,c^R),C_1) & \xrightarrow{\psi} & s^1 \\ \vdots & & \vdots \\ (N,\psi_r(R,E,c^R),C_r) & \xrightarrow{\psi} & s^r \end{array} \right\} \arrow[u]
\end{tikzcd}
$$

2.4.1 The constrained equal awards rule in two stages

In this part we provide a characterization of the two-stage constrained equal awards rule, which is defined by considering $\psi = CEA$. The two-stage rule Ψ^{CEA} is then given, for all (R, N, E, C) and for all $k \in R, i \in N$, by

$$\Psi_{ki}^{CEA}(R, N, E, C) = \min\{\beta_k, c_{ki}\}$$

where for all $k \in R$, β_k is such that $\sum\limits_{i \in N} \min\{\beta_k, c_{ki}\} = \min\{\lambda, c_k^R\}$ and λ is such that $\sum\limits_{k \in R} \min\{\lambda, c_k^R\} = E$.

Below, we list some properties which we will use to characterize the two-stage CEA rule.

Composition. A rule Ψ satisfies composition if for all (R, N, E, C) and for all $0 \leq E' \leq E$ we have that

$$\Psi(R, N, E, C) = \Psi(R, N, E', C) + \Psi(R, N, E - E', C - \Psi(R, N, E', C)).$$

According to this property, we can divide the total estate among the issues and the agents using two different procedures, which result in the same outcome. In the first procedure, we divide the total estate directly using Ψ. In the other procedure, we first divide a part E' of the estate and then divide remainder $E - E'$ on the basis of the remaining claims, both times using Ψ.

Invariance under claims truncation. A rule Ψ satisfies invariance under claims truncation if for each MIA situation (R, N, E, C) we have

$$\Psi(R, N, E, C) = \Psi(R, N, E, C^E),$$

where $C^E \in \mathbb{R}_+^{R\times N}$ is such that $c_{ki}^E = \min\{c_{ki}, E\}$ for all $k \in R$ and $i \in N$.

This property says that truncating each claim to the estate does not influence the outcome.

Equal treatment for the agents within an issue. A rule Ψ satisfies the property of equal treatment for the agents within an issue if for each MIA situation (R, N, E, C), for all $k \in R$ and $i, j \in N$ such that $c_{ki} = c_{kj}$, $\Psi_{ki}(R, N, E, C) = \Psi_{kj}(R, N, E, C)$.

If two agents claim the same quantity in an issue, they will receive the same.

Equal treatment for the issues. A rule Ψ satisfies equal treatment for the issues if for all MIA situation (R, N, E, C), for all $k, k' \in R$ satisfying that $c_k^R = c_{k'}^R$, $\Psi_{k1}(R, \{1\}, E, c^R) = \Psi_{k'1}(R, \{1\}, E, c^R)$.

If the total claim in two issues is the same, the quantity assigned to them in the quotient MIA situation coincides. The quotient MIA situation is the MIA situation where there is only one agent and the claim of each issue equals the sum of all the claims of the agents in that issue.

Quotient property. A rule Ψ satisfies the quotient property if for each MIA situation (R, N, E, C) and for all $k \in R$,

$$\sum_{i\in N} \Psi_{ki}(R, N, E, C) = \Psi_{k1}(R, \{1\}, E, c^R).$$

This property means that the total quantity assigned to an issue in a MIA situation is equal to the amount assigned to the same issue in the so-called quotient MIA situation.

Proposition 2.1 *The two-stage constrained equal awards rule, Ψ^{CEA}, satisfies the properties of composition, invariance under claims truncation, equal treatment for the agents within an issue, equal treatment for the issues, and the quotient property.*

Proof. We will only check that Ψ^{CEA} satisfies the properties of composition and invariance under claims truncation. The remaining properties follow immediately from the definitions. Let (R, N, E, C) be a MIA situation.

(a) Composition.

As composition is satisfied by CEA for bankruptcy problems, taking the MIA situation $(R, \{1\}, E, c^R)$, we have that for all $0 \leq E' \leq E$:

$$\Psi^{CEA}(R, \{1\}, E, c^R) = \Psi^{CEA}(R, \{1\}, E', c^R)$$

$$+\Psi^{CEA}(R, \{1\}, E - E', c^R - \Psi^{CEA}(R, \{1\}, E', c^R)).$$

Let $k \in R$. We apply composition again, with $0 \leq \Psi_k^{CEA}(R, \{1\}, E', c^R) \leq \Psi_k^{CEA}(R, \{1\}, E, c^R)$:

$$\Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E, c^R), C_k)$$

$$= \Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E', c^R), C_k)$$

$$+\Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E, c^R) - \Psi_k^{CEA}(R, \{1\}, E', c^R), C_k'),$$

where $C_k' = C_k - \Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E', c^R), C_k)$.

Therefore,

$$\Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E, c^R), C_k)$$

$$= \Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E', c^R), C_k)$$

$$+\Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E - E', c^R - \Psi^{CEA}(R, \{1\}, E', c^R)), C_k'),$$

where $C_k' = C_k - \Psi^{CEA}(\{k\}, N, \Psi_k^{CEA}(R, \{1\}, E', c^R), C_k)$.

Finally, using the definition of Ψ^{CEA} and gathering the results of all issues in a matrix, we obtain

$$\Psi^{CEA}(R, N, E, C) = \Psi^{CEA}(R, N, E', C)$$

$$+\Psi^{CEA}(R, N, E - E', C - \Psi^{CEA}(R, N, E', C)).$$

(b) Invariance under claims truncation.

Using the definition of Ψ^{CEA} and the fact that the *CEA* rule for bankruptcy problems satisfies invariance under claims truncation, we get for all $k \in R$ and

$i \in N$, $\Psi_{ki}^{CEA}(R,N,E,C^E) =$

$$\begin{aligned}
&= \Psi_{ki}^{CEA}(\{k\}, N, \Psi_{k1}^{CEA}(R,\{1\},E,(c^E)^R), C_k^E) \\
&= \Psi_{ki}^{CEA}(\{k\}, N, \Psi_{k1}^{CEA}(R,\{1\},E,((c^E)^R)^E), C_k^E) \\
&= \Psi_{ki}^{CEA}(\{k\}, N, \Psi_{k1}^{CEA}(R,\{1\},E,(c^R)^E), C_k^E) \\
&= \Psi_{ki}^{CEA}(\{k\}, N, \Psi_{k1}^{CEA}(R,\{1\},E,c^R), C_k^E) \\
&= \Psi_{ki}^{CEA}(\{k\}, N, \Psi_{k1}^{CEA}(R,\{1\},E,c^R), (C_k^E)^{\Psi_{k1}^{CEA}(R,\{1\},E,c^R)}) \\
&= \Psi_{ki}^{CEA}(\{k\}, N, \Psi_{k1}^{CEA}(R,\{1\},E,c^R), C_k^{\Psi_{k1}^{CEA}(R,\{1\},E,c^R)}) \\
&= \Psi_{ki}^{CEA}(\{k\}, N, \Psi_{k1}^{CEA}(R,\{1\},E,c^R), C_k) \\
&= \Psi_{ki}^{CEA}(R,N,E,C).
\end{aligned}$$

■

Theorem 2.1 *There is only one rule which satisfies the properties of composition, invariance under claims truncation, equal treatment for the agents within an issue, equal treatment for the issues, and the quotient property: the two-stage constrained equal awards rule* Ψ^{CEA}.

Proof. In view of Proposition 2.1, we only need to prove uniqueness. Let (R,N,E,C) be a MIA situation and let Ψ be a rule satisfying the five properties. We divide the proof into two parts.

In part I, we show that CEA should be applied in the quotient problem, i.e.,

$$\Psi(R,\{1\},E,c^R) = \Psi^{CEA}(R,\{1\},E,c^R). \tag{2.1}$$

In part II, we show that CEA should be used within each issue. Part I is a direct extension of the proof of Dagan (1996) and is given here in full to facilitate part II.

Part I.

We consider the corresponding quotient problem $(R,\{1\},E,c^R)$ and assume, without loss of generality, that $0 \leq c_1^R \leq c_2^R \leq \ldots \leq c_r^R$.

If $E \leq c_1^R := E^1$, because of the properties of invariance under claims truncation and equal treatment for the issues, the estate is split into equal parts and, as a result, (2.1) holds.

If $E^1 < E \leq c_1^R + c_1^R(1-\frac{1}{r}) := E^2$, the first case together with composition yields

$$\begin{aligned}\Psi(R,\{1\},E,c^R) &= \Psi^{CEA}(R,\{1\},E^1,c^R) + \\ &\quad +\Psi(R,\{1\},E-E^1,c^R-\Psi^{CEA}(R,\{1\},E^1,c^R))\end{aligned}$$

and $E - E^1 \leq c_1^R(1-\frac{1}{r}) = c_1^R - \Psi_{11}^{CEA}(R,\{1\},E^1,c^R)$. Again, because of invariance under claims truncation and equal treatment for the issues, we divide the amount $E - E^1$ into identical parts and hence

$$\Psi(R,\{1\},E-E^1,c^R-\Psi^{CEA}(R,\{1\},E^1,c^R)) =$$

$$= \Psi^{CEA}(R,\{1\},E-E^1,c^R-\Psi^{CEA}(R,\{1\},E^1,c^R))$$

and, given that Ψ^{CEA} satisfies composition, (2.1) holds. Continuing this procedure with $E^t := E^{t-1} + c_1^R(1-\frac{1}{r})^{t-1}$, $t \geq 2$, we obtain (2.1) for all $E < rc_1^R$. Furthermore, since a rule that satisfies composition is continuous in the estate, the statement also holds for $E = rc_1^R$.

The next step is to show (2.1) if $rc_1^R < E \leq rc_1^R + (r-1)(c_2^R - c_1^R)$. To show this, we repeat the previous procedure, with $rc_1^R + (c_2^R - c_1^R)$ taking the role of E^1, $rc_1^R + (c_2^R - c_1^R) + (c_2^R - c_1^R)(1-\frac{1}{r-1})$ taking the role of E^2, and so on. Using composition, we first divide rc_1^R according to the first step and next we divide the remainder equally among issues $\{2,\ldots,r\}$. We use the same limit argument as in the first step to obtain (2.1) for all $rc_1^R < E \leq rc_1^R + (r-1)(c_2^R - c_1^R)$.

We repeat this procedure, making one issue drop out in each step, until (2.1) is shown for all possible estates. Then, by the quotient property,

$$\sum_{i\in N} \Psi_{ki}(R,N,E,C) = \Psi_k^{CEA}(R,\{1\},E,c^R)$$

for all $k \in R$.

Part II.

Define $N_k = \{i \in N : c_{ki} > 0\}$ and $\rho_k = |N_k| \min_{i\in N_k} c_{ki}$ for all $k \in R$ and define $\rho = \min_{k\in R} \rho_k$. Without loss of generality, assume that claim c_{11} determines this minimum. Then $r\rho$ is the minimum estate to fully sustain claim c_{11} according to

Ψ^{CEA}. We first show that

$$\Psi(R,N,E,C) = \Psi^{CEA}(R,N,E,C) \tag{2.2}$$

if $E \leq r\rho$ and afterwards show it for $E > r\rho$.

1. In case $E \leq r\rho$, take $E^1 = \min_{k\in R} \min_{i\in N_k} c_{ki}$. Note that this minimum is not necessarily attained by c_{11}. First, suppose that it is.

   If $E \leq E^1$, then part I, invariance under claims truncation and equal treatment for the agents within an issue imply (2.2). Thus, $\Psi_{ki}(R,N,E,C) = \frac{E}{r|N_k|}$ for all $k \in R, i \in N$.

   Next, take $E^2 = E^1 + \min_{k\in R} \min_{i\in N_k} \left(c_{ki} - \Psi^{CEA}_{ki}(R,N,E^1,C)\right)$. If $E^1 < E \leq E^2$, then by composition and the previous step, we have

   $$\begin{aligned} \Psi(R,N,E,C) &= \Psi^{CEA}(R,N,E^1,C) + \\ &\quad +\Psi(R,N,E-E^1,C-\Psi^{CEA}(R,N,E^1,C)). \end{aligned}$$

   Then

   $$\begin{aligned} E - E^1 &\leq \min_{k\in R} \min_{i\in N_k} \left(c_{ki} - \Psi^{CEA}_{ki}(R,N,E^1,C)\right) \\ &= c_{11} - \Psi^{CEA}_{11}(R,N,E^1,C), \end{aligned}$$

   where the equality follows from the fact that the minimum locations for ρ and E^1 coincide. So, $E^2 = c_{11} + c_{11}(1 - \frac{1}{r|N_1|})$.

   Hence, we can continue the procedure as described in part I to show that (2.2) holds if $E \leq r|N_1|c_{11} = r\rho$.

   Next, suppose that the minimum locations for ρ and E^1 do not coincide. Then we start with the same procedure as in the previous case. The difference is that now we cannot conclude that in each step, the minimum remaining claim is in the same position. Two things can happen. The easy case occurs if after a finite number of steps the estate is smaller than all remaining claims. In this case, the procedure stops and we have (2.2) as a consequence of invariance under claims truncation, part I and equal treatment of the agents within an issue.

In the other case, we need to apply a similar limit argument as before. From the construction of ρ it follows that the remaining claim for agent 1 in issue 1 is the first to reach zero in our procedure, wherever the minimal remaining claim is located in each step. So, at some stage in the procedure, the minimal remaining claim must shift to the (1,1)-position and stay there. Note that it is possible that it first shifts to other positions, but this does not affect the argument. From this point on, we are faced with the easy situation with stationary minimum location as described above and we conclude, using the same limit argument, that (2.2) holds for all $E \leq r\rho$. In fact, the only difference with the previous situation is that, at the start of the procedure, the estate is divided at a different pace. This does not effect the total division in the limit.

2. If $E > r\rho$, we use composition to obtain

$$\begin{aligned}\Psi(R,N,E,C) &= \Psi^{CEA}(R,N,r\rho,C)+\\ &\quad +\Psi(R,N,E-r\rho,C-\Psi^{CEA}(R,N,r\rho,C)).\end{aligned}$$

Note that in the matrix of remaining claims, there is at least one more zero than in the original claims matrix. For these remaining claims, we again define N_k and ρ_k for all $k \in R$ and ρ in the same way and we reapply the procedure. The only thing that we have to take care of is that there might be an issue with only zero claims remaining. In this case, this issue is not taken into account when determining ρ. When the estate is allocated, the agents automatically receive zero in this issue because of reasonability of Ψ and the estate is divided among the issues according to part I.

Hence, in each step, we can apply the procedure to show that (2.2) holds for ever increasing estates. Since in each step at least one remaining claim becomes zero, we finish in a finite number of steps and conclude that (2.2) holds for all possible estates.

■

Remark 2.1 *The properties in Theorem 2.1 are independent.*

- The two-stage proportional rule (the two-stages rule with $\psi = P$) is not invariant under claims truncation but satisfies all the other properties mentioned in the theorem.

- The two-stage Talmud rule (the two-stages rule with $\psi = T$) does not satisfy the property of composition but it does the other properties.

- Let (R, N, E, C) be a multi-issue allocation situation and $\pi \in \Pi(N)$. We define a rule using the following procedure.

  First, we consider the quotient bankruptcy problem (R, E, c^R) and divide the amount E among the issues using the bankruptcy rule CEA. In this way, we obtain $CEA_k(R, E, c^R)$ for every $k \in R$.

  Second, for all $k \in R$, consider a new bankruptcy problem for the agents, $(N, CEA_k(R, E, c^R), C_k)$, and apply the bankruptcy rule f^π, defined for each bankruptcy problem (N, E, c) and $i \in N$ as

  $f_i^\pi(N, E, c) = \min\left\{\max\left\{0, E - \sum_{j:\pi(j)<\pi(i)} c_j\right\}, c_i\right\}.$

  In this way, we have that for all $i \in N$ and for all $k \in R$,

  $$\Psi_{ki}(R, N, E, C) = \min\left\{\max\left\{0, \min\{\lambda, c_k^R\} - \sum_{j:\pi(j)<\pi(i)} c_{kj}\right\}, c_{ki}\right\},$$

  with λ such that $\sum_{k \in R} \min\{\lambda, c_k^R\} = E$.

  This multi-issue allocation solution satisfies all the properties in Theorem 2.1, except equal treatment for the agents within an issue. We can easily check that this rule satisfies the properties of invariance under claims truncation, equal treatment for the issues, and the quotient property. To prove that this rule satisfies composition, it suffices to show that f satisfies composition in the context of bankruptcy problems. A similar argument as for Ψ^{CEA} can then be made to show that this MIA solution satisfies composition. Take a bankruptcy problem (N, E, c) and an estate E' with $0 \leq E' \leq E$. For every $i \in N$, we have that

  $$f_i^\pi(N, E - E', c - f^\pi(N, E', c))$$

$$= \min\left\{\max\left\{0, E - E' - \sum_{j,\pi(j)<\pi(i)} [c_j - f_j^\pi(N,E',c)]\right\},\right.$$
$$\left. c_i - f_i^\pi(N,E',c)\right\}$$

with

$$f_i^\pi(N,E',c) = \min\left\{\max\left\{0, E' - \sum_{j,\pi(j)<\pi(i)} c_j\right\}, c_i\right\}.$$

Moreover, we know that

$$\sum_{j,\pi(j)<\pi(i)} f_j^\pi(N,E',c)$$
$$= \begin{cases} E' & \text{if } E' - \sum_{j,\pi(j)<\pi(i)} c_j \leq 0, \\ \sum_{j,\pi(j)<\pi(i)} c_j & \text{if } E' - \sum_{j,\pi(j)<\pi(i)} c_j > 0, \end{cases}$$

and then,

$$\max\left\{0, E - E' - \sum_{j:\pi(j)<\pi(i)} [c_j - f_j^\pi(N,E',c)]\right\}$$
$$= \max\left\{0, E - \max\left\{E', \sum_{j,\pi(j)<\pi(i)} c_j\right\}\right\}$$
$$= \max\left\{0, E - \sum_{j,\pi(j)<\pi(i)} c_j\right\} - \max\left\{0, E' - \sum_{j,\pi(j)<\pi(i)} c_j\right\}.$$

Thus,

$$f_i^\pi(N, E-E', c - f^\pi(N,E',c))$$
$$= \min\left\{\max\left\{0, E - \sum_{j,\pi(j)<\pi(i)} c_j\right\}\right.$$

$$- \max\left\{0, E' - \sum_{j,\pi(j)<\pi(i)} c_j\right\}, c_i - f_i^\pi(N,E',c)\Bigg\}$$

$$= \min\left\{\max\left\{0, E - \sum_{j,\pi(j)<\pi(i)} c_j\right\} - f_i^\pi(N,E',c), c_i - f_i^\pi(N,E',c)\right\}$$

$$= \min\left\{\max\left\{0, E - \sum_{j,\pi(j)<\pi(i)} c_j\right\}, c_i\right\} - f_i^\pi(N,E',c)$$

$$= f_i^\pi(N,E,c) - f_i^\pi(N,E',c),$$

where the second equality follows from distinguishing between cases. Hence, we conclude that f^π satisfies composition.

- Let (R,N,E,C) be a multi-issue allocation situation and let $\tau \in \Pi(R)$. We define a rule using the following procedure. First, we consider the bankruptcy problem (R,E,c^R) and divide E among the issues using the bankruptcy rule f^τ. In this way, we obtain $f_k(R,E,c^R)$ for every $k \in R$. Second, for all $k \in R$, consider the new bankruptcy problem for the agents, $(N, f_k(R,E,c^R), C_k)$, and apply the bankruptcy rule CEA. In this way, we have that for all $i \in N$ and for all $k \in R$,

$$\Psi_{ki}(R,N,E,C) = \min\{\beta_k, c_{ki}\}$$

with β_k such that

$$\sum_{i\in N} \min\{\beta_k, c_{ki}\} = \min\left\{\max\left\{0, E - \sum_{\ell,\tau(\ell)<\tau(k)} c_\ell^R\right\}, c_k^R\right\}.$$

This rule satisfies all the properties in Theorem 2.1 except equal treatment for the issues.

- The rule given by $\Psi_{ki}(R,N,E,C) = \min\{\lambda, c_{ki}\}$ for every $i \in N$ and $k \in R$ with λ such that $\sum_{k\in R}\sum_{i\in N} \min\{\lambda, c_{ki}\} = E$ satisfies all the properties except the quotient property.

2.4.2 The constrained equal losses rule in two stages

In this case, the constrained equal losses rule in two stages takes into account the bankruptcy rule $\psi = CEL$. The two-stage extension Ψ^{CEL} is then given, for all (R, N, E, C) and all $k \in R$ and $i \in N$, by

$$\Psi_{ki}^{CEL}(R, N, E, C) = \max\{0, c_{ki} - \beta_k\},$$

where for all $k \in R$, β_k is such that $\sum_{i \in N} \max\{0, c_{ki} - \beta_k\} = \max\{0, c_k^R - \lambda\}$ and λ is such that $\sum_{k \in R} \max\{0, c_k^R - \lambda\} = E$.

We now mention some properties for MIA solutions, which we use to characterize the two-stage constrained equal losses rule.

Path independence. A MIA solution Ψ satisfies path independence if for each MIA situation (R, N, E, C) and for all $E' \in \mathbb{R}$ such that $E' \geq E$,

$$\Psi(R, N, E, C) = \Psi(R, N, E, \Psi(R, N, E', C)).$$

If a rule Ψ satisfies path independence, we can divide the estate using two procedures yielding the same result. The first procedure is to divide the money directly using Ψ. In the second procedure, we first divide a bigger estate $E' \geq E$ and then use the outcome $\Psi(N, R, E', C)$ as the matrix of claims to divide the real estate E, both times using Ψ.

Composition from minimal rights. A MIA solution Ψ satisfies composition of minimal rights if for each MIA situation (R, N, E, C)

$$\Psi(R, N, E, C) = m(R, N, E, C)$$
$$+\Psi\left(R, N, E - \sum_{k \in R} \sum_{i \in N} m_{ki}(R, N, E, C), C - m(R, N, E, C)\right),$$

where the minimum right of agent i in issue k is given by

$$m_{ki}(R, N, E, C) = \max\left\{0, E - \sum_{\ell \in R, j \in N, (\ell, j) \neq (k, i)} c_{\ell j}\right\}.$$

This property says that the agents in the issues will receive the same quantity according to the rule if they are firstly given their corresponding minimal rights and then the remainder part of the estate is divided taking into account the remainder demands. The minimal right of an agent in an issue is the part of the estate that is left after the remainder demands have been satisfied if it is positive and zero otherwise.

Duality. A MIA solution Ψ^* is the dual of another MIA solution Ψ if for each MIA situation (R, N, E, C)

$$\Psi^*(R, N, E, C) = C - \Psi\left(R, N, \sum_{k\in R}\sum_{i\in N} c_{ki} - E, C\right).$$

This property generalizes the property by Aumann and Maschler (1985) for bankruptcy problems and asserts that one rule is the dual of another one if it assigns to each agent in each issue what this agent demands minus what this agent obtains dividing the total losses according to the other rule. A rule Ψ is called self-dual if $\Psi^* = \Psi$.

The following lemma follows immediately from the observation that both CEA and CEL are dual rules for bankruptcy situations.

Lemma 2.1 $(\Psi^{CEA})^* = \Psi^{CEL}$.

The property P^* is the *dual property* of P if for all MIA solution Ψ, it satisfies property P if and only if the dual MIA solution Ψ^* satisfies property P^*. A property is called self-dual when it is dual of itself. The next two lemmas are extensions of results from Herrero and Villar (2001). The proofs are similar.

Lemma 2.2 *If the MIA solution Ψ is characterized by means of independent properties, then the dual MIA solution Ψ^* is characterized by the dual properties. Moreover, these dual properties are independent.*

Lemma 2.3

(a) *Composition and path independence are dual properties.*

(b) *Invariance under claims truncation and composition of minimal rights are dual properties.*

(c) *The quotient property, equal treatment for the agents within an issue, and equal treatment in the issues are self-dual.*

Using the previous lemmas and the characterization of the two-stage constrained equal awards rule, we characterize the constrained equal losses rule in the following theorem.

Theorem 2.2 *There is only one rule which satisfies the properties of path independence, composition of minimal rights, equal treatment for the agents within an issue, equal treatment for the issues, and the quotient property. This rule is the two-stage constrained equal losses rule Ψ^{CEL}.*

2.5 The run-to-the-bank rules with awards

In this section we extend the classical model of TU games to the TU games with awards. Moreover, we define a rule for this class of games, the run-to-the-bank rule with awards, and characterize it in terms of a property of balanced contributions. Finally, we apply this result to the cases of bankruptcy problems and MIA situations with awards.

A *TU game with awards* is a 3-tuple (N, v, μ), where (N, v) is a superadditive TU game and $\mu \in \mathbb{R}^F$ represents an award vector related to the coalition $F \subset N$. We assume that any solution must give the agents in F their predetermined award μ. Hence, we assume this award vector μ to satisfy $\sum\limits_{i \in F} \mu_i \leq v(N)$ and $\sum\limits_{i \in F} \mu_i = v(N)$ if $F = N$.

A *solution for TU games with awards*, G, is a function which associates with every TU game with awards (N, v, μ) a vector $G(N, v, \mu) \in \mathbb{R}^N$ such that $G_F(N, v, \mu) = \mu$.

Let (N, v, μ) be a TU game with awards. Moreover, assume that the grand coalition N forms and we want to distribute $v(N)$ among the players. For this purpose, we assume that the agents in F obtain their corresponding awards and we define for all $i \in N \setminus F$,

$$r_i(\mu) = v(N) - v(N \backslash (F \cup \{i\})) - \sum_{k \in F} \mu_k,$$

which is the contribution to the grand coalition of player i.

Note that in the case of $F = \emptyset$, the TU game with awards (N, v, μ) is a TU game and in this case $r_i(\mu) = v(N) - v(N \setminus \{i\})$. This quantity, the marginal contribution of agent i to the grand coalition, is usually called the utopia point of agent i (see Tijs (1981)).

Let $\mu^i \in \mathbb{R}^{F \cup \{i\}}$ be the extension such that $\mu^i_F = \mu$ and $\mu^i_i = r_i(\mu)$. For repeated extensions, we will use the notation $(\mu^i)^j = \mu^{i,j}$ and so on.

Consider a TU game with awards (N, v, μ), $\gamma \in \Pi(F)$, $\pi \in \Pi^{\gamma}(N) = \{\pi \in \Pi(N)$ such that $\pi^{-1}(q) = \gamma^{-1}(q),\ q = 1, \ldots, |F|\}$, and suppose that agents receive their allocation following the order given by π.

Then, we can define an allocation rule $\epsilon(\pi, \mu)$ inductively by
$\epsilon_F(\pi, \mu) = \mu$,

$\epsilon_{\pi^{-1}(|F|+1)}(\pi, \mu) = r_{\pi^{-1}(|F|+1)}(\mu)$, and, for all $p = 2, \cdots, |\ N \setminus F\ |$,

$$\begin{aligned}
\epsilon_{\pi^{-1}(|F|+p)}(\pi, \mu) &= r_{\pi^{-1}(|F|+p)}(\mu^{\pi^{-1}(|F|+1), \cdots, \pi^{-1}(|F|+p-1)}) \\
&= v(N) - \sum_{k \in F} \mu_k - \sum_{i=\pi^{-1}(|F|+1)}^{\pi^{-1}(|F|+p-1)} \epsilon_i(\pi, \mu) - v\left(\bigcup_{i=\pi^{-1}(|F|+p+1)}^{\pi^{-1}(|N|)} \{i\} \right).
\end{aligned}$$

Note that with this notation $\mu_i^{\pi^{-1}(1), \cdots, \pi^{-1}(|N|)} = \epsilon_i(\mu, \pi)$.

The interpretation is that agents arrive in order with awarded agents in front. When an agent $i = \pi^{-1}(p) \in N \setminus F$ arrives, he obtains the most he could get, taking into account that the quantity $\sum\limits_{k \in F} \mu_k + \sum\limits_{j=\pi^{-1}(|F|+1)}^{\pi^{-1}(|F|+p-1)} \epsilon_j(\pi, \mu)$ has been already allocated and the remaining agents can guarantee $v\left(\bigcup\limits_{j=\pi^{-1}(|F|+p+1)}^{\pi^{-1}(|N|)} \{j\} \right)$.

To define the run-to-the-bank rule, we take the average over all the possible orders. Given a TU game with awards (N, v, μ), the *run-to-the-bank rule with awards* ,ϵ, is the solution for TU games with awards defined by

$$\epsilon(N, v, \mu) \equiv \epsilon(\mu) = \frac{1}{|\ N \backslash F\ |!} \sum_{\pi \in \Pi^{\gamma}(N)} \epsilon(\pi, \mu).$$

In the next lemma, we prove that the run-to-the-bank rule with awards of a TU game with awards coincides with the Shapley value of an associated TU game. Note that the Shapley value of a TU game (N, v) is defined for all $i \in N$ as $Sh_i(N, v) = \frac{1}{|N|!} \sum\limits_{\pi \in \Pi(N)} [v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi))]$, with $Pre(i, \pi) = \{j \in N, \pi(j) < \pi(i)\}$.

Lemma 2.4 *Let (N, v, μ) be a TU game with awards. We have that*

$$\epsilon(\mu) = Sh(N \setminus F, w),$$

where $(N \setminus F, w)$ *is the TU game defined by*

$$w(S) = \begin{cases} v(S) & \text{if } S \subsetneq N \backslash F \\ v(N) - \sum\limits_{k \in F} \mu_k & \text{if } S = N \backslash F \end{cases}$$

Proof.

Given $i \in N \backslash F$ and $\pi \in \Pi^{\gamma}(N)$, we define the order $\alpha \in \Pi(N \setminus F)$ by $\alpha^{-1}(p) = \pi^{-1}(|N| - p + 1)$ for all $p \in \{1, \ldots, |N \backslash F|\}$.

Take $i \in N \backslash F$. We distinguish two cases:

1. $\pi(i) = |F|+1$.

$$\begin{aligned} \epsilon_i(\pi, \mu) &= r_i(\mu) = v(N) - v(N \backslash (F \cup \{i\})) - \sum_{k \in F} \mu_k \\ &= w(N \setminus F) - w(\{\pi^{-1}(|F| + 2), \ldots, \pi^{-1}(|N|)\}) \\ &= w(\{\alpha^{-1}(1), \ldots, \alpha^{-1}(|N \backslash F|)\}) \\ &- w(\{\alpha^{-1}(1), \ldots, \alpha^{-1}(|N \backslash F| - 1)\}) \\ &= w(Pre(i, \alpha) \cup \{i\}) - w(Pre(i, \alpha)). \end{aligned}$$

2. $\pi(i) \neq |F|+1$, i.e., there exists $s \in \{2, \ldots, |N \backslash F|\}$ such that $\pi(i) = |F|+s$.

$$\begin{aligned} \epsilon_i(\pi, \mu) &= r_i(\mu^{\pi^{-1}(|F|+1), \ldots, \pi^{-1}(|F|+s-1)}) \\ &= v(N) - \sum_{k \in F} \mu_k - \sum_{j=\pi^{-1}(|F|+1)}^{\pi^{-1}(|F|+s-1)} \epsilon_j(\pi, \mu) - v(\cup_{j=\pi^{-1}(|F|+s+1)}^{\pi^{-1}(|N|)} \{j\}) \\ &= v(\{\pi^{-1}(|F| + s), \ldots, \pi^{-1}(|N|)\}) \\ &- v(\{\pi^{-1}(|F| + s + 1), \ldots, \pi^{-1}(|N|)\}) \\ &= w(\{\pi^{-1}(|F| + s), \ldots, \pi^{-1}(|N|)\}) \\ &- w(\{\pi^{-1}(|F| + s + 1), \ldots, \pi^{-1}(|N|)\}) \\ &= w(\{\alpha^{-1}(1), \ldots, \alpha^{-1}(|N \backslash F| - s + 1)\}) \\ &- w(\{\alpha^{-1}(1), \ldots, \alpha^{-1}(|N \backslash F| - s)\}) \\ &= w(Pre(i, \alpha) \cup \{i\}) - w(Pre(i, \alpha)). \end{aligned}$$

Thus,

$$\begin{aligned}\epsilon_i(\mu) &= \frac{1}{|N \setminus F|!}\sum_{\pi\in\Pi^{\gamma}(N)}\epsilon_i(\pi,\mu)\\ &= \frac{1}{|N \setminus F|!}\sum_{\alpha\in\Pi(N\setminus F)}[w(Pre(i,\alpha)\cup\{i\}) - w(Pre(i,\alpha))]\\ &= Sh_i(N\setminus F, w).\end{aligned}$$

■

Now, we introduce the two properties of solutions for TU games with awards we will use to characterize the run-to-the-bank rule.

Efficiency. A solution for TU games with awards, G, satisfies efficiency if for all TU game with awards (N, v, μ), $\sum_{i\in N} G_i(N, v, \mu) = v(N)$.

Balanced contributions. A solution for TU games with awards, G, satisfies balanced contributions if for all TU game with awards (N, v, μ) and for all $i, j \in N\backslash F$, we have that

$$G_i(N, v, \mu) - G_i(N, v, \mu^j) = G_j(N, v, \mu) - G_j(N, v, \mu^i),$$

where, for all $l \in N\backslash F$, $\mu^l \in \mathbb{R}^{F\cup\{l\}}$ is such that $\mu^l_F = \mu$ and $\mu^l_l = v(N) - \sum_{k\in F}\mu_k - v(N\backslash\{F\cup\{l\}\})$.

This property says that the loss or gain for agent i when agent j receives the quantity he can guarantee himself and becomes a member in the coalition related to the award vector is the same as the loss or gain for agent j when agent i receives the quantity he can guarantee himself and becomes a member in the coalition related to the awards vector.

Theorem 2.3 *The run-to-the-bank rule with awards is the unique solution for TU games with awards that satisfies efficiency and balanced contributions.*

Proof.

Existence.

Let (N, v, μ) be a TU game with awards and consider $\gamma \in \Pi(F)$. Taking into account Lemma 2.4, the definition of the associated TU game $(N\setminus F, w)$, and the fact that the Shapley value is efficient on the class of TU games, we obtain that the run-to-the-bank rule is efficient.

Now, we show that the rule satisfies balanced contributions. Myerson (1980) proved that in TU games the Shapley value satisfies a property of balanced contributions. Applying this result to w, we obtain that for all $i, j \in N \setminus F$ we have

$$Sh_i(N \setminus F, w) - Sh_i(N \setminus (F \cup \{j\}), w) = Sh_j(N \setminus F, w) - Sh_j(N \setminus (F \cup \{i\}), w).$$

In the same way, as in Lemma 2.4, using the definition of μ^j,

$$\epsilon_i(\mu^j) = Sh_i(N \setminus (F \cup \{j\}), w) \text{ for all } i,\, j \in N \setminus F.$$

Finally, as a result of Lemma 2.4, for all $i, j \in N \setminus F$ we have

$$\epsilon_i(\mu) - \epsilon_i(\mu^j) = Sh_i(N \setminus F, w) - Sh_i(N \setminus (F \cup \{j\}), w)$$

$$= Sh_j(N \setminus F, w) - Sh_j(N \setminus (F \cup \{i\}), w) = \epsilon_j(\mu) - \epsilon_j(\mu^i).$$

Uniqueness.

We show uniqueness by induction on the size of F. Suppose that G^1 and G^2 are two solutions for TU games with awards satisfying efficiency and balanced contributions, and take a TU game with awards (N, v, μ).

If $F = N$, by the definition of MIA solution with awards, $G^1(N, v, \mu) = \mu = G^2(N, v, \mu)$.

If $|F| = |N| - 1$, on account of the definition of the solution for TU games with awards, we know that $G^1_k(N, v, \mu) = \mu_k = G^2_k(N, v, \mu)$ for all $k \in F$. In view of the fact that G_1 and G_2 satisfy efficiency, we conclude that in this case $G^1(N, v, \mu) = G^2(N, v, \mu)$.

Let $t \in \{0, \ldots, |N| - 2\}$ and assume that $G^1(N, v, \mu) = G^2(N, v, \mu)$ when $|F| = t + 1$.

Consider now that $|F| = t$ and let $i, j \in N \backslash F$. Then, by balanced contributions, we have

$$G^1_i(N, v, \mu) - G^1_j(N, v, \mu) = G^1_i(N, v, \mu^j) - G^1_j(N, v, \mu^i)$$

$$= G^2_i(N, v, \mu^j) - G^2_j(N, v, \mu^i) = G^2_i(N, v, \mu) - G^2_j(N, v, \mu),$$

where the second equality follows from the induction hypothesis.

Due to the definition of a solution for TU games with awards,

$$G_k^1(N, v, \mu) = \mu_k = G_k^2(N, v, \mu) \text{ for all } k \in F.$$

Thus, by efficiency, we have that

$$\sum_{k \in N \backslash F} G_k^1(N, v, \mu) = v(N) - \sum_{k \in F} \mu_k = \sum_{k \in N \backslash F} G_k^2(N, v, \mu),$$

and, hence, we have that $G^1 = G^2$. ■

2.5.1 Application to bankruptcy problems

In Bergantiños and Méndez-Naya (1999), a characterization of the random arrival rule was obtained with a property of balanced contributions. This result can be seen as a particular case of Theorem 2.3.

A bankruptcy rule ψ satisfies *balanced contributions* if for all bankruptcy problem (N, E, c) and for all $i, j \in N$, we have that

$$\psi_i(N, E, c) - \psi_i(N \backslash \{j\}, E_{-j}, c_{-j}) = \psi_j(N, E, c) - \psi_j(N \backslash \{i\}, E_{-i}, c_{-i}),$$

where for all $i \in N$, $E_{-i} = \max\{E - c_i, 0\}$ and $c_{-i} \in \mathbb{R}^{N \backslash \{i\}}$ is the vector of claims given by $(c_{-i})_k = c_k$ for all $k \in N \backslash \{i\}$.

This property says that the loss or gain for agent i when agent j leaves the problem and receives his claim is the same as the loss or gain for agent i when agent j receives his claim and leaves the problem.

Theorem 2.4 *The random arrival rule is the unique rule for bankruptcy problems that satisfies balanced contributions.*

Proof. If (N, E, c) is a bankruptcy problem, O'Neill (1982) associates with this problem the TU game $(N, v^{E,c})$ where $v^{E,c}(S) = \max \left\{ E - \sum_{j \in N \backslash S} c_j, 0 \right\}$ for all $S \subset N$. If we apply Theorem 2.3 to the game $(N, v^{E,c})$ and take $F = \emptyset$, we obtain the result. ■

2.5.2 Application to MIA situations

In Calleja et al. (2005), the run-to-the-bank rule for MIA situations was defined. Moreover, they characterized this solution by means of a property of consistency. We will characterize this solution with the property of balanced contributions.

The idea behind balanced contributions is to compare reduced situations, in which one of the agents has been "sent away" with a particular payoff. But, in the case of MIA situations, one cannot "send away" an agent with a payoff, since it is unclear what the claims matrix in the reduced situation should be. Simply removing this agent from the claims matrix does not work, because this ignores the interdependence of the issues. Then, in order to accommodate the idea of balanced contributions to the MIA situations, we use the same extension of the domain that Calleja et al. (2005) used for their characterization of the run-to-the-bank rules using consistency. Next, we give the definition.

A *MIA situation with awards* (Calleja et al. (2005)) is a 5-tuple (R, N, E, C, μ) where (R, N, E, C) is a MIA situation and $\mu \in \mathbb{R}^F$ represents an award vector related to the coalition $F \subset N$. The idea is that all agents are still part of the game, but any solution must give the agents in F their predetermined award μ. Hence, we assume this award vector μ to satisfy $\sum_{i \in F} \mu_i \leq E$ and $\sum_{i \in F} \mu_i = E$ if $F = N$.

Note that a MIA situation is a MIA situation with awards with $F = \emptyset$. So, indeed, introducing awards extends the domain and any characterization of a rule on the class of MIA situations with awards uniquely determines the restriction of this rule on the class of MIA situations with awards.

A *MIA solution with awards* Ψ is a function which associates with every MIA situation with awards (R, N, E, C, μ) a vector $\Psi(R, N, E, C, \mu) \in \mathbb{R}^N$ such that $\Psi_i(N, R, E, C, \mu) = \mu_i$ for all $i \in F$ and $\sum_{i \in N} \Psi_i(N, R, E, C, \mu) = E$.

Let (R, N, E, C) be a MIA situation and consider an order on the issues $\tau \in \Pi(R)$. We denote by $c_{k,S} = \sum_{i \in S} c_{ki}$ the total of claims of coalition $S \subset N$ according to issue $k \in R$. If $S = N$, we will consider c_k instead of $c_{k,N}$.

Suppose that only the first t issues in the order τ can be fully satisfied, with $t = \max\left\{ t' \mid \sum_{s=1}^{t'} c_{\tau^{-1}(s)} \leq E \right\}$, and let $E' = E - \sum_{s=1}^{t} c_{\tau^{-1}(s)}$ be the remaining estate.

Next, this remaining estate E' is distributed among the agents in the issue in the position $t+1$, $\tau^{-1}(t+1)$, according to the order $\pi \in \Pi(N)$. Thus, only the first q agents obtain their total claim, where $q = \max \left\{ q' \mid \sum_{p=1}^{q'} c_{\tau^{-1}(t+1)\pi^{-1}(p)} \leq E' \right\}$.

Let us consider the function

$$g(S, \tau^{-1}(t+1), \pi, E') = \begin{cases} E' - \sum\limits_{\substack{p=1 \\ \pi^{-1}(p) \in N \setminus S}}^{q} c_{\tau^{-1}(t+1)\pi^{-1}(p)} & \pi^{-1}(q+1) \in S \\ \\ \sum\limits_{\substack{p=1 \\ \pi^{-1}(p) \in S}}^{q} c_{\tau^{-1}(t+1)\pi^{-1}(p)} & \pi^{-1}(q+1) \notin S, \end{cases}$$

which describes the amount that the agents in $S \subset N$ obtain according to issue $\tau^{-1}(t+1)$ if the order on the agents is $\pi \in \Pi(N)$ and the remaining estate is E'.

Given the orders τ and π, the total payoff to coalition $S \subset N$ is given by

$$f_S(\pi, \tau) = \sum_{s=1}^{t} c_{\tau^{-1}(s),S} + g(S, \tau^{-1}(t+1), \pi, E').$$

The corresponding MIA game $(N, v^{(E,C)})$ is defined by

$$\begin{aligned} v^{(E,C)}(S) &= \min_{\tau \in \Pi(R)} \min_{\pi \in \Pi(N)} f_S(\pi, \tau) = E - \max_{\tau \in \Pi(R)} \max_{\pi \in \Pi(N)} f_{N \setminus S}(\pi, \tau) \\ &= E - \max_{\tau \in \Pi(R)} f_{N \setminus S}(\tau), \end{aligned}$$

where $f_{N\setminus S}(\tau) = f_{N\setminus S}(\hat{\pi}, \tau) = \sum_{s=1}^{t} c_{\tau^{-1}(s),N\setminus S} + \min\{c_{\tau^{-1}(t+1),N\setminus S}, E'\}$ and $\hat{\pi} \in \Pi(N)$ is such that $\hat{\pi}^{-1}(N \setminus S) = \{1, \ldots, |N \setminus S|\}$.

This game assigns to each coalition $S \subset N$ the quantity which is left after coalition $N \setminus S$ gets the maximal payoff by choosing an order in the issues and an order in the agents. An optimal order for the agents in $N \setminus S$ obviously puts them at the front of the queue.

Let (R, N, E, C, μ) be a MIA situation with awards and take $\gamma \in \Pi(F)$. The *run-to-the-bank rule with awards (ρ)* is the MIA solution with awards defined by

$$\rho(\mu) = \frac{1}{|N \setminus F|!} \sum_{\pi \in \Pi^{\gamma}(N)} \rho(\pi, \mu),$$

with $\Pi^{\gamma}(N) = \{\pi \in \Pi(N) \,|\, \forall q \in \{1, \ldots, |F|\} : \pi^{-1}(q) = \gamma^{-1}(q)\}$.

For all $\pi \in \Pi^{\gamma}(N)$, $\rho(\pi, \mu) \in \mathbb{R}^N$ is defined by $\rho_F(\pi, \mu) = \mu$ and, for the agents in $N \setminus F$, is defined recursively by

$$\rho_{\pi^{-1}(p)}(\pi, \mu) = \max_{\tau \in \Pi(R)} \left\{ f_{\pi^{-1}(p)}(\pi, \tau) - \sum_{q=1}^{p-1} \left[\rho_{\pi^{-1}(q)}(\pi, \mu) - f_{\pi^{-1}(q)}(\pi, \tau) \right] \right\}$$

for all $p \in \{1, \ldots, |N|\}$ with $\pi^{-1}(p) \notin F$.

The vector $\rho_{\pi^{-1}(p)}(\pi, \mu)$ is interpreted as follows. Firstly, all the agents in F receive their awards and get a position at the front of the order π. Then, each agent in $N \setminus F$ receives the maximal payoff by choosing an order on the issues, keeping in mind that he has to compensate all the preceding agents in the order π for the difference between the assignment they have received and what they receive when the order on the issues is the order that the agent chooses.

If $F = \emptyset$, then there are no fixed agents to put at the front of the queue and the definition boils down to the run-to-the-bank rule for MIA situations. Next, we provide a characterization of the rule on the wider domain of MIA situations with awards, which of course uniquely determines the run-to-the-bank rule for MIA situations without awards as well.

A MIA solution with awards Ψ satisfies *balanced contributions* if for all MIA situations with awards (R, N, E, C, μ) and for all $i, j \in N \setminus F$ we have that

$$\Psi_i(R, N, E, C, \mu) - \Psi_i(R, N, E, C, \mu^j) = \Psi_j(R, N, E, C, \mu) - \Psi_j(R, N, E, C, \mu^i),$$

where for all $\ell \in N \setminus F$, $\mu^\ell \in \mathbb{R}^{F \cup \{\ell\}}$ is such that $\mu^\ell_F = \mu$ and

$$\mu^\ell_\ell = \max_{\tau \in \Pi(R)} \left[f_{F \cup \{\ell\}}(\pi, \tau) - \sum_{k \in F} \mu_k \right] = \max_{\tau \in \Pi(R)} f_{F \cup \{\ell\}}(\pi, \tau) - \sum_{k \in F} \mu_k,$$

with $\pi \in \Pi^{\gamma}(N)$ for arbitrary $\gamma \in \Pi(F)$ such that $\pi^{-1}(|F| + 1) = \ell$.

This property has a similar flavour to the the property of balanced contributions of solutions for TU games with awards.

Theorem 2.5 *The run-to-the-bank rule with awards is the unique MIA solution with awards that satisfies balanced contributions.*

We omit the proof because it is enough to take the MIA game $(N, v^{(E,C)})$ introduced before and apply Theorem 2.3 to this game.

2.6 Concluding remarks

This chapter is mainly devoted to the study of multi-issue allocation situations. We define and characterize two different solutions for these problems based on two well-known bankruptcy rules. Moreover, we introduce the concept of cooperative games with awards and define and characterize a solution for these games. This result can be applied to both bankruptcy problems and multi-issue allocation situations.

The two multi-issue allocation solutions we define and characterize in this chapter, the CEA and CEL rules in two stages, are defined following a procedure in two stages: firstly, we use a bankruptcy rule in the issues and then, we use the same bankruptcy rule in the agents for each issue. It could be interesting to study more multi-issue allocation solutions defined in the same way by using other bankruptcy rules in the procedure. Another idea that it could be even more interesting is to study multi-issue allocation solutions defined using a procedure similar but where the bankruptcy rule used in each stage can be different.

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Chapter 3

Power indices

3.1 Introduction

One of the most important and suggestive elements in Political Science is power. Even though there is not a total consensus about the definition of power, we can understand political power of a member in a committee as the ability to influence outcomes according to his preferences. A power index gives a measure of this power, but power is such an extremely difficult concept to measure, that no agreement has been reached concerning the choice of a power index. In fact, in many papers, the choice of the power index is not theoretically justified and several power indices are employed.

The main power indices that we can find in the literature are the Shapley-Shubik index (Shapley and Shubik, 1954), the Banzhaf index (Banzhaf, 1965), the Johnston index (Johnston, 1978), the Deegan-Packel index (Deegan and Packel, 1979), and the Public Good Index (Holler, 1982). These measures of voting power are based on an evaluation of an actor's relative importance to coalition formation. Permutations of players play a decisive role in the calculation of Shapley-Shubik index, whereas other indices completely overlook permutations concentrating exclusively on groups of players.

Simple games can be used to model voting procedures. In such games, we say that a winning coalition is vulnerable when it has at least one member whose removal would cause the resulting coalition to be a losing coalition. An agent is considered critical when his elimination from a winning coalition turns this coalition into a losing coalition. A minimal winning coalition is a winning coalition such that all its members are critical.

In Banzhaf's model, the power of one agent is proportional to the number of coalitions in which he is critical. Johnston argued that the Banzhaf index, which is based on the idea of the removal of a critical voter from a winning coalition, does not take into account the total number of critical members in each coalition. Clearly, if a voter is the only critical agent in a coalition, this gives a stronger sign of power than in the case where all the agents are critical. This is the main idea underlying the Johnston index.

According to Deegan and Packel and Public Good indices, only minimal winning coalitions should be considered in establishing the power of a voter. Deegan and Packel proposed an index under the assumptions that all minimal winning coalitions are equiprobable and all the voters belonging to the same minimal winning coalition should obtain the same power. On the other hand, the Public Good Index is determined by the number of minimal winning coalitions containing the voter divided by the sum of such numbers across all the voters.

In Lorenzo-Freire et al. (2005), we do not discuss which is the best index. However, to facilitate the choice of a power index, some desirable properties have been introduced in the context of power indices. In Lorenzo-Freire et al. (2005), we mention some of these properties, as well as some characterizations of the main power indices according to them. Moreover, we give new characterizations of the Deegan-Packel and Johnston indices by means of monotonicity and mergeability properties.

One of the main difficulties with these indices is that computation generally requires the sum of a very large number of terms. Owen (1972) defined the multilinear extension of a game. It gives the expected utility of a random coalition. The multilinear extension has been used by Owen to compute the Shapley value (Shapley, 1953) and the Banzhaf value (Owen, 1975). Both values are probabilistic values (Weber, 1988), i.e., values that satisfy the property of additivity. The multilinear extensions are useful for computing the power of large games such as the Presidential Election Game and the Electoral College Game studied by Owen (1972). The multilinear extension approach has two advantages: thanks to its probabilistic interpretation, the central limit theorem of probability can be used, and, further, it is applied to composition of games.

The main objective of Alonso-Meijide et al. (2006) is to analyze whether some modification of the multilinear extension technique might be used to calculate the indices of Johnston, Deegan-Packel, and Public Good. As far as we know, this is the first time that the multilinear extension is applied to values which are not probabilistic values. These three indices have definitions that use vulnerable coalitions (in the case of Johnston index) or minimal winning coalitions (in the case of Deegan-Packel index and Public Good Index). Sometimes, however, we do not know a priori which coalitions are vulnerable or minimal winning, even though we know the game, specially in games with a large number of players. Obviously, in such a case it is very difficult to compute these indices. The advantage of our procedures is that if we know the multilinear extension of the game, we can provide an algorithm to easily compute Johnston index, Deegan-Packel index, and the Public Good Index.

In Section 2 we introduce some concepts for TU games. Section 3 is devoted to a review of the main power indices in simple games and its axiomatic characterizations. In Section 4 we give new characterizations for the Johnston, Deegan-Packel, and Public Good indices. In Section 5 we introduce the procedures to calculate the Johnston index, the Deegan-Packel index, and the Public Good Index by means of the multilinear extensions and, in Section 6, we apply them to some political examples: the Basque Country Parliament emerged from elections in April 2005 and the Victoria Proposal for Amendments to the Canadian Constitution.

3.2 Concepts for TU games

In this section we define some concepts in the context of TU games as well as several solutions for these games. Let us remember that a TU game is a pair (N, v) where N is the set of players and v is a function which assigns to every coalition $S \subset N$ the value of the coalition when its members cooperate, given by $v(S)$. Moreover, we assume that $v(\emptyset) = 0$. We will denote the class of TU games for N as $TU(N)$.

A *null player* in a TU game (N, v) is a player $i \in N$ such that $v(S \cup \{i\}) = v(S)$ for all $S \subset N \setminus \{i\}$.

Two players $i, j \in N$ are *symmetric* in a TU game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$.

Given a family $H \subset TU(N)$, a *solution* on H is a function f that assigns to a game $(N, v) \in H$ a vector $(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^N$, where the real number $f_i(N, v)$ is the payoff of the player i in the game (N, v) according to f.

To select a solution, we can take into account desirable properties. Next, we list out some of them.

Null player. A solution on H f satisfies the null player property if $f_i(N, v) = 0$ for all $(N, v) \in H$ and every null player $i \in N$.

Symmetry. A solution on H f is symmetric if $f_i(N, v) = f_j(N, v)$ for all $(N, v) \in H$ and for every pair of symmetric players $i, j \in N$.

Efficiency. A solution on H f is efficient if $\sum\limits_{i \in N} f_i(N, v) = v(N)$ for all $(N, v) \in H$.

Total power. A solution on H f satisfies the total power property if $\sum\limits_{i \in N} f_i(N, v) = \frac{1}{2^{n-1}} \sum\limits_{i=1}^{n} \sum\limits_{S \subset N \setminus \{i\}} [v(S \cup \{i\}) - v(S)]$ for all $(N, v) \in H$.

Additivity. A solution f is additive if $f(N, v+w) = f(N, v) + f(N, w)$ for every pair of games $(N, v), (N, w) \in H$ such that $(N, v + w) \in H$, where $(v + w)(S) = v(S) + w(S)$ for every $S \subset N$.

Strong monotonicity. A solution f satisfies the strong monotonicity property if $f_i(N, v) \geq f_i(N, w)$ for every pair of games $(N, v), (N, w) \in H$ and for all $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subset N \setminus \{i\}$.

Well-known solutions for TU games are the Shapley value (Shapley, 1953) and the Banzhaf value (Owen, 1975).

The Shapley value. Given a TU game (N, v), the Shapley value assigns to each player $i \in N$ the real number

$$Sh_i(N, v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n - s - 1)!}{n!} [v(S \cup \{i\}) - v(S)].$$

Note that the Shapley value for a TU game has already been defined in Chapters 1 and 2 by taking into account all the possible orders of the players. Both definitions lead to the same value.

The Banzhaf value. Given a TU game (N, v), the Banzhaf value assigns to each player $i \in N$ the real number

$$B_i(N,v) = \frac{1}{2^{n-1}} \sum_{S \subset N \setminus \{i\}} [v(S \cup \{i\}) - v(S)].$$

Shapley (1953) characterized the Shapley value and Feltkamp (1995) the Banzhaf value using some of the previous properties. Only one property differentiates both characterizations; the Shapley value satisfies efficiency meanwhile the Banzhaf value satisfies total power.

- The only solution for TU games that satisfies additivity, null player, symmetry, and efficiency is the Shapley value.
- The only solution for TU games that satisfies additivity, null player, symmetry, and total power is the Banzhaf value.

Using the strong monotonicity property, Young (1985) proposed another characterization of the Shapley value.

- The unique solution f defined on $TU(N)$ that satisfies strong monotonicity, symmetry, and efficiency is the Shapley value.

Similarly to Feltkamp, in Alonso-Meijide (2002) and in Lorenzo-Freire et al. (2005), a new characterization of the Banzhaf value is presented.

Theorem 3.1 *The unique solution f defined on $TU(N)$ that satisfies strong monotonicity, symmetry, and total power is the Banzhaf value.*

The proof immediately follows from a similar reasoning to that found in Young (1985).

3.3 Power indices for simple games

A *simple game* is a TU game (N, v) such that
(a) $v(S) = 0$ or $v(S) = 1$ for all $S \subset N$.
(b) v is a monotone function, that is, $v(S) \leq v(T)$ for all $S \subset T \subset N$.
(c) $v(N) = 1$.
We will denote the class of simple games with set of players N by $SI(N)$.

Given a simple game (N, v), a coalition $S \subset N$ is *winning* if $v(S) = 1$ and *losing* if $v(S) = 0$. We will consider $W(v)$ as the set of winning coalitions in this game, i.e., $W(v) = \{S \subset N : v(S) = 1\}$. A simple game can also be interpreted as a pair (N, W), where N is a coalition and W is the set of winning coalitions.

A winning coalition $S \subset N$ is a *minimal winning* coalition if $v(T) = 0$ for any $T \subsetneq S$. We will denote by $M(v)$ the set of minimal winning coalitions in the game (N, v) and by $M_i(v)$ the set of minimal winning coalitions player $i \in N$ belongs to, that is, $M_i(v) = \{S \in M(v) : i \in S\}$. A simple game can be interpreted not only as a pair (N, W) but also as a pair (N, M) where N is a coalition and M is the set of minimal winning coalitions.

We say that a winning coalition $S \subset N$ is a *quasi-minimal winning* coalition (or vulnerable coalition) if there exists a player $i \in S$ such that T is a losing coalition, for every coalition $T \subset S \setminus \{i\}$. For instance, a minimal winning coalition is a quasi-minimal winning coalition. We denote by $G(v)$ the set of quasi-minimal winning coalitions of the simple game (N, v). In the same way, we can identify the simple game with the set of the quasi-minimal winning coalitions.

It is clear that for every simple game (N, v),

$$M(v) \subset G(v) \subset W(v).$$

Given a simple game (N, v), a *swing* for a player $i \in N$ is a coalition $S \subset N$ such that $i \in S$, S is a winning coalition, and T is a losing one, for every coalition $T \subset S \setminus \{i\}$. We denote by $\eta_i(v)$ the set of swings for player $i \in N$. A winning coalition $S \subset N$ is a minimal winning coalition if and only if $S \in \eta_i(v)$ for every $i \in S$. A winning coalition $S \subset N$ is a quasi-minimal winning coalition if $S \in \eta_i(v)$ for some player $i \in S$.

Given $S \subset N$, we denote by $\chi(S)$ the set of *critical players* of S (we will name it *a critical coalition*), which is the set of players i of $S \subset N$ such that S is a swing for i. We denote by $G(S, v)$ the set of quasi-minimal coalitions such that the set of players $S \subset N$ are critical, that is to say, the set of quasi-minimal winning coalitions $T \subset N$, such that $\chi(T) = S$. It is important to point out that $\chi(S) = S$ means that S belongs to $M(v)$. For every $i \in N$, we denote by $G_i(v)$ the subset of $G(v)$ which consists of the coalitions $S \subset N$ such that $i \in \chi(S)$.

Next example explains these concepts.

Example 3.1 *Consider the simple game* (N, v)*, where* $N = \{1, 2, 3, 4\}$ *and*

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = v(\{2,3\}) = v(\{2,4\}) = v(\{3,4\}) = 0$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = v(\{1,2,3\}) = v(\{1,2,4\})$$
$$= v(\{1,3,4\}) = v(\{2,3,4\}) = v(N) = 1.$$

The set of winning coalitions is:

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,2,3,4\}\}.$$

For this simple game, the set of minimal winning coalitions is:

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\},$$

and the set of quasi-minimal winning coalitions is:

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}.$$

Thus, for this game the set of minimal winning coalitions is a strict subset of the set of quasi-minimal winning coalitions, and this is a strict subset of the set of winning coalitions.

The swings for player 1 are:

$$\eta_1(v) = \{\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}.$$

The subset of critical players of the coalition $\{1,2,3\}$ *is* $\chi(\{1,2,3\}) = \{1\}$ *and, given the coalition* $\{1\}$*, the set of quasi-minimal coalitions such that this coalition is critical, is* $G(\{1\}, v) = G_1(v) = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$*.*

Given a family of games $H \subset SI\,(N)$, a *power index* on H is a function f, which assigns to a simple game $(N, v) \in H$ a vector $(f_1\,(N, v), \ldots, f_n\,(N, v)) \in \mathbb{R}^N$, where the real number $f_i\,(N, v)$ is the "power" of the player i in the game (N, v) according to f. The power index of a simple game can be interpreted as a measure of the ability of the different players to turn a losing coalition into a winning one.

We consider the next power indices in $SI\,(N)$: the Shapley-Shubik index, the Banzhaf index, the Deegan-Packel index, the Public Good Index, and the Johnston index.

The Shapley-Shubik power index

The Shapley-Shubik power index was introduced by Shapley and Shubik (1954) and it is the restriction of the Shapley value to the family of simple games. In this index, each voter gets a weight for each swing, which depends on the number of possible voter orders. Thus, Shapley-Shubik power index depends on the number of permutations and size of each swing.

Given a simple game (N, v), the *Shapley-Shubik power index* assigns to each player $i \in N$ the real number

$$Sh_i\left(N, v\right) = \sum_{S \in \eta_i(v)} \frac{(s-1)!\,(n-s)!}{n!},$$

where s is the number of members in S.

In the class of simple games, the additivity property introduced by Shapley (1953) cannot be applied because the sum of two simple games is not a simple game. Dubey (1975) proposed the transfer property as a substitute of the additivity property and characterized the Shapley value in this class of games.

Transfer. A power index f defined on $H \subset SI\left(N\right)$ satisfies the transfer property if for all $\left(N, v\right), \left(N, w\right) \in H$ such that $\left(N, v \vee w\right)$, $\left(N, v \wedge w\right) \in H$, $f\left(N, v \vee w\right) + f\left(N, v \wedge w\right) = f\left(N, v\right) + f\left(N, w\right)$ where for all $S \subset N$

$$\left(v \vee w\right)\left(S\right) = \max\left\{v\left(S\right), w\left(S\right)\right\} \text{ and } \left(v \wedge w\right)\left(S\right) = \min\left\{v\left(S\right), w\left(S\right)\right\}.$$

The characterization is presented below.

- The unique power index f defined on $SI\left(N\right)$ that satisfies transfer, null player, symmetry, and efficiency is the Shapley-Shubik index.

In Alonso-Meijide (2002), an equivalent property to transfer is obtained; this property involves the solution for the unanimity games of the minimal winning coalitions and their unions. The resulting expression has the same flavour than the formula of the probability related to the union of n events in a random experiment.

Lemma 3.1 *A power index f defined on $SI(N)$ satisfies the transfer property if and only if, for every simple game (N,v) such that $M(v) = \{S_1, \cdots, S_m\}$, f assigns to every $i \in N$, the quantity*

$$\begin{aligned} f_i(N,v) &= \sum_{j_1 \in M} f_i(N, u_{S_{j_1}}) - \sum_{j_1 \in M} \sum_{j_2 \in M : j_2 > j_1} f_i(N, u_{S_{j_1} \cup S_{j_2}}) \\ &+ \sum_{j_1 \in M} \sum_{j_2 \in M : j_2 > j_1} \sum_{j_3 \in M : j_3 > j_2} f_i(N, u_{S_{j_1} \cup S_{j_2} \cup S_{j_3}}) \\ &- \cdots + (-1)^{m+1} f_i(N, u_{S_1 \cup \cdots \cup S_m}), \end{aligned}$$

where $M = \{1, \cdots, m\}$ and if $S \subset N$, u_S denotes the unanimity game of the coalition S, i.e., $u_S(T) = 1$ if $S \subset T$ and $u_S(T) = 0$ otherwise.

The Banzhaf power index

The Banzhaf power index appears in Banzhaf (1965), although Penrose (1946) defines a measure which is the half of the Banzhaf's power index. The Banzhaf power index it is the restriction of the Banzhaf value to the family of simple games. In this index, each voter gets the sum of his swings divided by the number of coalitions the voter belongs to. Contrary to the Shapley-Shubik index, the Banzhaf index considers that power is not directly associated with the order of voters, that is, depends only on the number of groups of voters.

Given a simple game (N,v), the Banzhaf index assigns to each player $i \in N$ the real number

$$B_i\left(N,v\right) = \frac{\left|\eta_i\left(v\right)\right|}{2^{n-1}}.$$

In the context of simple games, the total power property for the class of TU games can be rewritten; it states that the power of a player adds up to the total number of swings divided by the number of coalitions he can join.

Total power. A power index f defined on $H \subset SI(N)$ satisfies the total power property if

$$\sum_{i \in N} f_i\left(N,v\right) = \frac{\overline{\eta}\left(v\right)}{2^{n-1}},$$

for every simple game $(N,v) \in H$, where $\overline{\eta}\left(v\right) = \sum_{i \in N} \left|\eta_i\left(v\right)\right|$. Dubey and Shapley (1979) characterized the Banzhaf index as follows.

- The unique power index f defined on $SI\left(N\right)$ that satisfies transfer, null player, symmetry, and total power is the Banzhaf index.

The Deegan-Packel index

Deegan and Packel (1979) defined a new power index. They assume:

(a) Only minimal winning coalitions will emerge victorious.

(b) Each minimal winning coalition has an equal probability of forming.

(c) Players in a minimal winning coalition divide the “spoils” equally.

These assumptions seem reasonable in a wide variety of situations. According to these, they define the Deegan-Packel index. Given a simple game (N, v), this index assigns to each player $i \in N$ the real number

$$DP_i(N, v) = \frac{1}{|M(v)|} \sum_{S \in M_i(v)} \frac{1}{|S|}.$$

The Deegan-Packel index does not satisfy transfer property, but it satisfies the property of DP-mergeability.

Two simple games (N, v) and (N, w) are *mergeable* if for all pair of coalitions $S \in M(v)$ and $T \in M(w)$, it holds that $S \not\subset T$ and $T \not\subset S$. The minimal winning coalitions in game $(N, v \vee w)$ are precisely the union of the minimal winning coalitions in games (N, v) and (N, w). If two games (N, v) and (N, w) are mergeable, the mergeability condition guarantees that $|M(v \vee w)| = |M(v)| + |M(w)|$.

DP-mergeability. A power index f on $H \subset SI(N)$ satisfies DP-mergeability if for any pair of mergeable simple games (N, v), $(N, w) \in H$ such that $(N, v \vee w) \in H$, it holds that

$$f(N, v \vee w) = \frac{|M(v)| f(N, v) + |M(w)| f(N, w)}{|M(v \vee w)|}.$$

This property states that power in a merged game is a weighted mean of power of the two component games, where the weights come from the number of minimal winning coalitions in each component game, divided by the number of minimal winning coalitions in the merged game. Deegan and Packel (1979) characterized DP as follows.

- The unique power index f on $SI(N)$ satisfying DP-mergeability, null player, symmetry, and efficiency is the Deegan-Packel power index.

The Public Good Index

In the Public Good Index, introduced by Holler (1982), only minimal winning coalitions are considered relevant when it comes to measuring power. Then, given a simple game (N, v), the Public Good Index assigns to each player $i \in N$ the real number

$$PGI_i(N, v) = \frac{|M_i(v)|}{\sum_{j \in N} |M_j(v)|}.$$

An axiomatic characterization of this index can be found in Holler and Packel (1983). This characterization follows the spirit of the characterization of the Deegan-Packel index.

PGI-mergeability. A power index f on $H \subset SI(N)$ satisfies PGI-mergeability if for any pair of mergeable simple games $(N, v), (N, w) \in H$ such that $(N, v \vee w) \in H$, it holds that, for all player $i \in N$,

$$f_i(N, v \vee w) = \frac{f_i(N, v) \sum_{j \in N} |M_j(v)| + f_i(N, w) \sum_{j \in N} |M_j(w)|}{\sum_{j \in N} |M_j(v \vee w)|}.$$

- The unique power index f defined on $SI(N)$ satisfying PGI-mergeability, null player, symmetry, and efficiency is the Public Good Index.

Remark 3.1 *In Fishburn and Brams (1996), a power index based on minimal winning coalitions was introduced as the Member Bargaining Power. Holler (1998) showed that this index coincides with the Public Good Index.*

The Johnston index

The idea behind the Johnston index (Johnston, 1978) is quite similar to the one in Deegan-Packel index. Johnston power index takes into account how many swings there are in a single voter group. This index divides the spoils equally among the swingers. The Johnston index assigns to a player $i \in N$ in a simple game (N, v) the amount given by the expression

$$J_i(N, v) = \frac{1}{|G(v)|} \sum_{S \in G_i(v)} \frac{1}{|\chi(S)|}.$$

This index coincides with the Deegan-Packel index when $G(v) = M(v)$ (in this case, $S = \chi(S)$ for all $S \in G(v)$). Johnston assumes that not only minimal winning coalitions but also quasi-minimal winning coalitions emerge victorious, that is, each quasi-minimal winning coalition has an equal probability of forming and players in a quasi-minimal winning coalition divide the spoils equally among the swingers.

3.4 New characterizations of power indices for simple games

In this section, we provide new characterizations for the Deegan-Packel, Public Good, and Johnston power indices for simple games.

3.4.1 Characterization of the Deegan-Packel power index

The Deegan-Packel power index is characterized by means of a property, among others, of mergeability. A new characterization of the Deegan-Packel index is provided in Alonso-Meijide (2002) and Lorenzo-Freire et al. (2005), using a similar property to strong monotonicity (Young, 1985) instead of mergeability. This property is called minimal monotonicity. In the formulation of this property, we take into account a relation between two simple games v and w given in terms of the cardinality of the sets of minimal winning coalitions.

Minimal monotonicity. A power index f on $H \subset SI(N)$ satisfies the property of minimal monotonicity if for any pair of games $(N,v), (N,w) \in H$, it holds that

$$f_i(N,w)\,|M(w)| \geq f_i(N,v)\,|M(v)|\,,$$

for all player $i \in N$ such that $M_i(v) \subset M_i(w)$.

This property states that if the set of minimal winning coalitions containing a player i in game (N,v) is a subset of minimal winning coalitions containing this player in game (N,w), then the power of player i in game (N,w) is not less than power of player i in game (N,v) (previously, we must normalize this power by the number of minimal winning coalitions of games (N,v) and (N,w)).

Note that minimal monotonicity implies

$$f_i(N,w)\,|M(w)| = f_i(N,v)\,|M(v)|\,,$$

for any two games (N, v) and $(N, w) \in H \subset SI\,(N)$, and for all $i \in N$ such that $M_i\,(v) = M_i\,(w)$.

- The unique power index f on $SI(N)$ satisfying minimal monotonicity, null player, symmetry, and efficiency is the Deegan-Packel power index.

3.4.2 Characterization of the Public Good Index

A new characterization of Public Good Index is provided here, using a property similar to strong monotonicity instead of PGI-mergeability. We name this property PGI-minimal monotonicity. It takes into account a relation between two simple games (N, v) and (N, w), in terms of the cardinality of the sets of minimal winning coalitions.

PGI-minimal monotonicity. A power index f on $H \subset SI(N)$ satisfies the property of PGI-minimal monotonicity if for any pair of games (N, v), $(N, w) \in H$, it holds that:

$$f_i(N, w) \sum_{j \in N} |M_j(w)| \geq f_i(N, v) \sum_{j \in N} |M_j(v)|\,,$$

for all player $i \in N$ such that $M_i(v) \subset M_i(w)$.

This property states that if the set of minimal winning coalitions containing a player i in game (N, v) is a subset of minimal winning coalitions containing this player in game (N, w), then the power of player i in game (N, w) is not less than power of player i in game (N, v) (taking into account that we must normalize this power by the number of minimal winning coalitions of every player in the games (N, v) and (N, w)).

Note that PGI-minimal monotonicity implies

$$f_i(N, w) \sum_{j \in N} |M_j(w)| = f_i(N, v) \sum_{j \in N} |M_j(v)|\,,$$

for any two simple games (N, v) and (N, w), and for all $i \in N$ such that $M_i(v) = M_i(w)$.

In the next result, we propose a new characterization of the Public Good Index.

Theorem 3.2 *The unique power index f on $SI(N)$ satisfying PGI-minimal monotonicity, null player, symmetry, and efficiency is the Public Good Index.*

Proof.

Existence.

In Holler and Packel (1983), it is proved that the Public Good Index satisfies null player, symmetry, and efficiency properties. To prove that it satisfies PGI-minimal monotonicity, suppose two simple games $(N, v), (N, w)$, and a player $i \in N$ such that $M_i(v) \subset M_i(w)$. Then

$$PGI_i(N, v) = \frac{|M_i(v)|}{\sum_{j \in N} |M_j(v)|},$$

and,

$$\begin{aligned} PGI_i(N, w) &= \frac{|M_i(w)|}{\sum_{j \in N} |M_j(w)|} \\ &= \frac{|M_i(v)|}{\sum_{j \in N} |M_j(w)|} + \frac{|M_i(w) - M_i(v)|}{\sum_{j \in N} |M_j(w)|}. \end{aligned}$$

Then,

$$\begin{aligned} PGI_i(N, w) \sum_{j \in N} |M_j(w)| &= |M_i(v)| + |M_i(w) - M_i(v)| \\ &\geq |M_i(v)| = PGI_i(N, v) \sum_{j \in N} |M_j(v)|. \end{aligned}$$

Uniqueness.

We prove the uniqueness by induction on $|M(v)|$. If $|M(v)| = 1$, then $v = u_S$ for a coalition $S \subset N$. Here, $M(v) = \{S\}$. If a power index f satisfies the properties of efficiency, symmetry, and null player, it holds that

$$f_i(N, v) = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Then, the solution is unique when $|M(v)| = 1$. Assume uniqueness whenever $|M(v)|$ is at most $m - 1$, where $m > 1$, and let (N, v) be a simple game such that

$|M(v)| = m$. Suppose that $M(v) = \{S_1, S_2, \ldots, S_m\}$.

Consider $R = S_1 \cap S_2 \cap \ldots \cap S_m$ and suppose that $i \notin R$. We define the simple game (N, v') where $M(v') = \{S \in M(v) : i \in S\}$.

Taking into account that $M_i(v) = M_i(v')$, by the property of PGI-minimal monotonicity, it holds that

$$f_i(N, v) \sum_{j \in N} |M_j(v)| = f_i(N, v') \sum_{j \in N} |M_j(v')|.$$

So, by induction, $f_i(N, v)$ is unique when $i \notin R$.

It remains to show the uniqueness when $i \in R = S_1 \cap S_2 \cap \ldots \cap S_m$.

By symmetry, $f_i(N, v)$ is a constant c for all members in R. Since the solution is efficient and is unique for all i not in R, it follows that c must be unique. ■

3.4.3 Characterization of the Johnston power index

In this subsection, we give a characterization of the Johnston power index in the class of simple games. We modify the transfer property; more specifically, we adapt the equivalent property to transfer property, which is introduced in Alonso-Meijide (2002), and formulated in Lemma 3.1. This property identifies the power of a simple game with the power of unanimity games in minimal winning coalitions. In this case, the value of every simple game can be identified with the value for the unanimity games of critical coalitions.

To prove the main result of this subsection and to formulate our new property, next lemma will be useful. This lemma says that if the players of a coalition are critical for at least one quasi-minimal winning coalition, then such a coalition can be expressed as an intersection of minimal winning coalitions.

Lemma 3.2 *Given a simple game (N, v), where $M(v) = \{S_1, \cdots, S_m\}$, and $T \in G(v)$, it holds that $\chi(T) = \cap\{S_j \in M(v) : S_j \subset T\}$.*

Proof.

Let (N, v) be a simple game, where $M(v) = \{S_1, \cdots, S_m\}$, $M = \{1, \cdots, m\}$, and $T \in G(v)$.

We distinguish two possibilities:

1. $T \in M(v)$. In this case, we know that $\chi(T) = T$.

2. $T \notin M(v)$. Consider $R_T = \{j \in M : S_j \in M(v), S_j \subset T\}$. Note that $\cup_{j\in R_T} S_j \subset T$. Then, $\chi(T) = \cap_{j\in R_T} S_j$.

   Take $i \in \chi(T)$. By definition, $v(T) = 1$ and $v(T \setminus \{i\}) = 0$, and then it is obvious that $i \in \cap_{j\in R_T} S_j$.

   Take now $i \in \cap_{j\in R_T} S_j$. We have that $i \in T$. If $i \notin \chi(T)$, $v(T \setminus \{i\}) = 1$, and we can consider $S_j \in M(v)$, with $S_j \subset T \setminus \{i\} \subset T$, and such that $i \notin S_j$. This is a contradiction. Therefore, we have that $\cap_{j\in R_T} S_j \subset \chi(T)$.

■

In the next example, we show that not all the intersections of minimal coalitions are critical for at least one quasi-minimal coalition.

Example 3.2 *Consider the game defined in Example 3.1.*

If we choose $S = \{2\}$, we know that $\{2\} = \{1,2\} \cap \{2,3,4\}$ but, however, $G(\{2\}, v) = \{T \in G(v) : \chi(T) = \{2\}\} = \emptyset$.

Next, we introduce a new property called critical mergeability. It states that the power in a game (N, v) is a weighted mean of the power of the unanimity games in critical coalitions. The weight of a component unanimity game, corresponding to a critical coalition S, is the proportion of quasi-minimal coalitions where S is critical. By Lemma 3.2, the set of critical coalitions is a subset of the set of intersections of minimal winning coalitions.

Critical mergeability. A power index f defined on $H \subset SI(N)$ satisfies the property of critical mergeability if for any game $(N, v) \in H$ such that $M(v) = \{S_1, \cdots, S_m\}$, and $M = \{1, \cdots, m\}$, it holds that

$$f(N, v) = \sum_{S\in\mathcal{F}} \frac{\mid G(S, v) \mid}{\mid G(v) \mid} f(N, u_S)$$

where $\mathcal{F} = \{\cap_{j\in R} S_j : \cap_{j\in R} S_j \neq \emptyset, R \subset M\}$.

In the definition above, the power of an agent in a game is obtained as a weighted mean of power of unanimity games of critical coalitions.

In the next theorem, we propose a new characterization of the Johnston index.

Theorem 3.3 *The unique power index f on $SI(N)$ satisfying critical mergeability, null player, symmetry, and efficiency is the Johnston power index.*

Proof.

Existence.

We can prove that the Johnston power index satisfies null player, symmetry, and efficiency in a similar way than in Deegan and Packel (1979) for the Deegan-Packel power index. To prove that the Johnston power index satisfies critical mergeability, we fix a simple game (N, v) and a player $i \in N$. Then, by the definition of the Johnston power index, Lemma 3.2, and taking $\mathcal{F}$ in the conditions of the property of critical mergeability,

$$\begin{aligned} \mid G(v) \mid J_i(N, v) &= \sum_{S \in G_i(v)} \frac{1}{|\chi(S)|} \\ &= \sum_{S \in \mathcal{F}} \sum_{T \in G(v) : \chi(T) = S} J_i(N, u_S) \\ &= \sum_{S \in \mathcal{F}} \mid G(S, v) \mid J_i(N, u_S). \end{aligned}$$

Uniqueness.

Suppose we have a power index, f, on $SI(N)$ satisfying critical mergeability, null player, symmetry, and efficiency. Let (N, v) be a simple game. Then, by critical mergeability,

$$f(N, v) = \sum_{S \in \mathcal{F}} \frac{\mid G(S, v) \mid}{\mid G(v) \mid} f(N, u_S).$$

Note that the weights do not depend on the power index and the two indices, f and J, satisfy null player, symmetry, and efficiency. Thus,

$$f(N, v) = \sum_{S \in \mathcal{F}} \frac{\mid G(S, v) \mid}{\mid G(v) \mid} J(N, u_S).$$

By the property of critical mergeability applied to the Johnston power index, $f(N, v) = J(N, v)$. ■

Remark 3.2 *In Lorenzo-Freire et al. (2005), the Johnston power index is characterized in the class of* $\{0, 1\}$*-games. This is a class which contains the simple games. The Johnston power index is characterized by means of the null player property, symmetry, maximal efficiency (a property of efficiency adapted to the class of* $\{0, 1\}$*-games), and a property called composition (this property is similar* *to the property of mergeability). This characterization does not hold in the class of simple games, because the composition of two simple games is not a simple game. We say that a $\{0,1\}$-game (N,v) is composition of two $\{0,1\}$-games (N,v_1) and (N,v_2) if for every $S \subset N$,*

$$| G(S,v) | = | G(S,v_1) | + | G(S,v_2) |,$$

i.e., for every coalition S, the number of quasi-minimal coalitions in the game (N,v) such that S is the set of critical players is precisely the sum of the number of quasi-minimal coalitions in games (N,v_1) and (N,v_2) where S is critical.

In the next table, we provide an overview of the properties for power indices in simple games that we have mentioned before and, for each property, we indicate whether it is satisfied (✓) by the Shapley-Shubik, Banzhaf, Deegan-Packel, Public Good, and Johnston indices or not (−).

Property	*Shapley Shubik*	*Banzhaf*	*Deegan Packel*	*Public Good*	*Johnston*
Null player	✓	✓	✓	✓	✓
Symmetry	✓	✓	✓	✓	✓
Transfer	✓	✓	−	−	−
Efficiency	✓	−	✓	✓	✓
Total power	−	✓	−	−	−
Strong monotonicity	✓	✓	−	−	−
DP-mergeability	−	−	✓	−	−
Minimal monotonicity	−	−	✓	−	−
PGI-minimal monotonicity	−	−	−	✓	−
PGI-mergeability	−	−	−	✓	−
Critical mergeability	−	−	−	−	✓

3.5 Multilinear extensions

Sometimes, the computation of the Shapley and Banzhaf values is not easy, due to the fact that their computation requires the sum of a very large number of terms. To facilitate the computation of these values, Owen (1972) defined the multilinear extension of a game.

The *multilinear extension* of a TU game (N, v) is given by:

$$h(x_1, \ldots, x_n) = \sum_{S \subset N} \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j)\, v(S),$$

for $0 \leq x_i \leq 1$, $i = 1, \cdots, n$.

Heuristically, $h(x_1, \cdots, x_n)$ can be thought of as the mathematical expectation related to the formation of a winning coalition. Moreover, it is known that composition of games corresponds to composition of their multilinear extensions.

The multilinear extension of a game is useful for computation of values. Indeed, the Shapley value of a game can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0, 1]^N$ (see (Owen, 1972)). In turn, the derivatives of that multilinear extension, evaluated at point $(1/2, 1/2, \ldots, 1/2)$, give the Banzhaf value of the game (Owen, 1975). These results, joint with the above mentioned properties of multilinear extensions, allow us to simplify the calculation of Shapley and Banzhaf values for games with a large number of players.

In the same way, we can use this procedure to make possible the computation of the Deegan-Packel index, the Public Good Index, and the Johnston index for games with a large number of players. The next lemma (Owen (1972)), which provides a simple method to compute the multilinear extension of a game by using the expression of the game as a linear combination of unanimity games, is useful to obtain the computations of the three indices.

Lemma 3.3 *If a characteristic function game* (N, v) *can be expressed as* $\sum_{S \subset N} c_S u_S$, *the multilinear extension of this game is*

$$h\left(x_1, \ldots, x_n\right) = \sum_{S \subset N} c_S \prod_{i \in S} x_i,$$

for $0 \leq x_i \leq 1$, $i = 1, \cdots, n$, *where* c_S *is a constant for all* $S \subset N$ *and* u_S *the corresponding unanimity game.*

It is well known that every game $(N, v) \in TU\left(N\right)$ can be written as a linear combination of unanimity games. Moreover, if the game (N, v) is a simple game, it holds that:

$$v = \sum_{\substack{S \subset N \\ S \in W}} c_S u_S.$$

By the previous lemma, the multilinear extension of a simple game (N, v) is:

$$h(x_1, \ldots, x_n) = \sum_{\substack{S \subset N \\ S \in W}} c_S \prod_{i \in S} x_i,$$

where it holds that $c_S = 1$ if $S \in M(v)$.

3.5.1 The multilinear extension of the Deegan-Packel index

In Alonso-Meijide (2002) and Lorenzo-Freire et al. (2006), the computation of the Deegan-Packel power index of a simple game (N, v) using the multilinear extension procedure is given. Next, we describe the procedure (using the multilinear extension of the game) to obtain this power index. The advantage of this result resides in the fact that provides us with an effective method to compute this index when the set of minimal winning coalition is not known a priori. This could be the case of games where the number of players is large or when the game can be written as a composition of several games (see Owen (1995)).

Theorem 3.4 *Let (N, v) be a simple game. We can compute the Deegan-Packel power index for every player $i \in N$ by the following procedure:*

(1). *Obtain the multilinear extension $h(x_1, \ldots, x_n)$ of the game (N, v).*

(2). *In the previous expression, eliminate the monomials $c_S \prod_{i \in S} x_i$ where $S \subset N$ and $c_S \neq 1$. We obtain a new multilinear function $l(x_1, \ldots, x_n)$.*

(3). *Let p be the minimum degree of the monomials $\prod_{i \in S} x_i$ of the function l. From $k = p + 1$ to $k = n$, eliminate those monomials of degree k which are divisible by some monomials of the function l with degree from p to $k - 1$. Then, we obtain a function g.*

(4). *Finally, to obtain the Deegan-Packel power index of a player $i \in N$, we compute*

$$DP_i(N, v) = \frac{1}{g(1, \ldots, 1)} \int_0^1 \frac{\partial g}{\partial x_i}(t, \ldots, t)\, dt.$$

Proof.

Let(N, v) be a simple game and consider $i \in N$. In Steps (2) and (3), we eliminate those terms corresponding to winning coalitions that are not the minimal winning ones. Then, it is clear that function g after Step (3) is

$$g\left(x_1, \ldots, x_n\right) = \sum_{S \in M(v)} \prod_{k \in S} x_k.$$

It holds that $g\left(1, \ldots, 1\right) = \sum\limits_{S \in M(v)} 1 = \left| M\left(v\right) \right|.$

Taking into account that,

$$\int_0^1 \frac{\partial g}{\partial x_i}\left(t, \ldots, t\right) dt = \int_0^1 \sum_{S \in M_i(v)} t^{|S|-1} dt = \sum_{S \in M_i(v)} \frac{1}{|S|},$$

the proof is finished. ■

3.5.2 The multilinear extension of the Public Good Index

The next procedure gives a method to compute the Public Good Index.

Theorem 3.5 *Let (N, v) be a simple game. We can compute the Public Good Index for every player $i \in N$ by taking into account the following procedure:*

(1). *We follow Steps (1), (2), and (3) of Theorem 3.4.*

(2). *Obtain n functions g_i, $i = 1, \ldots, n$ by calculating*

$$g_i\left(x_i\right) = g\left(1, \ldots, 1, x_i, 1, \ldots 1\right), \text{ for } 0 \leq x_i \leq 1.$$

(3). *Finally, compute the derivatives, $g_i'(x_i)$, of the previous functions, and we obtain that the Public Good Index for a player $i \in N$ is*

$$PGI_i\left(N, v\right) = \frac{g_i'(x_i)}{\sum_{j \in N} g_j'(x_j)},$$

with $0 \leq x_j \leq 1$ for all $j \in N$.

Proof. Let (N, v) be a simple game. In a similar way to Theorem 3.4, the function g after Step (1) is

$$g(x_1, \ldots, x_n) = \sum_{S \in M(v)} \prod_{k \in S} x_k.$$

Moreover, the functions g_i for every $i \in N$, after Step (2) are

$$g_i(x_i) = |M_i(v)|\, x_i + k_i, \text{ where } k_i \in \mathbb{R}.$$

Taking into account that

$$g_i'(x_i) = |M_i(v)|,$$

the proof is finished. ■

3.5.3 The multilinear extension of the Johnston index

We now describe a procedure to obtain the Johnston power index of a simple game. In this case, the advantage of the approach resides in the fact that the set of quasi-minimal winning coalitions and its corresponding critical coalitions are not necessary to obtain the index.

Theorem 3.6 *Let (N, v) be a simple game. We can compute the Johnston power index for every player $i \in N$, by the following procedure:*

(1). *We follow the computations in Steps* (1) *and* (2) *in Theorem 3.4.*

(2). *Consider p as the minimum degree of the monomials $\prod_{i \in S} x_i$ in the function l. Take $r = 1$.*

(2.1). *If $p + r > n$, let us denote by g the function we built and go to Step* (3).

(2.2). *Eliminate all the monomials of degree $p+r$ in l if all its divisors with $p + r - 1$ factors and degree $p + r - 1$ are in l.*

(2.3). *Add (with a positive sign) all the monomials of degree $p+r$ and $p+r$ factors in case that they are not in l and only a strict subset of its divisors with p, $p + 1$, $\cdots$, $p + r - 1$ factors and degree p, $p + 1$, $\cdots$, $p + r - 1$, respectively, which are in l.*

(2.4). *If there are no monomials of degree* $p+r$ *in this function, go to Step* (2.6).

(2.5). *Replace each monomial of degree* $p+r$ *by the highest common factor of the set of its divisors in* l *whose degree is between* p *and* $p+r-1$.

(2.6). *Take* $r = r+1$ *and go to step* (2.1).

(3). *Finally, to obtain the Johnston power index of a player* $i \in N$, *we compute*

$$J_i(N,v) = \frac{1}{g(1,\ldots,1)} \int_0^1 \frac{\partial g}{\partial x_i}(t,\ldots,t)dt.$$

Proof.

Consider $(N,v) \in SI(N)$ and $i \in N$. In Steps (1) and (2), we obtain the terms corresponding to the critical coalitions of all quasi-minimal winning coalitions. Then, it is clear that the function g after Step (2) is

$$g(x_1, x_2, \ldots, x_n) = \sum_{S \in G(v)} \prod_{k \in \chi(S)} x_k.$$

Moreover, $g(1,\ldots,1) = \sum_{S \in G(v)} 1 = \mid G(v) \mid$.

Taking into account that

$$\int_0^1 \frac{\partial g}{\partial x_i}(t,\ldots,t)dt = \int_0^1 \sum_{S \in G_i(v)} t^{|\chi(S)|-1}dt = \sum_{S \in G_i(v)} \frac{1}{\mid \chi(S) \mid},$$

the proof is finished. ■

3.6 Examples

In this section, the previous procedures are applied to two examples. In the first one, the Basque Country Parliament, it is shown how the algorithms work. In the second one, the Victoria proposal, the multilinear extension of the corresponding game is obtained by taking into account the expression of the previous game from several games in order to apply the algorithms to calculate the power indices.

3.6.1 The Basque Country Parliament

The Basque Country is one of the seventeen Spanish autonomous communities. The Basque Country Parliament is made up of 75 members. We will study the power indices in the case of the elections held on April 2005. Another study of the Basque Country Parliament appears in Carreras and Owen (1996). They obtain results for the elections held on November 1986.

According to the elections of April, 2005, the Parliament was composed of 29 members of the middle-of-the-road regional party PNV (party 1), 18 members of the socialist party PSE-EE (party 2), 15 members of the conservative party PP (party 3), 3 members of the communist party IU-EB (party 5), and 9 and 1 members, respectively, of the left-wing regionalist parties PCTV (party 4), and ARALAR (party 6).

A simple game is a *weighted majority game* if there exist a quota q and a weight $\alpha_i > 0$ for each player i $(i = 1, \cdots, n)$, where $q \leq \sum_{i \in N} \alpha_i$, such that

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} \alpha_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

Analyzing this Parliament as a weighted majority game, the quota is $q = 38$ and the vector of weights is $(29, 18, 15, 9, 3, 1)$.

It is easy to show that this simple game can be written in terms of unanimity games, taking $S_1 = \{1\}$ and $S_2 = \{2, 3, 4\}$, by

$$v = \sum_{\substack{R \subset S_2 \\ |R|=1}} u_{S_1 \cup R} - \sum_{\substack{R \subset S_2 \\ |R|=2}} u_{S_1 \cup R} + u_{S_2}.$$

Applying Lemma 3.3, the multilinear extension of this game is

$$h(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_2 + x_1x_3 + x_1x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 + x_2x_3x_4.$$

Eliminating those monomials with coefficients different from 1, we obtain the function l

$$l(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4.$$

The Deegan-Packel index.

Once we have the function l, by Step (3) we eliminate those monomials that can be divided by any other monomial of l, obtaining the function g

$$g\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4.$$

Finally, to compute Deegan-Packel index, we apply Step (4) of Theorem 3.4, calculating:

$$g\left(1,1,1,1,1,1\right) = 4, \quad \int_0^1 \frac{\partial g}{\partial x_1}\left(t,\ldots,t\right)dt = \int_0^1 3tdt = \frac{3}{2},$$

$$\int_0^1 \frac{\partial g}{\partial x_i}\left(t,\ldots,t\right)dt = \frac{5}{6}, \text{ for all } i \in \{2,3,4\}, \quad \text{and}$$

$$\int_0^1 \frac{\partial g}{\partial x_i}\left(t,\ldots,t\right)dt = 0, \text{ for all } i \in \{5,6\}.$$

The Deegan-Packel index is $DP\left(N,v\right) = \left(\frac{9}{24}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24}, 0, 0\right)$.

The Public Good Index.

As Step (1) is similar for the Public Good Index to Steps (1), (2), and (3) for the Deegan-Packel index, we take into account the function g and, due to the Step (2), we get

$$g_1(x_1) = 3x_1 + 1, \quad g_2(x_2) = 2x_2 + 2, \quad g_3(x_3) = 2x_3 + 2, \quad g_4(x_4) = 2x_4 + 2,$$

$$g_5(x_5) = 4, \text{ and } g_6(x_6) = 4.$$

And finally, by Step (3) of Theorem 3.5, the Public Good Index is given by

$$PGI\left(N,v\right) = \left(\frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, 0, 0\right).$$

The Johnston index.

We consider the function l obtained in Step (2) of Theorem 3.4, when we computed Deegan-Packel power index. If we take into account Step (2) of Theorem 3.6, we have several stages:

First stage. We consider the monomials with degree 2 of the function l

$$x_1x_2 + x_1x_3 + x_1x_4.$$

Second stage. Using the monomials, we add all the possible monomials of degree 3 and 3 factors such that only a strict subset of its divisors with 2 factors and degree 2 are in l.

Then, we have

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4+$$

$$x_1x_2x_5 + x_1x_2x_6 + x_1x_3x_4 + x_1x_3x_5 + x_1x_3x_6 + x_1x_4x_5 + x_1x_4x_6$$

and, replacing the monomials of degree 3 and 3 factors by the highest common factor of the set of its divisors in l whose degree is 2, we obtain

$$3x_1 + 3x_1x_2 + 3x_1x_3 + 3x_1x_4 + x_2x_3x_4.$$

Third stage. By the same procedure for the monomials of degree 4, we get

$$9x_1 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 3x_2x_3x_4.$$

Fourth stage. Taking into account the monomials of degree 5, the next result is obtained:

$$12x_1 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 4x_2x_3x_4.$$

We stop, obtaining that

$$g(x_1, x_2, x_3, x_4, x_5, x_6) = 12x_1 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 4x_2x_3x_4.$$

Therefore, by Step (3) of Theorem 3.6,

$$g(1, 1, 1, 1, 1, 1) = 28,$$

$$\int_0^1 \frac{\partial g}{\partial x_1}(t, \dots, t)dt = \int_0^1 (12 + 12t)dt = 18,$$

$$\int_0^1 \frac{\partial g}{\partial x_i}(t, \dots, t)dt = \int_0^1 (4t + 4t^2)dt = \frac{10}{3} \text{ for all } i \in \{2, 3, 4\},$$

and

$$\int_0^1 \frac{\partial g}{\partial x_i}(t,\ldots,t)dt = 0 \text{ for all } i \in \{5,6\}.$$

Then, the Johnston index is

$$J(N,v) = \left(\frac{27}{42}, \frac{5}{42}, \frac{5}{42}, \frac{5}{42}, 0, 0\right).$$

3.6.2 The Victoria Proposal

To ratify the amendments to the Canadian Constitution, in accordance with the suggestions made by the Victoria Proposal, it is necessary that they are approved by at least the next Provinces:

1. Ontario (1) and Quebec (2),

2. Two of the four Maritime Provinces: New Brunswick (3), Nova Scotia (4), Newfoundland (5), and Prince Edward Island (6).

3. Either British Columbia (7) and one of the Prairie Provinces, or all three of the Prairie Provinces. The Prairie Provinces are Alberta (8), Saskatchewan (9), and Manitoba (10).

We analyze this situation as a simple game v. Moreover, we have a natural partition of the Provinces into three subsets P_1, P_2, and P_3, where $P_1 = \{1,2\}$ ={Ontario, Quebec}, $P_2 = \{3,4,5,6\}$ ={Maritime Provinces}, and $P_3 = \{7,8,9,10\}$.

We note that the game v can be expressed as a composition of games

$$v = u[v_1, v_2, v_3],$$

where v_1 is a two-player game in which $\{1,2\}$ is the only winning coalition, v_2 is a four-player game in which any two-player coalition (or larger) wins, v_3 is a four-player game in which a coalition S wins if

(a) S has two players, and $1 \in S$

or

(b) S has three or four players.

Finally, u is a three-person simple game in which only the three-person coalition wins. For more details about this game and composition of games, the reader can see Owen (1995).

Taking into account Lemma 3.3, we obtain that for u the multilinear extension is $f(y_1, y_2, y_3) = y_1 y_2 y_3$. For v_1, v_2 and v_3, we have that

$g_1(x_1, x_2) = x_1 x_2$,

$g_2(x_3, x_4, x_5, x_6) = x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6 - 2x_3x_4x_5 - 2x_3x_4x_6 - 2x_3x_5x_6 - 2x_4x_5x_6 + 3x_3x_4x_5x_6$,

and

$g_3(x_7, x_8, x_9, x_{10}) = x_7x_8 + x_7x_9 + x_7x_{10} + x_8x_9x_{10} - x_7x_8x_9 - x_7x_8x_{10} - x_7x_9x_{10}$, respectively.

We know that h, the multilinear extension of v, is

$$\begin{aligned} h(x_1, \dots, x_{10}) &= f(g_1(x_1, x_2), g_2(x_3, x_4, x_5, x_6), g_3(x_7, x_8, x_9, x_{10})) \\ &= g_1(x_1, x_2) \times g_2(x_3, x_4, x_5, x_6) \times g_3(x_7, x_8, x_9, x_{10}). \end{aligned}$$

Finally, if we apply Theorems 3.4, 3.5, and 3.6 as in the example of Basque Country Parliament, we obtain the Deegan-Packel , Public Good, and Johnston power indices. We show them, jointly with Shapley-Shubik and Banzhaf power indices, in the next table. Taking into account that all these indices are symmetric, we only present the results for representatives of the four types of players.

Provinces	*Sh*	*B*	*DP*	*PGI*	*J*
Ontario and Quebec	0.3155	0.1718	0.1607	0.1600	0.2410
Maritime Provinces	0.0298	0.0469	0.0803	0.0800	0.0509
British Columbia	0.1250	0.1289	0.1250	0.1200	0.1744
Prairie Provinces	0.0417	0.0430	0.0773	0.0800	0.0466

It is interesting to note that the Banzhaf, Deegan-Packel, and Johnston indices assign greater power to the Maritime Provinces than to the Prairie Provinces. By contrast, the Shapley-Shubik index favored the Prairie Provinces more than the Maritimes. The Public Good Index assigns the same power to the Prairie Provinces and to the Maritime Provinces since the number of minimal winning coalitions containing a Praire Province or a Maritime Province is the same.

3.7 Concluding remarks

In this chapter we study some power indices and give a way to calculate these power indices by means of the multilinear extension.

Owen (1977) defined a solution for TU games where the set of players is divided in a priori unions, due to the existence of affinities among them (for example, the players can be divided according to political or geographical affinities). In Alonso-Meijide and Fiestras-Janeiro (2004) two modifications of the Deegan-Packel index for this kind of games are proposed. An interesting research issue could be the modification of the Public Good Index and Johnston power index to the case where players are divided in a priori unions, as well as the study of its properties and possible applications.

Bergantiños et al. (1993) defined a modification of the Shapley-Shubik index for situations where some players are incompatible. In the same way, we could study what happens in the case of other power indices.

As far as the multilinear extension is concerned, it could be interesting to implement with the computer both procedures: the computation of the power index with the definition and the computation of the power index with the multilinear extension. Then, we can compare the time of computation. It can be applied not only to the Deegan-Packel, Johnston, and Public Good indices but also to other power indices such as Shapley-Shubik and Banzhaf.

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Resumen en español

La Teoría de Juegos es una rama de las Matemáticas que estudia la toma de decisiones en situaciones en las que hay varios decisores y el resultado depende de la elección que haya hecho cada uno de ellos. La importancia de esta disciplina radica en su aplicación a muchos otros ámbitos académicos, tales como la Economía, la Politología, la Sociología, la Filosofía, la Informática, y la Biología.

A pesar de que se conoce algún trabajo anterior relacionado con la Teoría de Juegos, se podría decir que la Teoría de Juegos nace como disciplina en el año 1928 con la publicación del artículo "Zur Theorie der Gesellschaftsspiele", donde John von Neumann demuestra el Teorema Minimax para juegos bipersonales de suma nula. El trabajo de von Neumann culmina en el año 1944 con la publicación del libro "Theory of Games and Economic Behavior ", en colaboración con Oskar Morgenstern. En 1950, John Nash definió el equilibro que lleva su nombre y que es considerado como uno de los conceptos más importantes de la Teoría de Juegos. Desde ese momento, las contribuciones a la Teoría de Juegos experimentan un aumento considerable. En el año 1994, los teóricos de juegos John Harsanyi, John Nash, y Reinhard Selten ganan el premio Nobel de Economía. Más tarde, en el año 2005, la contribución de otros dos teóricos de juegos en el ámbito de la Economía, Robert Aumann y Thomas Schelling, es de nuevo reconocida al otorgarles el premio Nobel.

La Teoría de Juegos se divide en dos importantes partes: los juegos no cooperativos y los juegos cooperativos. En el caso de los juegos no cooperativos, un juego es un modelo que describe todos los posibles movimientos de los jugadores. En cambio, en el caso de los juegos cooperativos, se asume que se puede llegar a acuerdos vinculantes entre jugadores y se describen únicamente los resultados que se obtienen en todas las posibles coaliciones de jugadores.

En esta tesis nos centramos en los juegos cooperativos. Además, se divide en tres partes diferentes e independientes entre sí. El Capítulo 1 se centra en los problemas de árboles de coste. En el Capítulo 2 se introducen nuevos resultados sobre problemas de bancarrota y situaciones de asignación con varios asuntos. Finalmente, el Capítulo 3 estudia los índices de poder. Mientras que los problemas de árboles de coste y de bancarrota se pueden vincular al campo de la Economía, los índices de poder son herramientas útiles en el campo de la Politología.

Resumen del Capítulo 1

Consideremos la siguiente situación: un grupo de agentes quiere un servicio determinado que únicamente puede ser proporcionado por un proveedor, llamado fuente. Los agentes serán servidos por medio de conexiones que conllevan un coste. Además, se pueden conectar a la fuente tanto directa como indirectamente. Esta clase de problemas son los llamados problemas de árboles de mínimo coste. Nótese que hay una gran variedad de situaciones que se podrían modelar de esta forma. Por ejemplo, Bergantiños y Lorenzo (2004) estudiaron una situación real donde los habitantes de un pueblo tenían que pagar el coste de construcción de tuberías de sus respectivas casas a un suministro de agua. Otros ejemplos son los sistemas de comunicación, tales como el teléfono, Internet, las radiotelecomunicaciones, o la televisión por cable.

Un tema relevante en esta literatura es la definición de algoritmos para construir árboles de mínimo coste. Kruskal (1956) y Prim (1957) definieron dos algoritmos para encontrar los árboles de mínimo coste. Bird (1976), Kar (2002), y Dutta y Kar (2004) introdujeron varias reglas para estos problemas. Además, Bird (1976) asoció un juego cooperativo con utilidad transferible a cada problema de árboles de mínimo coste. Kar (2002) estudió el valor de Shapley de este juego, mientras que Granot y Huberman (1981 y 1984) estudiaron el núcleo y el nucleolo. Feltkamp et al. (1994) introdujeron la regla ERO, que fue estudiada por Bergantiños y Vidal-Puga (2004, 2005a, 2005b, y 2005c).

Todas las reglas mencionadas anteriormente reparten el coste entre los agentes teniendo en cuenta únicamente la matriz de costes. En algunas situaciones, podría tener sentido usar más información. Por ejemplo, en el caso de Bergantiños y Lorenzo (2004), podemos tener también en cuenta la renta de cada habitante del pueblo, que se puede representar con un sistema de pesos. Uno de los principales objetivos de este capítulo consiste en estudiar buenas reglas que asignen el coste total de conexión entre los agentes, usando tanto la matriz de costes como el sistema de pesos. Nosotros lo haremos considerando varias familias de valores de Shapley ponderados.

Otras reglas que no sólo dependen de la matriz de costes aparecen en Tijs et al. (2005). Estas reglas son las llamadas reglas de obligación y están asociadas a funciones de obligación. Tijs et al. (2005) demostraron que las reglas de obligación satisfacen dos propiedades interesantes: la propiedad de monotonía en la población (si un nuevo agente se conecta a los que ya estaban conectados previamente, nadie va a empeorar) y monotonía fuerte en los costes (si el coste de conexión entre dos agentes aumenta, nadie va a mejorar). Las reglas de obligación también fueron estudiadas en Moretti et al. (2005). En Bergantiños y Lorenzo-Freire (2006) y Lorenzo-Freire y Lorenzo (2006), demostramos que algunas familias de reglas de Shapley ponderadas son reglas de obligación. Éste es un resultado sorprendente ya que se definen de una forma totalmente diferente. Como consecuencia, estas familias también satisfacen monotonía en la población y monotonía fuerte en los costes.

Bergantiños y Vidal-Puga (2005a) demostraron que la regla ERO, que es una regla de obligación, es la única regla que satisface las propiedades de monotonía en la población, monotonía fuerte en los costes, e igual reparto de coste extra. En Bergantiños y Lorenzo-Freire (2006), modificamos la propiedad de igual reparto de coste extra, considerando un sistema de pesos y definiendo la propiedad de reparto ponderado de coste extra con respecto al sistema de pesos. Además, demostramos que hay una única regla en problemas de árboles de mínimo coste que satisface monotonía en la población, monotonía fuerte en los costes, y reparto ponderado de coste extra con respecto al sistema de pesos. Esta regla es el valor de Shapley ponderado de un juego para este sistema de pesos y la llamamos regla optimista de Shapley ponderada.

En Lorenzo-Freire y Lorenzo (2006), damos la primera caracterización de las reglas de obligación por medio de dos propiedades: monotonía en la población y una propiedad de aditividad adecuada para los problemas de árboles de mínimo coste, llamada aditividad restringida. Este resultado no es únicamente relevante por la caracterización en sí misma, sino que también proporciona una forma sencilla de calcular las funciones de obligación.

El Capítulo 1 se organiza de la siguiente forma. En la Sección 2 introducimos los problemas de árboles de mínimo coste. En la Sección 3 introducimos varias familias de reglas de Shapley ponderadas y en la Sección 4 las reglas de obligación. En la Sección 5 estudiamos la relación de las reglas optimistas de Shapley ponderadas con las reglas de obligación. En la Sección 6 presentamos la caracterización axiomática de las reglas optimistas de Shapley ponderadas. Finalmente, la Sección 7 aborda la caracterización de la familia de las reglas de obligación.

Resumen del Capítulo 2

Los problemas de bancarrota son situaciones donde tenemos que dividir un bien entre un conjunto de agentes, pero este bien no es suficiente para satisfacer todas las cantidades que los agentes demandan. Debido a esta insuficiencia, se introducen las reglas de bancarrota. La definición de estas reglas depende del contexto. Hay muchas situaciones que se pueden describir por medio de los modelos de bancarrota. Uno de los ejemplos más clásicos es el caso de una empresa que cae en bancarrota y tiene que dividir el activo (bien) entre los demandantes (agentes).

A pesar de ser ya estudiados en el Talmud, el primer análisis formal de estos problemas aparece en O'Neill (1982). En este artículo O'Neill no sólo asocia un juego cooperativo a cada problema de bancarrota, sino que también define la regla de llegada aleatoria, que coincide con el valor de Shapley de este juego y se caracteriza con una propiedad de consistencia. Otros trabajos interesantes relacionados con el estudio de estos problemas son los de Aumann and Maschler (1985), Curiel et al. (1987), Moulin (1987), Young (1988), Dagan (1996), y Herrero y Villar (2001). Aunque sólo hemos mencionado estos trabajos, hay una amplia literatura sobre bancarrota. Thomson (2003) es una buena recopilación de resultados en problemas de bancarrota.

En Lorenzo-Freire et al. (2005a), estudiamos una extensión de los problemas de bancarrota: las situaciones de asignación con varios asuntos. Las situaciones de asignación con varios asuntos fueron introducidas por Calleja et al. (2005) para modelar problemas similares a los de bancarrota, en los que el bien no se divide teniendo en cuenta una única demanda por parte de cada agente, sino que hay varias demandas por cada agente.

En Calleja et al. (2005), una solución de asignación con varios asuntos es una función que asigna a cada situación de asignación con varios asuntos un vector. Pero, debido al hecho de que en las situaciones de asignación con varios asuntos hay una matriz de demandas, creemos que también sería natural definir las reglas como funciones donde a cada situación de asignación con varios asuntos se le asigna una matriz. En Lorenzo-Freire et al. (2005b), introducimos una nueva definición para las soluciones de asignación con varios asuntos. Teniendo en cuenta esta nueva definición, definimos una nueva familia de reglas: las reglas en dos etapas. En estas reglas, consideramos dos etapas: en la primera etapa, distribuimos el bien entre los asuntos usando cualquier regla de bancarrota, dependiendo del contexto, y, en la segunda, dividimos la asignación obtenida para cada asunto entre los agentes, usando la misma regla de bancarrota. Caracterizamos dos reglas de esta familia, la regla CEA en dos etapas y la regla CEL en dos etapas. Estas reglas se definen teniendo en cuenta las reglas CEA y CEL en problemas de bancarrota, respectivamente.

En cuanto a la regla CEA en problemas de bancarrota, esta regla se caracterizó (Dagan, 1996) con las propiedades de igual tratamiento (véase O'Neill (1982)), composición (véase Young (1988)), e independencia de demandas truncadas. En Lorenzo-Freire et al. (2005b), caracterizamos la regla CEA en dos etapas con tres propiedades que siguen la filosofía de las propiedades anteriores en el contexto de las situaciones de asignación con varios asuntos y una propiedad del cociente. Esta propiedad del cociente es similar a la propiedad introducida por Owen (1977) en el contexto de los juegos TU con uniones a priori.

Por otra parte, la regla CEL se caracterizó en el trabajo de Herrero y Villar (2001) con las propiedades de igual tratamiento, composición de derechos mínimos (véase Curiel et al. (1987)), e independencia de caminos (véase Moulin (1987)). También caracterizamos la regla CEL con tres propiedades en la misma línea y la propiedad del cociente.

En la literatura relativa al valor de Shapley se utiliza un principio de reciprocidad entre los agentes. Este principio se introduce en Myerson (1980). El principio de Myerson de contribuciones equilibradas afirma que, dados dos agentes, la ganancia o pérdida de cada agente cuando el otro abandona el juego debería ser igual para los dos agentes.

En su trabajo, Calleja et al. (2005) generalizaron la regla de llegada aleatoria en la clase de situaciones de asignación con varios asuntos, definiendo la regla RTB proporcional y la regla RTB de la cola. Además, definen dos juegos cooperativos, el juego proporcional y el juego de la cola y obtienen que las reglas proporcional y de la cola coinciden con el valor de Shapley en los juegos proporcional y de la cola, respectivamente. Teniendo en cuenta la caracterización de O'Neill de la regla de llegada aleatoria para problemas de bancarrota usando la propiedad de consistencia, Calleja et al. (2005) caracterizan las reglas proporcional y de la cola de forma similar. Debido al hecho de que la idea de que un agente deje el juego no es fácil de implementar en situaciones de asignación con varios asuntos, tienen que extender estas situaciones a las situaciones de asignación con varios asuntos y adjudicaciones. En tales situaciones, los agentes que han recibido sus correspondientes adjudicaciones son todavía parte del juego y cualquier solución les debe otorgar sus correspondientes adjudicaciones.

La consideración de un conjunto de agentes con adjudicaciones tiene gran sentido en una amplia variedad de situaciones, no sólo en las situaciones de asignación con varios asuntos. Supongamos, por ejemplo, el problema de asignación de costes de un proyecto en el cual algunos agentes están invitados a participar con el compromiso de recibir unas asignaciones fijas. De esta forma, en Lorenzo-Freire et al. (2005a), extendemos el modelo clásico de juegos cooperativos con utilidad transferible al modelo más general de juegos TU con adjudicaciones, en los cuales cualquier solución debe otorgar a algunos agentes sus adjudicaciones fijas. Definimos la regla RTB en esta clase de juegos y la caracterizamos por medio de una propiedad de contribuciones equilibradas. Como aplicación de nuestro principal resultado, nos centramos en el caso más pesimista, la regla de la cola para situaciones de asignación con varios asuntos, caracterizando la regla RTB de la cola con una propiedad de contribuciones equilibradas. Resultados similares se podrían obtener para el caso proporcional.

La estructura de este capítulo es como sigue. En la Sección 2 introducimos los problemas de bancarrota. En la Sección 3 definimos situaciones de asignación con varios asuntos. La Sección 4 se centra en la definición y caracterización de la regla CEA y la regla CEL en dos etapas. Finalmente, en la Sección 5, se presenta la nueva caracterización de las reglas RTB para juegos TU con adjudicaciones.

Resumen del Capítulo 3

Uno de los elementos más importantes en las Ciencias Políticas es el poder. Aún cuando no hay un consenso total sobre la definición de poder, podemos interpretar el poder de un miembro en un comité como su habilidad para cambiar los resultados de acuerdo a sus preferencias. Un índice de poder es una medida de este poder, pero el poder es un concepto tan extremadamente difícil de medir, que no se ha llegado a un consenso sobre cuál es el índice de poder más adecuado.

Los principales índices de poder que podemos encontrar en la literatura son el índice de Shapley-Shubik (Shapley y Shubik, 1954), el índice de Banzhaf (Banzhaf, 1965), el índice de Johnston (Johnston, 1978), el índice de Deegan-Packel (Deegan y Packel, 1979), y el Índice del Bien Público (Holler, 1982). Estas medidas de poder están basadas en la importancia relativa de un jugador en la formación de coaliciones. Las permutaciones de los jugadores juegan un papel decisivo en el cálculo del índice de Shapley-Shubik, mientras que los otros índices se centran exclusivamente en grupos de jugadores.

Los juegos simples se pueden usar para modelizar votaciones. En estos juegos, se dice que una coalición ganadora es vulnerable cuando tiene al menos un miembro cuya eliminación implicaría que la coalición resultante se convirtiera en perdedora. Un agente es denominado crítico cuando es el causante de que una coalición ganadora se convierta en perdedora. Una coalición ganadora minimal es una coalición ganadora en la que sus miembros son críticos.

En el modelo de Banzhaf, el poder de un agente es proporcional al número de coaliciones en las que es crítico. Johnston argumentó que el índice de Banzhaf, que se basa en la idea de eliminar los votantes críticos de la coalición ganadora, no tiene en cuenta el número total de miembros críticos en cada coalición. Para definir este índice, Johnston considera que si un votante es el único agente crítico en una coalición, tiene un poder más fuerte que cuando todos los agentes son críticos.

Según Deegan y Packel, sólo las coaliciones minimales ganadoras deberían ser tenidas en cuenta a la hora de obtener el poder de un votante. De esta forma, propusieron un índice asumiendo que todos las coaliciones minimales ganadoras son equiprobables y que todos los votantes que pertenecen a la misma coalición minimal ganadora deberían obtener el mismo poder. Por otra parte, el Índice del Bien Público viene dado por el número de coaliciones minimales ganadoras que contienen al votante, dividido por la suma de las mismas para todos los votantes.

En Lorenzo-Freire et al. (2005), no se discute qué índice debemos elegir, puesto que la elección de un índice depende de las propiedades asociadas a la situación que estamos estudiando. Sin embargo, para falicitar esta elección, se introducen algunas propiedades deseables en el contexto de los índices de poder.

En Lorenzo-Freire et al. (2005), mencionamos algunas de estas propiedades, así como algunas caracterizaciones de los principales índices de poder. Además, damos nuevas caracterizaciones de los índices de Deegan-Packel y Johnston por medio de propiedades de monotonía y fusión.

Una de las principales dificultades cuando se trabaja con estos índices viene dada por el hecho de que su cálculo generalmente requiere la suma de un gran número de términos. Owen (1972) definió la extensión multilineal de un juego. Dicha extensión nos da la utilidad esperada de una coalición cualquiera. Owen la utilizó para calcular el valor de Shapley (Shapley, 1953) y el valor de Banzhaf (Owen, 1975). Ambos valores son valores probabilísticos (Weber, 1988), es decir, valores que satisfacen la propiedad de aditividad. Las extensiones multilineales son útiles para el cálculo del poder de juegos tales como el juego de las Elecciones Presidenciales y el juego del Colegio Electoral, estudiados por Owen (1972). La extensión multilineal tiene dos ventajas: gracias a su interpretación probabilística, se puede utilizar el teorema central del límite y, además, se aplica a la composición de juegos.

El principal objetivo de Alonso-Meijide et al. (2006) consiste en analizar si alguna modificación de la extensión multilineal se puede utilizar para calcular los índices de Johnston, Deegan-Packel, y Bien Público. La ventaja de estos métodos consiste en que, si sabemos cuál es la extensión multilineal del juego, podemos dar un algoritmo para calcular fácilmente estos tres índices.

En la Sección 2 introducimos algunos conceptos para juegos TU. La Sección 3 se dedica a una revisión de los principales índices de poder y sus caracterizaciones axiomáticas. En la Sección 4 damos nuevas caracterizaciones de los índices de Johnston, Deegan-Packel, y Bien Público. En la Sección 5 introducimos los procedimientos para calcular los tres índices a partir de las extensiones multilineales y en la Sección 6 se consideran dos ejemplos: el ejemplo del Parlamento del País Vasco, correspondiente a las elecciones celebradas en Abril del 2005, y la propuesta de enmiendas a la Constitución de Canadá.