UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Departamento de Estatística e Investigación Operativa

ESSAYS ON COOPERATIVE GAMES WITH RESTRICTED COOPERATION AND SIMPLE GAMES

("Aportaciones al estudio de juegos cooperativos con cooperación restringida y juegos simples")

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Realizado el acto público de defensa y mantenimiento de esta tesis doctoral el día 23 de marzo de 2012, en la Facultad de Matemáticas de la Universidad de Santiago de Compostela, ante el tribunal formado por:

Presidente: Dr. D. Ignacio García Jurado

Vocales: Dr.ª D.ª Miren Josune Albizuri Irigoyen
Dr. D. Josep Freixas Bosch
Dr. D. Gerard van der Laan

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siendo directores el Dr. D. José María Alonso Meijide y la Dr.ª D.ª María Gloria Fiestras Janeiro, obtuvo la máxima calificación APTO CUM LAUDE. Además, esta tesis ha cumplido los requisitos necesarios para la obtención del DOCTORADO EUROPEO.

Familiari eskainia

Ogi gogorrari...hagin zorrotza

Preface

This work is the result of my first years as a researcher in Mathematics, in particular in the field of Game Theory. Since high school I have been certain to like Mathematics, by that time facing problems became a hobby for me. I would like to thank Luisa Berri-Otxoa, an excellent high school teacher for helping me develop a mathematical way of thinking and encouraging me to start a bachelor in Mathematics. During the time spent studying Mathematics at the University of the Basque Country I realized that my particular hobby would turn endless. It was during the fifth course of the bachelor when I had the opportunity to know the field of Game Theory. This happened in Santiago the Compostela where I came following an exchange program between both universities. During that year I had the luck to follow two courses thought by Ignacio García Jurado, the first one on Decision Theory and the second one on Game Theory.

Ignacio is one of the persons I am in debt with for being such a good teacher. In addition, he helped me a lot when I made the decision to start out as a researcher, thanks Ignacio for all your help. My PhD started under his supervision and with the huge support of the whole Santiago Game Theory research group (SaGaTh). Soon after that I meet my current supervisors José María Alonso Meijide and María Gloria Fiestras Janeiro who have been my greatest academic supports so far. I would like to express my gratitude to them for all the time spent on me and for putting up with my stubbornness so many times.

During these four years I had the opportunity to collaborate with excellent researchers and to discover other ways of working during my stays in prestigious research centers. The first such research center is the University of Tilburg in The Netherlands, where I went invited by Professor Peter Borm. I would like to thank his hospitality and the nice atmosphere I found there. During my visit to Tilburg I had the opportunity to share the office with Oriol Tejada Punyol, now at the ETH-Zürich. By that time both of us were PhD students and our talks late in the afternoon gave rise to a fruitful scientific collaboration that still continues. Two of the chapters of this dissertation prove this assertion. I would like to thank Oriol very sincerely for all the interesting discussions that we had and for giving us the opportunity to carry our teamwork out. My second research stay was at the Department of Applied Economics IV of the University of the Basque Country under the supervision of Miren Josune Albizuri Irigoien. Again, I would like to thank her hospitality and the chance to know this active research group. The work done with Josune gave me the opportunity to broaden my research lines and has proved to be very fruitful.

There is a lot of people to whom I am grateful from a professional point of view. First, I would like to thank Alberto Pinto and Flavio Ferreira (from the University of Porto and the Polytechnic Institute of Porto, respectively) for giving me the opportunity to work with them. I am much obliged to them for their hospitality and friendliness. Second, I am also grateful to René van den Brink and Gerard van der Laan (from the Free University of Amsterdam) for their courtesy and cordiality. Most of the results of our collaboration were obtained during a visit of Oriol and me to the Tinbergen Institute of Amsterdam, so all the support from the Institute is gratefully acknowledged. Finally, I would also like to thank all the members of the SaGaTh research group and all the members of the Department of Statistics and Operations Research for all their help and support.

I would also like to mention all the PhD students from the Faculty of Mathematics, in particular all of my office mates.

Finally, I would also like to thank all the nice people I have met during these years in Santiago. Specially, Tito, Sara, Faia, Iván, and Eva, many thanks for your friendship and support. My warmest appreciation goes to my dearest friend Anne, for all the wonderful moments spent together.

Last, but not least, I want to thank my family for being my main emotional support. With all of you close by everything is much easier.

Contents

Notations

This thesis is a collection of independent chapters and most of the notation used is introduced in each of the chapters. However, it is worth introducing the following symbols and notations that are common throughout the thesis.

$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of non negative real numbers
$\emptyset$	The empty set
$T \subseteq S$	T is a subset of S
$T \subsetneq S$	T is a subset of S and T is not equal to S
$T \cup S$	The set of elements which are contained in T or S
$T \cap S$	The set of elements which are contained in T and S
2^N	The set of all subsets of N
$\|S\|$	The number of elements in S
$\mathbb{R}^N$	The $\|N\|$-dimensional real vector space, where the coordinates are indexed by the elements of N
$\Pi(N)$	The set of permutations over the finite set N
$\square$	End of a Proof
$\Diamond$	End of an Example
$\triangleleft$	End of a Remark

Introduction

The goal of Game Theory is the analysis of conflictive situations in which more than one player interact. In such situations the agents or players have different preferences over the outcomes of the game. This research branch studies how rational individuals should behave when they have to face different kinds of conflictive situations. Game Theory classifies such situations in two big groups. A situation is modelled as a non cooperative game when the players do not have mechanisms to make binding agreements before the game is played, in this first group each players' "best" strategies are studied. A situation is modelled as a cooperative game when players have mechanisms to make binding agreements before the game itself is played. The class of cooperative games is divided into transferable and non transferable utility games. We assume that players obtain utility from each possible outcome of the game depending on their preferences. In transferable utility cooperative games, TU games from now on, the utility that players get can be divided and transferred among other players without any loss. A main objective of the research in TU games is the study of values that can be used for different purposes. Most of the times, values are used for sharing the utility obtained from the cooperation. However, there are many situations in which values are used to measure the relative importance of each of the players in the coalition that has emerged.

This dissertation is a collection of contributions to particular classes of TU games. Classes of games that generalize the basic model, the so-called games with restricted cooperation, and simple games, which constitute an important subclass of games. The main contributions presented in this essay concern the proposal and characterization of values in these classes of games. Characterizing a value by means of properties is interesting for at least two reasons. First, it may be more appealing to define a value by means of properties instead of just giving its explicit definition because in this way the features of the value can be summarized. Second, characterization results may help on deciding whether to use one value or another in a particular situation since the properties may have implications which are easy to interpret.

Chapter 1 starts introducing some basic concepts and notation dealing with the basic model of TU games. Next, some of the existing solution concepts are briefly described making an especial emphasis on the Shapley and Banzhaf values. These two values are the basis of most of the subsequent chapters. Chapters 2 through 5 study different types of games with restricted cooperation. In general, games with restricted cooperation are built introducing additional information to enrich the model. This information is represented by some structure that describes the way in which the agents are allowed to cooperate. These games are said to generalize TU games since there is always a trivial structure which indicates that the cooperation is not restricted at all.

Chapter 2 is devoted to the study of games with levels structure of cooperation. The games with levels structure of cooperation generalize the games with a priori unions. A game with a priori unions assumes that the players are organized in groups and that the cooperation among them must "respect" this group structure. Hence, the external information is in this case described by a partition of the set of players. This type of games are proposed in Aumann & Drèze (1974) and there is a vast literature related to them. In Chapter 2 some of the most important results in this framework are summarized first. Next, the model of games with levels structure of cooperation is introduced. This model is proposed by Winter (1989) and the literature related to it is quite limited. The contribution of Chapter 2 is to generalize some of the existing results of games with a priori unions to this more general model. More precisely, a value that generalizes the Banzhaf value is proposed and parallel characterizations of this value and a formerly existing value are presented. By parallel characterizations we mean characterizations that can be compared. Ideal parallel characterizations would share most of the properties. In this way the different properties would highlight the differences between the solution concepts. Finally, an example is proposed to illustrate the use of the studied values. Chapter 2 is the consequence of a joint work with Oriol Tejada and the main results contained on it have already been published in Decision Support Systems (Álvarez-Mozos & Tejada 2011).

Chapter 3 deals with share functions on several classes of games with restricted cooperation. Share functions are first introduced in van der Laan & van den Brink (1998) as an alternative way to study the Shapley and Banzhaf values. More precisely, the Shapley value and the Normalized Banzhaf value. The concept of share functions allows for a common approach to these two values, again highlighting the differences among them. Chapter 3 studies share functions and their generalizations in the classes of games that are considered in Chapter 2. That is, TU games, games with a priori unions, and games with levels structure of cooperation. The main results contained in this chapter are the result of my ongoing collaboration with Oriol Tejada, on the one hand, and René van den Brink and Gerard van der Laan, on the other hand. At the moment an article containing the results presented in this chapter is under the peer-reviewing process of an international journal and a working paper has been published in the reports series of the Department of Statistics and Operations Research of the University of Santiago de Compostela (Álvarez-Mozos et al. 2011).

Another class of games with restricted cooperation that has aroused much interest among game theorists is the class of games with graph restricted communication proposed by Myerson (1977). In Chapter 4 this restriction to the cooperation is considered together with the a priori unions structure. First, some important results of the literature related to the games with graph restricted communication are recalled. Then, games with graph restricted communication and a priori unions are studied. This class of games with restricted cooperation is introduced by Vázquez-Brage et al. (1996), where a generalization of the Shapley value is proposed and characterized. The main contribution of Chapter 4 is to define and characterize two generalizations of the Banzhaf value to this framework. The characterizations ease the comparison of the three values considered in this setting because they use similar properties. The chapter concludes illustrating the values with an example coming from the political field. The results contained in Chapter 4 constitute my first publication and are the main contribution of my MSc thesis. It is a joint work with my supervisors, José M. Alonso-Meijide and M. Gloria Fiestras-Janeiro and has been published in Mathematical Social Sciences (Alonso-Meijide et al. 2009a).

In Chapter 5 we study the model of games with incompatibilities. In this case the restrictions to the cooperation are given by means of a graph which describes the existing incompatibilities among players. To my knowledge, the existing literature on this topic is quite limited. Indeed, only a generalization of the Shapley value has been proposed. The main contribution of this chapter is to propose and characterize a generalization of the Banzhaf value to this class of games. This characterization is comparable with the characterization of the formerly existing value, and hence, it helps to compare both values. The chapter concludes studying a real example coming from the political field. As the previous chapter, the work contained in this chapter is a joint work with my supervisors, José M. Alonso-Meijide and M. Gloria Fiestras-Janeiro. It contains the results of my second article (Alonso-Meijide et al. 2009b) published in Homo Oeconomicus.

Chapter 6 is probably the most different one. This chapter focuses on simple games and power indices. Hence, this time we do not consider that the cooperation among the players is restricted, instead we deal with a particular subclass of TU games. Simple games are mainly used as tools to study decision making bodies, such as Parliaments or Committees. This time we propose and characterize two new power indices. Again, the characterizations allow several power indices to be compared based on the properties satisfied by each of them. This chapter closes with the study of the distribution of power in the Portuguese Parliament. The results contained in this chapter are a joint work with my supervisor José M. Alonso-Meijide on the one hand, and professors Alberto Pinto and Flávio Ferreira, on the other hand. The results contained in this chapter have already been published in the Journal of Difference Equations and Applications (Alonso-Meijide et al. 2011a) and as a chapter in the book Dynamics, Games, and Science II (Alonso-Meijide et al. 2011b). Moreover, another article has been recently submitted for its possible publication in an international journal (Álvarez-Mozos et al. submitted).

1

Preliminaries

This chapter is devoted to the introduction of the basic concepts of the Cooperative Transferable Utility Game Theory that will be used throughout the manuscript. In Section 1.1, the cooperative transferable utility games are introduced and some important properties of them are stated. Section 1.1.1 revises the different approaches that have been proposed in order to obtain solutions to this class of games. Section 1.1.2 recalls with detail the main characterizations of the Shapley and Banzhaf values. In Section 1.2 the family of simple games is formally introduced. Section 1.2.1 concludes introducing power indices related to the Shapley and Banzhaf values.

1.1 Transferable utility games

A *cooperative transferable utility game (game from now on)* is a pair (N, v), where N is the (finite) set of players, and $v : 2^N \to \mathbb{R}_+$ is the characteristic function of the game, which satisfies $v(\emptyset) = 0$. In general, we interpret $v(S)$ as the benefit that S can obtain on its own, i.e., independent to the decisions of players in $N \setminus S$. We denote by $\mathcal{G}^N$ the set of all games with set of players N and by $\mathcal{G}$ the set of all games with any finite set of players.

Note that we demand the worth of every coalition to be non negative, that is, we only consider benefits or savings games. This is done for the sake of exposition but most of the stated results hold also in the case in which some or all coalitions may have a negative worth. To avoid cumbersome notation braces will be omitted whenever it does not lead to confusion, for example we will write $v(S \cup i)$ or $v(S \setminus i)$ instead of $v(S \cup \{i\})$ or $v(S \setminus \{i\})$.

We can define the sum and the scalar product in the set $\mathcal{G}^N$ in the following way. Let $(N, v), (N, w) \in \mathcal{G}^N$ and $\lambda \in \mathbb{R}$.

- The sum game $(N, v + w) \in \mathcal{G}^N$ is defined for every $S \subseteq N$ by $(v + w)(S) = v(S) + w(S)$.
- The scalar product game $(N, \lambda v) \in \mathcal{G}^N$ is defined for every $S \subseteq N$ by $(\lambda v)(S) = \lambda v(S)$.

The set $\mathcal{G}^N$ together with the operations defined above has a vector space structure. The neutral element in this space is the null game $(N, v_0) \in \mathcal{G}^N$, defined for every $S \subseteq N$, by $v_0(S) = 0$. Shapley (1953) shows that this space has dimension $2^n - 1$ and that the family of unanimity games constitutes a basis of it. Given $S \subseteq N$, the unanimity game with carrier S is defined for every $T \subseteq N$ by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Then, any game can be uniquely written as a linear combination of this type of games. In other words, given $(N, v) \in \mathcal{G}^N$, there exists a unique set of scalars $\{\lambda_S \in \mathbb{R}\}_{\emptyset \neq S \subseteq N}$, for which

$$v = \sum_{\emptyset \neq S \subseteq N} \lambda_S u_S.$$

The scalars, λ_S, are known as the Harsanyi dividends (Harsanyi 1959, 1963) and defined for each $S \subseteq N$ by

$$\lambda_S = \sum_{T \subseteq S} (-1)^{s-t} v(T),$$

where s and t are the cardinalities of S and T respectively. Hence, any game can be decomposed in its positive and negative parts (N, v^+) and (N, v^-). Let $(N, v) \in \mathcal{G}$, then $v + v^- = v^+$ where

$$v^+ = \sum_{\substack{\emptyset \neq T \subseteq N \\ \lambda_T > 0}} \lambda_T u_T \quad \text{and} \quad v^- = \sum_{\substack{\emptyset \neq T \subseteq N \\ \lambda_T < 0}} -\lambda_T u_T.$$

There are important subclasses of games that play a crucial role in the literature. Moreover, some of the results presented in the subsequent chapters only hold in some of these classes of games. Each family of games is characterized by a property that constitutes a reasonable requirement in many real situations in which the game is to be used. Next, we enumerate some properties a game may satisfy.

Definition 1.1.1. Let $(N, v) \in \mathcal{G}$ be a game.

- (N, v) is called *additive* if for every $S, T \subseteq N$ such that $S \cap T = \emptyset$,

$$v(S \cup T) = v(S) + v(T).$$

- (N, v) is called *superadditive* if for every $S, T \subseteq N$ such that $S \cap T = \emptyset$,

$$v(S \cup T) \geq v(S) + v(T).$$

- (N, v) is called *monotone* if for every $S \subseteq T \subseteq N$,

$$v(S) \leq v(T).$$

  We denote by $\mathcal{M}^N$ the class of monotone games with set of players N and by $\mathcal{M}$ the set of monotone games with any finite set of players.

- (N, v) is called *convex* if for every $S \subseteq T \subseteq N$,

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T).$$

As one can expect, these classes of games are closely related. Indeed, the implications depicted in Figure 1.1 hold. Additive games are in a sense the

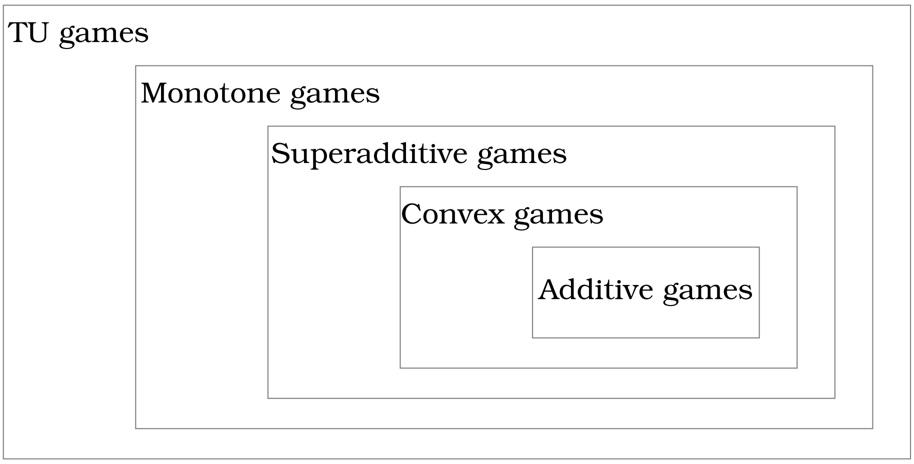

Figure 1.1: Representation of important sub classes of games

most basic type of games. They are defined by an $|N|$-dimensional vector whose coordinates represent the worth of each singleton coalition. The worth of any other coalition is obtained by simply adding up the individual worths of each of its members. Clearly, in such a situation players do not have incentives to form coalitions. In general in superadditive games players have incentives to cooperate, i.e., they benefit from the cooperation with the rest of players. Hence, in superadditive games it can be assumed that the grand coalition N is to be formed. The monotonicity property may be seen as a weakening of the superadditivity property. In a monotone game larger coalitions are worthier. However, in monotone games two coalitions can be better off on their own rather than together. Next, let $i \in N$ and $(N, v) \in \mathcal{G}$, player *i's marginal contribution to coalition* $S \subseteq N \setminus i$ is given by

$$v(S \cup i) - v(S).$$

In convex games the marginal contribution of every player is a monotone function in the sense that it does not decrease when the considered coalition enlarges. That is, in convex games a player's marginal contribution to a coalition is higher the larger the coalition is. The marginal contribution of a player to a coalition measures the change in the worth of the coalition when this player joins it. The class of convex games represents an important subclass of games.

Finally, we formally introduce some type of players which are specially interesting in a given game and which are the basis of many properties that are used to characterize different solution concepts. The particularities of these players are based on how their marginal contributions behave.

Definition 1.1.2. Let $(N, v) \in \mathcal{G}$ be a game.

- Player $i \in N$ is called a *dummy player* in (N, v) if for every $S \subseteq N \setminus i$,

$$v(S \cup i) = v(S) + v(i).$$

- Player $i \in N$ is called a *null player* in (N, v) if it is a dummy player and

$$v(i) = 0.$$

- Players $i, j \in N$ are called *symmetric* in (N, v) if for every $S \subseteq N \setminus \{i, j\}$,

$$v(S \cup i) = v(S \cup j)$$

Hence, a dummy player is a player, i, whose marginal contribution to any coalition coincides with the worth of her stand alone coalition, $v(i)$. That is, she contributes to every coalition with the worth she can obtain on her own. A null player is a player whose marginal contribution to any coalition equals zero. In other words, a null player is a dummy player such that the worth of her stand alone coalition equals zero. Finally, two players are symmetric whenever their marginal contributions to every coalition coincide.

1.1.1 Solution concepts

The situations modelled by a game have a cooperative approach. Therefore an implicit objective in any situation modelled by a game is the grand coalition, N, to be formed, and the generated benefits to be shared among the players. Hence, one of the goals of Cooperative Game Theory is to distribute the worth of the grand coalition, $v(N)$, among the players involved. An allocation is simply a vector $x \in \mathbb{R}^N$, where each coordinate represents the amount allotted to each player.

The aim is to provide a sharing rule which is "admissible" for the players. But, what do we mean by admissible? This is an open question in the literature and many different approaches have been developed in the last years in order to obtain an answer to it. Two of the most accepted principles are individual rationality and efficiency. We say that an allocation $x \in \mathbb{R}^N$ satisfies individual rationality if it allocates, to each player i, at least what she can obtain on her own, i.e., if $v(i) \leq x_i$. An allocation $x \in \mathbb{R}^N$ is efficient if it shares the worth of the grand coalition, i.e., if $\sum_{i \in N} x_i = v(N)$. The allocations which satisfy these two properties are called imputations. In general, a solution concept is a map that associates a set of allocations to every game.

Solution concepts can be classified in two big groups. In the first one we have the set-valued solutions. They are mainly based on stability, i.e, they provide a set of solutions on which players will possibly agree. In other words, this approach discards those payoffs which are not acceptable for a group of players. It depends on the game, but these kind of solutions can be unique, can be a set of different vectors or can even be empty. The most well known such a solution concept is the core, which is introduced by Gillies (1953). The idea behind the core is very simple. It follows a coalitional rationality principle, which states that no coalition should have incentives to break the grand coalition. Other set-valued solution concepts are the stable set (von Neumann & Morgenstern 1944), the bargaining set (Aumann & Maschler 1965), the kernel (Davis & Maschler 1965), the Harsanyi set (Hammer et al. 1977, Vasil'ev 1980), and the Weber set (Weber 1988). The second group is formed by the so called point-valued solution concepts or values. From now on we will focus our attention on them. Each value provides an allocation which is fair in some sense.

A main objective of Cooperative Game Theory is to characterize values by means of reasonable properties. For doing so, first desirable properties that the value satisfies must be identified and then, a set of them must be chosen in such a way that the value is the only one satisfying them. The most popular point-valued solution concept is the Shapley value (Shapley 1953). There exists a vast literature concerning this solution concept and many different characterizations have been provided. The characterizations help us to identify the basic properties that each value satisfies and eases the comparison among the different values. Another popular point-valued solution is the Banzhaf value (Banzhaf 1965). As we will later see in its explicit expression, the Banzhaf value is very similar to the Shapley value. Other point-valued solution concepts include the nucleolus (Schmeidler 1969), the τ-value (Tijs 1981), and the core center (González-Díaz & Sánchez-Rodríguez 2007).

1.1.2 The Shapley and Banzhaf values

This section is devoted to the description and comparison of the Shapley and Banzhaf values. We will first present their explicit definitions and next, review their characterization results. These values are the basis of most of the following chapters.

Although Shapley (1953) introduces his value axiomatically, i.e., he first states a set of desirable properties which a value should satisfy, and then proves that there is only one value satisfying them. Here, we will first give the explicit analytical definitions of both the Shapley and the Banzhaf values, to come to the discussion on the properties later.

A *value on $\mathcal{G}$* is a map f that assigns a vector $\mathsf{f}(N,v) \in \mathbb{R}^N$ to every game $(N,v) \in \mathcal{G}$.

Definition 1.1.3. (Shapley 1953). The *Shapley value*, Sh, is the value on $\mathcal{G}$ defined for every $(N,v) \in \mathcal{G}$ and $i \in N$ by

$$\mathsf{Sh}_i(N,v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \left[v(P_i^\pi \cup i) - v(P_i^\pi) \right],$$

where P_i^π denotes the set of predecessors of i at π, i.e., $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$.

Note that many permutations give rise to the same set of predecessors. It is an easy exercise to count how many times do they repeat and to provide an alternative expression of the Shapley value.

Remark 1.1.4. The *Shapley value*, Sh, is the value on $\mathcal{G}$ defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by

$$\mathsf{Sh}_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left[v(S \cup i) - v(S) \right],$$

where $n = |N|$ and $s = |S|$. $\triangleleft$

Definition 1.1.5. (Owen 1975). The *Banzhaf value*, Ba, is the value on $\mathcal{G}$ defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by

$$\mathsf{Ba}_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} \left[v(S \cup i) - v(S) \right].$$

The explicit forms of the Shapley and Banzhaf values show how both solutions concepts are based on the marginal contributions of a player to the coalitions not containing this player. The difference lies on the coefficients used to weight the addends.

The Shapley and Banzhaf values have a simple probabilistic interpretation. For doing so we will base our explanation on the work by Weber (1988) on probabilistic values. Fix a player $i \in N$, and let $\{p_S^i : S \subseteq N \setminus i\}$ be a probability distribution over the collection of coalitions not containing i. A value on $\mathcal{G}$, f, is a probabilistic value if for every $(N, v) \in \mathcal{G}$ and $i \in N$,

$$\mathsf{f}_i(N, v) = \sum_{S \subseteq N \setminus i} p_S^i \left[v(S \cup i) - v(S) \right].$$

Let i view her participation in a game as consisting merely of joining some coalition $S \subseteq N \setminus i$, and then receiving as a reward her marginal contribution to this coalition, $v(S \cup i) - v(S)$. If, for each $S \subseteq N \setminus i$, p_S^i is the (subjective) probability that she joins coalition S, then $\mathsf{f}_i(N, v)$ is simply i's expected payoff from participating in the game.

As it can be seen in Remark 1.1.4 and Definition 1.1.5, both Sh and Ba are instances of probabilistic values. The Banzhaf value arises from the subjective belief that each player is equally likely to join any coalition, that is, $p_S^i = \frac{1}{2^{n-1}}$ for all $S \subseteq N \setminus i$. On the other hand, the Shapley value arises from the belief that for every player, the coalition she joins is equally likely to be of any size $s \in \{0, \dots, n-1\}$ and that all coalitions of a given size are equally likely. That is, for every $S \subseteq N \setminus i$ such that $|S| = s$,

$$p_S^i = \frac{1}{n} \binom{n-1}{s}^{-1} = \frac{s!(n-s-1)!}{n!}.$$

To end with this section we state different characterizations of both the Shapley and the Banzhaf values. For doing this we need to define some properties a value on $\mathcal{G}$, f, could be asked to satisfy. At this point it is worth to make a note on the way the properties are denoted throughout the whole document. Small capital letters are used to depict an acronym for each property. Moreover, since similar properties are used in different frameworks, each acronym is preceded with some calligraphic letters that represent the class of games on which it applies.

$\mathcal{G}$:EFF A value on $\mathcal{G}$, f, satisfies *efficiency* if for every $(N, v) \in \mathcal{G}$,

$$\sum_{i \in N} \mathsf{f}_i(N, v) = v(N).$$

$\mathcal{G}$:DPP A value on $\mathcal{G}$, f, satisfies the *dummy player property* if for every $(N, v) \in \mathcal{G}$ and each dummy player $i \in N$ in (N, v),

$$\mathsf{f}_i(N, v) = v(i).$$

$\mathcal{G}$:NPP A value on $\mathcal{G}$, f, satisfies the *null player property* if for every $(N, v) \in \mathcal{G}$ and each null player $i \in N$ in (N, v),

$$\mathsf{f}_i(N, v) = 0.$$

$\mathcal{G}$:SYM A value on $\mathcal{G}$, f, satisfies *symmetry* if for every $(N, v) \in \mathcal{G}$ and each pair of symmetric players $i, j \in N$ in (N, v),

$$\mathsf{f}_i(N, v) = \mathsf{f}_j(N, v).$$

$\mathcal{G}$:ANO A value on $\mathcal{G}$, f, satisfies *anonymity* if for every $(N, v) \in \mathcal{G}$ and $\pi \in \Pi(N)$,

$$\mathsf{f}_{\pi(i)}(N, v) = \mathsf{f}_i(N, \pi v),$$

where the game $(N, \pi v)$ is defined for every $S \subseteq N$ by $\pi v(S) = v(\pi(S))$.

$\mathcal{G}$:ADD A value on $\mathcal{G}$, f, satisfies *additivity* if for every pair $(N, v), (N, w) \in \mathcal{G}$,

$$\mathsf{f}(N, v + w) = \mathsf{f}(N, v) + \mathsf{f}(N, w).$$

$\mathcal{G}$:TRP A value on $\mathcal{G}$, f, satisfies the *transfer property* if for every pair of games $(N, v), (N, w) \in \mathcal{G}$,

$$\mathsf{f}(N, v) + \mathsf{f}(N, w) = \mathsf{f}(N, v \vee w) + \mathsf{f}(N, v \wedge w),$$

where $(N, v \vee w), (N, v \wedge w) \in \mathcal{G}$ are defined for every $S \subseteq N$ by $(v \vee w)(S) = \max\{v(S), w(S)\}$ and $(v \wedge w)(S) = \min\{v(S), w(S)\}$.

$\mathcal{G}$:2-EFF A value on $\mathcal{G}$, f, satisfies *2-efficiency* if for every $(N, v) \in \mathcal{G}$ and each pair of players $i, j \in N$,

$$\mathsf{f}_i(N, v) + \mathsf{f}_j(N, v) = \mathsf{f}_p(N^{ij}, v^{ij}),$$

where (N^{ij}, v^{ij}) is the $\{ij\}$-*merged game* obtained from (N, v) when players i and j merge in a new player $p = \{i, j\}$, i.e., $N^{ij} = (N \setminus \{i, j\}) \cup \{p\}$ and for every $S \subseteq N^{ij}$

$$v^{ij}(S) = \begin{cases} v(S) & \text{if } p \notin S \\ v((S \setminus p) \cup \{i, j\}) & \text{if } p \in S \end{cases}$$

$\mathcal{G}$:2-EFF* A value on $\mathcal{G}$, f, satisfies *2-efficiency** if for every $(N, v) \in \mathcal{G}$ and each pair of players $i, j \in N$,

$$\mathsf{f}_i(N, v) + \mathsf{f}_j(N, v) \leq \mathsf{f}_p(N^{ij}, v^{ij}).$$

$\mathcal{G}$:2-AEF A value on $\mathcal{G}$, f, satisfies *2-amalgamation efficiency* if for every $(N, v) \in \mathcal{G}$,

$$\mathsf{f}_i(N \setminus j, v_{i \triangleleft j}) = \mathsf{f}_i(N, v) + \mathsf{f}_j(N, v),$$

where $(N \setminus j, v_{i \triangleleft j})$ is the $\{i \triangleleft j\}$-*amalgamation game* obtained from (N, v) when player j leaves the game and delegates her role to player i, i.e., for every $S \subseteq N \setminus j$,

$$v_{i \triangleleft j}(S) = \begin{cases} v(S) & \text{if } i \notin S \\ v(S \cup j) & \text{if } i \in S \end{cases}$$

$\mathcal{G}$:TPP A value on $\mathcal{G}$, f, satisfies the *total power property* if for every $(N, v) \in \mathcal{G}$,

$$\sum_{i \in N} \mathsf{f}_i(N, v) = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{S \subseteq N \setminus i} [v(S \cup i) - v(S)] = \sum_{S \subseteq N} (2s - n) v(S).$$

$\mathcal{G}$:SMO A value on $\mathcal{G}$, f, satisfies *strong monotonicity* if for every pair of games $(N,v),(N,w) \in \mathcal{G}$ and each $i \in N$ such that for all $S \subseteq N \setminus i$, $v(S \cup i) - v(S) \leq w(S \cup i) - w(S)$,

$$\mathsf{f}_i(N,v) \leq \mathsf{f}_i(N,w).$$

$\mathcal{G}$:EMC A value on $\mathcal{G}$, f, satisfies *equal marginal contributions* if for every pair of games $(N,v),(N,w) \in \mathcal{G}$ and each $i \in N$ such that for all $S \subseteq N \setminus i$, $v(S \cup i) - v(S) = w(S \cup i) - w(S)$,

$$\mathsf{f}_i(N,v) = \mathsf{f}_i(N,w).$$

A value is efficient (satisfies efficiency) if it completely shares the worth of the grand coalition, $v(N)$, among the players. When a value is used for sharing purposes, $\mathcal{G}$:EFF is an essential requirement.

$\mathcal{G}$:DPP states that a player whose marginal contribution to any coalition is always the worth she can obtain on her own, $v(i)$, should be allotted exactly that amount. $\mathcal{G}$:NPP is just $\mathcal{G}$:DPP only required for null players.

A value is symmetric (or satisfies symmetry) when it allocates the same amount to players whose marginal contribution to every coalition coincide. The anonymity property states that the amount that a player receives does not depend on her label or relative position inside N. $\mathcal{G}$:ANO implies $\mathcal{G}$:SYM while the reverse does not hold in general.

$\mathcal{G}$:ADD is a standard property in the literature, even if it has been criticized for the use of the sum game. It states that the payoff of the sum game equals the sum of payoffs in the original games. $\mathcal{G}$:TRP avoids the use of the sum game but it is very similar to $\mathcal{G}$:ADD.

The $\mathcal{G}$:2-EFF and $\mathcal{G}$:2-AEF properties, as well as $\mathcal{G}$:2-EFF*, describe how a value should behave when two players of the original game are not allowed to act independently anymore. Indeed, $\mathcal{G}$:2-EFF and $\mathcal{G}$:2-AEF state that a value should be immune against such changes. In a sense, they are quite similar properties which has lead some authors to identify them as equivalent properties (see Casajus (to appear) for instance). However, note that the $\{ij\}$-merged game and the $\{i \triangleleft j\}$-amalgamation game have different players' sets, and hence, they are different games. The difference lies on the name given to the merged or amalgamated player, in one case it is new player, p, which acts on behalf of players i and j, in the other case one player i stays in the game and player j leaves delegating her role to player i. Consequently, $\mathcal{G}$:2-EFF and $\mathcal{G}$:2-AEF are different properties.

This confusion between $\mathcal{G}$:2-EFF and $\mathcal{G}$:2-AE is addressed in Alonso-Meijide et al. (submitted). Using Theorem 1 of Casajus (to appear) it can be easily seen that $\mathcal{G}$:2-AEF implies $\mathcal{G}$:2-EFF. Nevertheless, the reverse implication does not hold as the following example shows.

Example 1.1.6. Let a, b be two distinct, fixed, and indivisible players. By indivisible we mean that there is no pair of players i, j such that $\{i, j\} = a$ or $\{i, j\} = b$. That is, a and b are not obtained after a merging of two players[1]. Let g be the value defined for every $(N, v) \in \mathcal{G}$ by

- If $N = \{a, b\}$, $\begin{cases} \mathsf{g}_a(N,v) = \frac{3}{4}\left[v(N) - v(b)\right] + \frac{1}{4}v(a) \\ \mathsf{g}_b(N,v) = \frac{1}{4}\left[v(N) - v(a)\right] + \frac{3}{4}v(b) \end{cases}$
- Otherwise $\mathsf{g}(N, v) = \mathsf{Ba}(N, v)$.

Then, g satisfies $\mathcal{G}$:2-EFF but not $\mathcal{G}$:2-AEF. To show that g does not satisfy $\mathcal{G}$:2-AEF, take $N = \{a, b, c\}$ and $v = u_{\{a,b\}}$. Then,

$$\begin{aligned} \mathsf{g}_a(N,v) + \mathsf{g}_c(N,v) &= \mathsf{Ba}_a(N,v) + \mathsf{Ba}_c(N,v) = \frac{1}{2} \qquad \text{and} \\ \mathsf{g}_a(N \setminus c, v_{a \lhd c}) &= \frac{3}{4}\left[v(N) - v(b)\right] + \frac{1}{4}v(a) = \frac{3}{4}. \end{aligned}$$

Finally, g satisfies $\mathcal{G}$:2-EFF. The proof is straightforward taking into account that a and b are singletons and hence, it is not possible to have a pair of players i, j such that $a = \{i, j\}$ or $b = \{i, j\}$. ◊

The $\{ij\}$-merged game is first introduced by Lehrer (1988) and used to propose the $\mathcal{G}$:2-EFF* property. Observe that $\mathcal{G}$:2-EFF* is just a weaker version of $\mathcal{G}$:2-EFF. It states that a value that satisfies it is immune against the artificial splitting of a player in two new players. To our knowledge the $\{i \lhd j\}$-amalgamation game is introduced in Casajus (to appear).

$\mathcal{G}$:TPP establishes that the total payoff obtained by the players is the sum of all marginal contributions of every player normalized by 2^{n-1}. Depending on the particular game this amount may be more, less or equal to $v(N)$.

The last two properties, $\mathcal{G}$:SMO and $\mathcal{G}$:EMC are logically related since $\mathcal{G}$:SMO implies $\mathcal{G}$:EMC. They link the payoffs of two games with the differences between the marginal contributions of the aforementioned games.

There is a vast literature concerning characterizations of the Shapley and Banzhaf values by means of properties. Next, we present the main such char-

[1]The concept of indivisible players will be used throughout this dissertation with slightly different meanings. However, the intuition behind it is always the same. An indivisible player is and has always been an individual player.

acterization results. In Shapley (1953) the Shapley value is introduced in an axiomatic way.

Theorem 1.1.7. (Shapley 1953). *The Shapley value,* Sh, *is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:EFF, $\mathcal{G}$:NPP, $\mathcal{G}$:SYM, *and* $\mathcal{G}$:ADD.

In Young (1985) the Shapley value is characterized without $\mathcal{G}$:ADD property, which is the most criticized one among the properties used in the characterization by Shapley.

Theorem 1.1.8. (Young 1985). *The Shapley value,* Sh, *is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:EFF, $\mathcal{G}$:SYM, *and* $\mathcal{G}$:SMO *(or* $\mathcal{G}$:EMC*).*

One can easily check that if $\mathcal{G}$:SMO is replaced by $\mathcal{G}$:EMC the uniqueness of the characterization above still holds. In general, characterizations by means of properties are tighter the weaker the properties are. Thus, characterizations by means of weaker properties are preferable. Henceforth, we will refer to the characterization of the Shapley value by Young as the characterization by means of $\mathcal{G}$:EFF, $\mathcal{G}$:SYM, and $\mathcal{G}$:EMC.

Feltkamp (1995) presents parallel characterizations of both the Shapley and Banzhaf values. The properties used in the characterizations either coincide or are alike. In this way a comparison of the properties that each solution concept satisfies can be easily done.

Theorem 1.1.9. (Feltkamp 1995). *The Shapley value,* Sh, *is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:EFF, $\mathcal{G}$:NPP, $\mathcal{G}$:ANO, *and* $\mathcal{G}$:TRP.

The first characterization of the Banzhaf value which makes use of $\mathcal{G}$:SYM, $\mathcal{G}$:ADD and $\mathcal{G}$:DPP is stated in Lehrer (1988). In this theorem Shapley's $\mathcal{G}$:EFF property is substituted by $\mathcal{G}$:2-EFF*.

Theorem 1.1.10. (Lehrer 1988). *The Banzhaf value,* Ba, *is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:2-EFF*, $\mathcal{G}$:NPP, $\mathcal{G}$:SYM, *and* $\mathcal{G}$:ADD.

Theorem 1.1.11. (Feltkamp 1995). *The Banzhaf value,* Ba, *is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:TPP, $\mathcal{G}$:NPP, $\mathcal{G}$:ANO, *and* $\mathcal{G}$:TRP.

Nowak (1997) shows that the Banzhaf value actually satisfies $\mathcal{G}$:2-EFF property, which is a strengthening of $\mathcal{G}$:2-EFF* property used in Theorem 1.1.10.

Theorem 1.1.12. (Nowak 1997). *The Banzhaf value,* Ba, *is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:2-EFF, $\mathcal{G}$:DPP, $\mathcal{G}$:SYM, *and* $\mathcal{G}$:EMC.

In a recent work Casajus (to appear) provides a new characterization of the Banzhaf value by means of only two properties.

Theorem 1.1.13. (Casajus to appear). *The Banzhaf value,* Ba*, is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:2-AEF *and* $\mathcal{G}$:DPP.

Another recent work which provides a characterization of the Banzhaf value is Lorenzo-Freire et al. (2007).

Theorem 1.1.14. (Lorenzo-Freire et al. 2007). *The Banzhaf value,* Ba*, is the unique value on* $\mathcal{G}$ *satisfying* $\mathcal{G}$:TPP*,* $\mathcal{G}$:SYM*, and* $\mathcal{G}$:SMO.

From the results presented above, four parallel characterizations may be considered, more precisely, Theorems 1.1.7 versus 1.1.10, 1.1.9 versus 1.1.11, 1.1.8 versus 1.1.12, and 1.1.8 versus 1.1.13. By parallel we mean characterizations of Sh and Ba by means of similar sets of properties. In this case this similarity is a consequence of having some of the properties in common and hence, the differences are restricted to one or two properties. These parallel characterizations are depicted in Table 1.1

	Sh	Ba	
Shapley (1953)	$\mathcal{G}$:EFF $\mathcal{G}$:NPP $\mathcal{G}$:SYM $\mathcal{G}$:ADD	$\mathcal{G}$:2-EFF* $\mathcal{G}$:NPP $\mathcal{G}$:SYM $\mathcal{G}$:ADD	Lehrer (1988)
Feltkamp (1995)	$\mathcal{G}$:EFF $\mathcal{G}$:NPP $\mathcal{G}$:ANO $\mathcal{G}$:TRP	$\mathcal{G}$:TPP $\mathcal{G}$:NPP $\mathcal{G}$:ANO $\mathcal{G}$:TRP	Feltkamp (1995)
Young (1985)	$\mathcal{G}$:EFF $\mathcal{G}$:SYM $\mathcal{G}$:EMC	$\mathcal{G}$:2-EFF $\mathcal{G}$:DPP $\mathcal{G}$:SYM $\mathcal{G}$:EMC	Nowak (1997)
Young (1985)	$\mathcal{G}$:EFF $\mathcal{G}$:SYM $\mathcal{G}$:EMC	$\mathcal{G}$:2-AEF $\mathcal{G}$:DPP	Casajus (to appear)

Table 1.1: Parallel characterizations of Sh and Ba

As it can be seen, there are few but important differences between Sh and Ba. The Shapley value is efficient while the Banzhaf value is not. The Banzhaf value divides the amount indicated by the total power property. The last property that makes the difference between Sh and Ba is the 2-efficiency. The Banzhaf value satisfies it while the Shapley value does not (in fact it does not satisfy the weaker 2-efficiency*). In many situations the efficiency is a basic requirement. However there are situations in which a value is not used for sharing purposes. For instance, we may use a value to compare the possibilities of the players to be influential in a given situation, in such a situation the efficiency is less crucial. Hence, the decision whether to use Sh or Ba depends on the situation under study.

If we want to use the Banzhaf value for sharing purposes we may re-scale it to guarantee that the sum of the payoffs equals $v(N)$. However, as has been argued by several authors, such a normalization is not as innocent as it seems (Dubey & Shapley 1979). In particular, the probabilistic interpretation of the Banzhaf value is lost and the dummy player and additivity properties are violated. We present the formal definition of the normalized Banzhaf value next.

Definition 1.1.15. (van den Brink & van der Laan 1998a). The *normalized Banzhaf value*, $\overline{\mathsf{Ba}}$, is a value on $\mathcal{G}$ defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by

$$\overline{\mathsf{Ba}}_i(N, v) = \frac{\mathsf{Ba}_i(N, v)}{\sum_{j \in N} \mathsf{Ba}_j(N, v)} v(N), \qquad \text{if } (N, v) \neq (N, v_0)$$

and $\overline{\mathsf{Ba}}_i(N, v_0) = 0$.

A characterization of the normalized Banzhaf value can be found in van den Brink & van der Laan (1998a) for the class of monotone games. The value is characterized by means of efficiency, null player out, additive game, independence of irrelevant permutations, and proportional proxy agreement properties. The last two properties may also be replaced by independence of irrelevant unanimity replacements and unanimity proxy properties which are slightly weaker. However, all properties but efficiency are quite different from the ones seen above in characterizations of both the Shapley and Banzhaf values, and hence, their formal definitions are omitted. The interested reader is referred to van den Brink & van der Laan (1998a) and van den Brink & van der Laan (1998b).

An interesting way to avoid the "efficiency issue" is to consider share functions. The concept of share functions is first introduced in a working paper back in 1995 which gave rise to van der Laan & van den Brink (1998). A share vector for a game $(N, v) \in \mathcal{G}$ is an $|N|$-dimensional real vector, $\rho \in \mathbb{R}^N$ such that, $\sum_{i \in N} \rho_i = 1$ and for every $i \in N$ $\rho_i \geq 0$. ρ_i represents player i's share of the worth to be distributed. By a *share function on* $\mathcal{G}$ we mean a map, ρ, that assigns to every game $(N, v) \in \mathcal{G}$ a share vector $\rho(N, v) \in \{x \in \mathbb{R}^N : \text{for every } i \in N, x_i \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$. Hence, in every game $(N, v) \in \mathcal{G}$, a share function ρ gives a payoff $\rho_i(N,v)v(N)$ to player $i \in N$. In the literature share functions are considered on the subclass of monotone games, which covers the most interesting situations modeled by games. The reason is that we want negative shares to be discarded. Consequently a share function for monotone games may be obtained from each value on $\mathcal{G}$ just by restricting it to $\mathcal{M}$ and normalizing it. Then, a *share function on* $\mathcal{M}$ is a map, ρ, that assigns to every monotone game $(N,v) \in \mathcal{M}$ a share vector $\rho(N,v) \in \{x \in \mathbb{R}^N : \text{ for every } i \in N, \rho_i(N,v) \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$. We denote by $\mathcal{M}^+$ the set of monotone games different from the null game, i.e., $\mathcal{M}^+ = \{(N,v) \in \mathcal{M} : v \neq v_0\}$.

The literature related to share functions includes van den Brink & van der Laan (2001), where the core is studied using share functions, and van den Brink & van der Laan (2007), where the concepts of potentials and reduced games are modified for share functions. Chapter 3 deals with share functions on different classes of games.

Next, we introduce the share functions associated with the Shapley and Banzhaf values.

Definition 1.1.16. The *Shapley share function*, ρ^{Sh}, is a share function on $\mathcal{M}$ defined for every $(N,v) \in \mathcal{M}$ and $i \in N$ by

$$\rho_i^{\mathsf{Sh}}(N,v) = \frac{\mathsf{Sh}_i(N,v)}{v(N)} \quad \text{if } (N,v) \in \mathcal{M}^+,$$

and $\rho_i^{\mathsf{Sh}}(N,v_0) = \frac{1}{n}$.

Definition 1.1.17. The *Banzhaf share function*, ρ^{Ba}, is a share function on $\mathcal{M}$ defined for every $(N,v) \in \mathcal{M}$ and $i \in N$ by

$$\rho_i^{\mathsf{Ba}}(N,v) = \frac{\mathsf{Ba}_i(N,v)}{\sum_{j \in N} \mathsf{Ba}_i(N,v)} = \frac{\overline{\mathsf{Ba}}_i(N,v)}{v(N)} \quad \text{if } (N,v) \in \mathcal{M}^+,$$

and $\rho_i^{\mathsf{Ba}}(N,v_0) = \frac{1}{n}$.

1.2 Simple games

In this section an important subclass of games is introduced and some basic results in this framework are revised. The origin of the idea behind simple games dates back to von Neumann & Morgenstern (1944). However, the concept is redefined by Shapley (1962) and most of the work done on the subject is based on this later definition. One of the most important features of simple games steams from its applications. In fact, simple games are widely accepted tools to model decision making bodies, like Parliaments or Committees. Therefore, these games are a main objective of Social Choice and very useful in Political Sciences. Felsenthal & Machover (1998) constitutes a good survey on the topic.

A *simple game* is a non-null monotone game such that the worth of every coalition is either 0 or 1. Formally, $(N, v) \in \mathcal{G}$ is a simple game if and only if

- $(N, v) \in \mathcal{M}$,
- for every $S \subseteq N$, $v(S) \in \{0, 1\}$, and
- $v(N) = 1$.

We denote by $\mathcal{SG}$ the set of all simple games.

In a simple game $(N, v) \in \mathcal{SG}$, a coalition $S \subseteq N$ is *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$. $W(v)$ denotes the set of winning coalitions of the simple game (N, v) and, given $i \in N$, $W_i(v)$ denotes the subset of $W(v)$ formed by coalitions containing player i, i.e., $W_i(v) = \{S \in W(v) : i \in S\}$. Given a simple game $(N, v) \in \mathcal{SG}$, a *swing* for a player $i \in N$ is a coalition $S \subseteq N$ such that $i \in S$, S is a winning coalition, and $S \setminus i$ is a losing coalition. The set of all swings for player $i \in N$ is denoted by $\eta_i(v)$. Any simple game $(N, v) \in \mathcal{SG}$ may be determined by its set of winning coalitions $W(v)$. Given a player set N and an arbitrary family of coalitions $W \subseteq 2^N$, we abuse notation slightly and write $(N, W) \in \mathcal{SG}$ if

- $\emptyset \notin W$,
- $N \in W$, and
- for every $S \subseteq T \subseteq N$, if $S \in W$ then $T \in W$.

A winning coalition $S \in W(v)$ is a *minimal winning* coalition if every proper subset of S is a losing coalition, that is, S is a minimal winning coalition in (N, v) if $v(S) = 1$ and $v(T) = 0$ for every $T \subsetneq S$. $W^m(v)$ denotes the set of minimal winning coalitions of the game (N, v) and $W_i^m(v)$ the subset of $W^m(v)$ formed by coalitions containing player i, i.e., $W_i^m(v) = \{S \in W^m(v) : i \in S\}$. Similar to the case of winning coalitions, a simple game may also be defined by its set of minimal winning coalitions $W^m(v)$. Given a player set N and an arbitrary family of coalitions $W^m \subseteq 2^N$, we abuse notation slightly and write $(N, W^m) \in \mathcal{SG}$ if

- $\emptyset \notin W^m$,
- $W^m \neq \emptyset$, and
- for every $S, T \in W^m$, $S \not\subset T$ and $T \not\subset S$.

It is easy to obtain the set of minimal winning coalitions from the set of winning coalitions and vice versa, i.e.,

$$W^m(v) = \{S \in W(v) : \text{for every } T \subsetneq S, \quad T \notin W(v)\},$$
$$W(v) = \{S \subseteq N : \text{there is } T \subseteq S, \quad T \in W^m(v)\}.$$

1.2.1 Power indices

When simple games are considered, the concept of value for general games is known as power index. As mentioned before, simple games and power indices constitute one of the most fruitful application of Mathematics to Social Sciences. Therefore, a vast literature exists and many power indices have been introduced so far. The features of each solution concept have been studied and the different power indices have been compared. In Chapter 6 simple games and power indices are taken back again, in order to present a proposal of two new power indices that have not been considered yet.

In this setting, power is understood as the ability of a player to influence the outcome of a ballot. A *power index* is a map f that assigns a vector $\mathsf{f}(N, v) \in \mathbb{R}^N$ to every simple game $(N, v) \in \mathcal{SG}$. Since simple games are a subclass of games, each value on $\mathcal{G}$ can be restricted to simple games giving rise to a power index. In the definitions below the restrictions of Sh and Ba to simple games are presented.

Definition 1.2.1. (Shapley & Shubik 1954). The *Shapley-Shubik power index*, SS, is a power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by

$$\mathsf{SS}_i(N, v) = \sum_{S \in \eta_i(v)} \frac{s!(n-s-1)!}{n!},$$

where $n = |N|$ and $s = |S|$.

Definition 1.2.2. (Banzhaf 1965, Coleman 1971, Penrose 1946). The *Penrose-Banzhaf-Coleman power index*, PBC, is a power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by

$$\mathsf{PBC}_i(N, v) = \frac{|\eta_i(v)|}{2^{n-1}},$$

where $n = |N|$.

It is worth to make a comment on the origin of these power indices. On the one hand, the Shapley-Shubik power index is introduced as a direct application of the Shapley value (introduced one year earlier) to simple games. On the other hand, the Penrose-Banzhaf-Coleman power index is introduced independently in the three papers cited above. Penrose (1946) constitutes one of the first properly scientific proposal to measure the a priori voting power, but it lay unnoticed by the scientific community for decades. In Banzhaf (1965), the ideas used by Penrose are reconsidered. Finally, Coleman (1971) measures the a priori voting power using the PBC power index apparently unaware of the previous works. Few years later, the PBC power index is generalized to the whole class of games by Owen (1975) and named the Banzhaf value (see Definition 1.1.5). The interested reader is referred to Felsenthal & Machover (2005). In Laruelle & Valenciano (2008) a critical survey of the literature is done reflecting in some sense the difficulties that arise when it comes to measure the power.

The power indices defined above have an interesting probabilistic interpretation that is explained next following Straffin (1988). Assume that p_i is the probability that player i votes in favour of a bill and that this probability follows a uniform distribution on $[0,1]$. In such a situation the Shapley-Shubik index is the probability of player i's vote to change the result under the homogeneity assumption, i.e., if $p_i = p$ for every $i \in N$. On the other hand, the Penrose-Banzhaf-Coleman index is the same probability under the independence assumption, i.e., p_i and p_j are independent for every $i \neq j \in N$.

The characterization results of Section 1.1.2 may not hold for the class of simple games. In particular those characterizations that use additivity do not hold in $\mathcal{SG}$. The problem is that the sum of two simple games is never a simple game, and hence, the additivity property becomes vacuous. To overcome this difficulty, additivity can be replaced by the transfer property. Note that if $(N,v),(N,w) \in \mathcal{SG}$, then $(N, v \wedge w),(N, v \vee w) \in \mathcal{SG}$. In order to present characterizations of SS and PBC formally, some of the properties stated in Section 1.1.2 need to be redefined for this context.

$\mathcal{SG}$:EFF A power index, f, satisfies *efficiency* if for every $(N,v) \in \mathcal{SG}$,

$$\sum_{i\in N} \mathsf{f}_i(N,v) = 1.$$

$\mathcal{SG}$:NPP A power index, f, satisfies the *null player property* if for every $(N,v) \in \mathcal{SG}$ and each null player $i \in N$ in (N,v),

$$\mathsf{f}_i(N,v) = 0.$$

$\mathcal{SG}$:SYM A power index, f, satisfies *symmetry* if for every $(N, v) \in \mathcal{SG}$ and each pair of symmetric players $i, j \in N$ in (N, v),

$$\mathsf{f}_i(N, v) = \mathsf{f}_j(N, v).$$

$\mathcal{SG}$:TRP A power index, f, satisfies the *transfer property* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$,

$$\mathsf{f}(N, v) + \mathsf{f}(N, w) = \mathsf{f}(N, v \vee w) + \mathsf{f}(N, v \wedge w).$$

$\mathcal{SG}$:TPP A power index, f, satisfies the *total power property* if for every simple game $(N, v) \in \mathcal{SG}$,

$$\sum_{i \in N} \mathsf{f}_i(N, v) = \frac{\sum_{i \in N} |\eta_i(v)|}{2^{n-1}}.$$

Next, in line with the characterizations of Sh and Ba presented in Section 1.1.2, parallel characterizations of SS and PBC are presented.

Theorem 1.2.3. (Dubey 1975) *The Shapley-Shubik power index,* SS*, is the unique power index satisfying* $\mathcal{SG}$:EFF*,* $\mathcal{SG}$:SYM*,* $\mathcal{SG}$:NPP*, and* $\mathcal{SG}$:TRP*.*

Theorem 1.2.4. (Dubey & Shapley 1979) *The Penrose-Banzhaf-Coleman power index,* PBC*, is the unique power index satisfying* $\mathcal{SG}$:TPP*,* $\mathcal{SG}$:SYM*,* $\mathcal{SG}$:NPP*, and* $\mathcal{SG}$:TRP*.*

As mentioned in Section 1.1.2, the main difference between Sh and Ba is that the former is efficient while the later satisfies the total power property. The characterizations above show that this difference is transferred when simple games are considered. Hence, the main difference between SS and PBC is that the former is efficient while the latter satisfies the total power property. Observe that when the goal is to compare the strength of two players the efficiency may not be essential. Besides, if in a voting body, the unanimity is needed to reach an agreement, the voting body itself would have less power than if we consider the majority rule, because it would be more difficult to make a decision. An efficient power index can make no difference between these two situations. As before, the parallel characterizations stated in Theorems 1.2.3 and 1.2.4 are summarized in Table 1.2.

Finally, it is worth to mention several papers that propose different characterizations of SS and PBC as an example of the large literature existing in this topic. See for instance, Owen (1978), Haller (1994), Albizuri & Ruiz (2001), Laru-

Dubey (1975)	SS	PBC	Dubey & Shapley (1979)
	$\mathcal{SG}$:EFF	$\mathcal{SG}$:TPP	
	$\mathcal{SG}$:NPP	$\mathcal{SG}$:NPP	
	$\mathcal{SG}$:SYM	$\mathcal{SG}$:SYM	
	$\mathcal{SG}$:TRP	$\mathcal{SG}$:TRP	

Table 1.2: Parallel characterizations of SS and PBC

elle & Valenciano (2001), and Barua et al. (2005). We omit the formal statement of these results for the shake of brevity.

2

Games with levels structure of cooperation

In the game theoretical models introduced in Chapter 1 there is no restriction to the cooperation, and the game is defined by the worth that any coalition can obtain on its own. However, there are many real situations in which there is a priori information about the behavior of the players or there are environmental restrictions and only partial cooperation occurs. Different approaches have been used to address this type of situations and different models of games with restricted cooperation have been studied so far.

Aumann & Drèze (1974) consider that the restrictions to the cooperation are given by a partition of the set of agents. This partition is capable of modelling the affinities among agents. The model including a game and such a partition is called a game with a priori unions. For this family of games, Owen (1977) proposes and characterizes a modification of the Shapley value to allocate the total gains, the so called Owen value. This value initially splits the total amount among the unions, according to the Shapley value in the induced game played by the unions (quotient game). Then, once again using the Shapley value within each union, its total reward is allocated among its members (quotient game property), taking into account their possibilities of joining other unions. Owen (1982) defines a modification of the Banzhaf value following a similar procedure, known as the Banzhaf-Owen value. The first characterization of the Banzhaf-Owen value is proposed in Amer et al. (2002). As argued in Amer et al. (2002) the Banzhaf-Owen value does not satisfy two interesting properties: symmetry among unions and the quotient game property. In order to solve such a drawback, Alonso-Meijide & Fiestras-Janeiro (2002) define and characterize the Symmetric coalitional Banzhaf value, a different modification of the Banzhaf value, that satisfies the two properties considered above. The Symmetric coalitional

Banzhaf value uses the Banzhaf value to allocate the payoff among the unions and the Shapley value to split this payoff within the members of each union. In Alonso-Meijide et al. (2007), a comparison among the three aforesaid values is presented.

Winter (1989) goes one step beyond by introducing games with many levels of cooperation, which extends the model of games with a priori unions. He proposes and characterizes an extension of the Owen value for this kind of situations, which we will call the Shapley levels value. As before, players are assumed to be organized in groups to bargain for the division of the worth available (first level of cooperation). Nevertheless, this time the formed unions may again organize themselves in larger groups (second level of cooperation) while they maintain their internal obligations of the first level, and so on and so forth. Hence, this time the restrictions to the cooperation are described by a sequence of partitions of the player set, each of them being coarser than the previous ones. Calvo et al. (1996) give an alternative characterization of the Shapley levels value using a balanced contributions property and Vidal-Puga (2005) implements the Shapley levels value in a subgame perfect equilibrium of a particular bidding mechanism. More recently, Alonso-Meijide & Carreras (2011) propose the so called Proportional coalitional Shapley value, which is a value for games with a priori unions and show that it can be easily extended to the levels structure framework. This new value also follows a two steps procedure. As the Owen value does, it shares according to the Shapley value among the unions but in a second step it shares the amount alloted to each union according to the Shapley values of its members in the original game.

This chapter is the consequence of a joint work with Oriol Tejada from ETH-Zürich and the main results contained here have been published in Decision Support Systems (Álvarez-Mozos & Tejada 2011). The remaining part of the chapter is organized as follows. In Section 2.1 we recall the model of games with a priori unions and the main results concerning the Owen, the Banzhaf-Owen, and the Symmetric coalitional Banzhaf values. In Section 2.2 the model of games with levels structure of cooperation is introduced. The main results of the literature concerning this model are revised and the so called Banzhaf levels value is introduced. Section 2.2.1 proceeds with the characterization of two values for this class of games. Finally, in Section 2.2.2 an example is presented to illustrate the studied values.

2.1 Games with a priori unions and values on $\mathcal{GU}$

Let us consider a finite set of agents, say, $N = \{1, \dots, n\}$. We denote the set of all partitions of N by $P(N)$. Each $P = \{P_1, \dots, P_m\} \in P(N)$ is called a *system of a priori unions or coalition structure* on N and each $P_k \in P$ is called a union. The so called *trivial coalition structures* are $P^n = \{\{1\}, \{2\}, \dots, \{n\}\}$, where each union is a singleton, and $P^N = \{N\}$, where the only union is the grand coalition. Let $P = \{P_1, \dots, P_m\} \in P(N)$ and consider that each union selects a representative. The set of such representatives is denoted by $M = \{1, \dots, m\}$. For $i \in P_k \in P$, P^{-i} will denote the partition obtained from P when player i leaves the union P_k and becomes a singleton, i.e.

$$P^{-i} = \{P_h \in P | h \neq k\} \cup \{P_k \setminus i, \{i\}\}.$$

A *game with a priori unions* is a triple (N, v, P) where $(N, v) \in \mathcal{G}$ and $P \in P(N)$. We denote by $\mathcal{GU}$ the set of all such games.

Definition 2.1.1. Given a game with a priori unions $(N, v, P) \in \mathcal{GU}$, the associated *quotient game* $(M, v^P) \in \mathcal{G}$ is the game played by the unions and defined for every $R \subseteq M$ by

$$v^P(R) = v(P_R),$$

where $P_R = \cup_{k \in R} P_k$. Note that if $P = P^n$, $v^P = v$.

Next, we recall some of the point-valued solution concepts existing in the literature for this class of games. A *value on $\mathcal{GU}$* is a map f that assigns a vector $\mathsf{f}(N, v, P) \in \mathbb{R}^N$ to every game with a priori unions $(N, v, P) \in \mathcal{GU}$. In this context we consider three possible extensions of the Shapley and Banzhaf values.

Definition 2.1.2. (Owen 1977). The *Owen value*, Ow, is the value on $\mathcal{GU}$ defined for every $(N, v, P) \in \mathcal{GU}$ and $i \in N$ by

$$\begin{aligned}\mathsf{Ow}_i(N, v, P) & \\ &= \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{r!(m-r-1)!}{m!} \frac{t!(p_k - t - 1)!}{p_k!} \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right],\end{aligned}$$

where $i \in P_k \in P$, $r = |R|$, $m = |M| = |P|$, $p_k = |P_k|$, and $t = |T|$.

Definition 2.1.3. (Owen 1982). The *Banzhaf-Owen value*, BO, is the value on $\mathcal{GU}$ defined for every $(N, v, P) \in \mathcal{GU}$ and $i \in N$ by

$$\mathsf{BO}_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right],$$

where $i \in P_k \in P$, $m = |M| = |P|$, and $p_k = |P_k|$.

Definition 2.1.4. (Alonso-Meijide & Fiestras-Janeiro 2002). The *Symmetric coalitional Banzhaf value*, SCB, is the value on $\mathcal{GU}$ defined for every $(N, v, P) \in \mathcal{GU}$ and $i \in N$ by

$$\mathsf{SCB}_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{t!(p_k - t - 1)!}{p_k!} \left[\begin{array}{c} v(P_R \cup T \cup i) \\ -v(P_R \cup T) \end{array} \right],$$

where $i \in P_k \in P$, $m = |M| = |P|$, $p_k = |P_k|$, and $t = |T|$.

The values on $\mathcal{GU}$ considered above follow a two-step procedure. In the first step the worth of the grand coalition is shared among the unions and in the second step the amount allotted to each union is shared among the members of the union. There exists a vast literature concerning values on $\mathcal{GU}$ and their characterizations by means of properties, mainly in the case of the Owen value. However, for the purpose of this work we do not need to present all of them in detail.

The first characterization of Ow is proposed in Owen (1977), in the paper where the value on $\mathcal{GU}$ is introduced. This first characterization is based on five properties, the carrier property, two anonymity properties, one for the unions and another one for the players, additivity, and one last property which is the basis of the quotient game property which will be presented soon. The carrier property together with the null player property is equivalent to efficiency. In Hart & Kurz (1983) three different characterizations of Ow are proposed. The three of them are based on efficiency, symmetry, and additivity but they differ in the fourth property. In Winter (1992) the Owen value is characterized making use of a consistency property which states that the payoff of any player $i \in P_k \in P$ can be derived from a reduced game whose player set is a subset of P_k. Amer & Carreras (1995b) obtain a characterization of Ow by means of only three properties although two of them are quite demanding. Another characterization of the Owen value can be found in Vázquez-Brage et al. (1997) and will be presented in this section. In Hamiache (1999), Albizuri & Zarzuelo (2004), and Albizuri (2008) new characterizations of the Owen value are proposed.

The first characterization of the Banzhaf-Owen value is proposed in Albizuri (2001), but only on the restricted domain of simple games. The first characterization of BO on the full domain of games with a priori unions is established in Amer et al. (2002). The authors use six properties, three well known properties in the literature (additivity, dummy player property, and symmetry), and three other properties which have (to my knowledge) never been used before, although they appear to be very interesting and easy to interpret. Two of these new properties are based on a delegation game, which is a game obtained from the original one, considering that a player delegates his role to another player, and a last property called Many null players whose definition is quite cumbersome. The aforementioned delegation game follows the same idea of the $\{i \triangleleft j\}$-amalgamation game game, $(N, v_{i \triangleleft j})$, introduced in Section 1.1.2 to define the $\mathcal{G}$:2-AEF property. However, the former maintains the players set N fixed even though player j becomes a null player while the later sends player j off. As they said in Remark 3.3(b) (Amer et al. 2002) their characterization is far from giving rise to an almost common (a parallel) characterization of both Ow and BO similar to Feltkamp's one for Sh and Ba. With that target Alonso-Meijide et al. (2007) propose a new characterization of the Banzhaf-Owen value, together with a survey of values on $\mathcal{GU}$ which helps to understand the differences among the three allocation rules presented here for games with a priori unions. This characterization is presented below.

The Symmetric coalitional Banzhaf value is characterized in Alonso-Meijide & Fiestras-Janeiro (2002) with two different sets of properties. We will present one of them in detail next.

Some of the properties we need only apply to games with a priori unions where the system of a priori unions is the trivial singleton coalition structure and others apply to the whole class of games with a priori unions. Next, the properties that apply only to games with a priori unions with the trivial singleton coalition structure are presented.

$\mathcal{GU}$:EFF A value on $\mathcal{GU}$, f, satisfies *efficiency* if for every $(N, v) \in \mathcal{G}$,

$$\sum_{i \in N} \mathsf{f}_i(N, v, P^n) = v(N).$$

$\mathcal{GU}$:2-EFF A value on $\mathcal{GU}$, f, satisfies *2-efficiency*[1] if for every $(N, v) \in \mathcal{G}$ and every pair of distinct players $i, j \in N$,

$$\mathsf{f}_i(N, v, P^n) + \mathsf{f}_j(N, v, P^n) = \mathsf{f}_p(N^{ij}, v^{ij}, P^{n-1}).$$

[1] Recall the formal definition of the merged game (N^{ij}, v^{ij}) in page 9

$\mathcal{GU}$:DPP A value on $\mathcal{GU}$, f, satisfies the *dummy player property* if for every $(N,v) \in \mathcal{G}$ and every $i \in N$ dummy player in (N,v),

$$\mathsf{f}_i(N,v,P^n) = v(i).$$

$\mathcal{GU}$:SYM A value on $\mathcal{GU}$, f, satisfies *symmetry* if for every $(N,v) \in \mathcal{G}$ and every pair of symmetric players $i,j \in N$ in (N,v),

$$\mathsf{f}_i(N,v,P^n) = \mathsf{f}_j(N,v,P^n).$$

$\mathcal{GU}$:EMC A value on $\mathcal{GU}$, f, satisfies *equal marginal contributions* if for every (N,v), $(N,w) \in \mathcal{G}$ and every $i \in N$ such that for every $S \subseteq N \setminus i$, $v(S \cup i) - v(S) = w(S \cup i) - w(S)$, then

$$\mathsf{f}_i(N,v,P^n) = \mathsf{f}_i(N,w,P^n).$$

The five properties presented above are based on $\mathcal{G}$:EFF, $\mathcal{G}$:2-EFF, $\mathcal{G}$:DPP, $\mathcal{G}$:SYM, and $\mathcal{G}$:EMC considered in Section 1.1.2. Indeed, since the properties are stated only for the trivial coalition structure, a value f on $\mathcal{GU}$ satisfies them whenever the value on $\mathcal{G}$ that f generalizes, satisfies the corresponding properties in $\mathcal{G}$. Next, properties that describe the performance of values with respect to the a priori unions structure are presented. These properties apply to games with a priori unions with an arbitrary coalition structure.

$\mathcal{GU}$:QGP A value on $\mathcal{GU}$, f, satisfies the *quotient game property* if for every $(N,v,P) \in \mathcal{GU}$ and every $P_k \in P$,

$$\sum_{i \in P_k} \mathsf{f}_i(N,v,P) = \mathsf{f}_k(M,v^P,P^m).$$

$\mathcal{GU}$:1-QGP A value on $\mathcal{GU}$, f, satisfies the *1-quotient game property* if for every $(N,v,P) \in \mathcal{GU}$ and every $i \in N$ such that there is $k \in M$ with $P_k = \{i\}$,

$$\mathsf{f}_i(N,v,P) = \mathsf{f}_k(M,v^P,P^m).$$

$\mathcal{GU}$:BCU A value on $\mathcal{GU}$, f, satisfies *balanced contributions within the unions* if for every $(N,v,P) \in \mathcal{GU}$ and every $i,j \in P_k \in P$,

$$\mathsf{f}_i(N,v,P) - \mathsf{f}_i(N,v,P^{-j}) = \mathsf{f}_j(N,v,P) - \mathsf{f}_j(N,v,P^{-i}).$$

$\mathcal{GU}$:NID A value on $\mathcal{GU}$, f, satisfies *neutrality under individual desertion* if for every $(N,v,P) \in \mathcal{GU}$ and every $i,j \in P_k \in P$,

$$\mathsf{f}_i(N,v,P) = \mathsf{f}_i(N,v,P^{-j}).$$

The $\mathcal{GU}$:QGP and $\mathcal{GU}$:BCU are introduced in Vázquez-Brage et al. (1997) and used to characterize the Owen value (see Theorem 2.1.5 below). The $\mathcal{GU}$:QGP states that the sum of the payoffs assigned to the individual players of a union coincides with the payoff assigned to the union in the quotient game. The $\mathcal{GU}$:1-QGP and $\mathcal{GU}$:NID properties are introduced in Alonso-Meijide et al. (2007). Note that $\mathcal{GU}$:1-QGP is a weaker version of $\mathcal{GU}$:QGP since it is only required for players that form singleton unions. The $\mathcal{GU}$:BCU property is an interesting reciprocity property. It states that if two players i and j are in the same a priori union, then the loss (or gain) that player i inflicts on player j when she decides to leave the union is the same loss (or gain) inflicted on player i when j leaves the union. This property is based on the idea that the benefits (or losses) obtained from constituting a union cannot reward only one player. The $\mathcal{GU}$:NID property is a stronger version of $\mathcal{GU}$:BCU since it states that when a player decides to leave a union the remaining players of that union are not affected by her decision. Finally, we state three characterization results, one for each of the values on $\mathcal{GU}$ presented in this section.

Theorem 2.1.5. (Vázquez-Brage et al. 1997). *The Owen value,* Ow, *is the unique value on $\mathcal{GU}$ satisfying $\mathcal{GU}$:EFF, $\mathcal{GU}$:SYM, $\mathcal{GU}$:EMC, $\mathcal{GU}$:QGP, and $\mathcal{GU}$:BCU.*

Theorem 2.1.6. Alonso-Meijide & Fiestras-Janeiro (2002). *The Symmetric coalitional Banzhaf value,* SCB, *is the unique value on $\mathcal{GU}$ satisfying $\mathcal{GU}$:2-EFF, $\mathcal{GU}$:DPP, $\mathcal{GU}$:SYM, $\mathcal{GU}$:EMC, $\mathcal{GU}$:QGP and $\mathcal{GU}$:BCU.*

Theorem 2.1.7. (Alonso-Meijide et al. 2007). *The Banzhaf-Owen value,* BO, *is the unique value on $\mathcal{GU}$ satisfying $\mathcal{GU}$:2-EFF, $\mathcal{GU}$:DPP, $\mathcal{GU}$:SYM, $\mathcal{GU}$:EMC, $\mathcal{GU}$:1-QGP, and $\mathcal{GU}$:NID.*

In Table 2.1 the above characterization results are summarized. These three results contribute to the understanding of the differences among the presented values on $\mathcal{GU}$. As it is seen, the only difference between Ow and SCB lies on the fact that the former is the Shapley value when the trivial singleton coalition structure is considered whereas the later is the Banzhaf value. Instead, the differences between Ow and BO arise in properties $\mathcal{GU}$:EFF/$\mathcal{GU}$:2-EFF,

Ow	SCB	BO
$\mathcal{GU}$:EFF	$\mathcal{GU}$:2-EFF $\mathcal{GU}$:DPP	$\mathcal{GU}$:2-EFF $\mathcal{GU}$:DPP
$\mathcal{GU}$:SYM $\mathcal{GU}$:EMC	$\mathcal{GU}$:SYM $\mathcal{GU}$:EMC	$\mathcal{GU}$:SYM $\mathcal{GU}$:EMC
$\mathcal{GU}$:QGP $\mathcal{GU}$:BCU	$\mathcal{GU}$:QGP $\mathcal{GU}$:BCU	$\mathcal{GU}$:1-QGP $\mathcal{GU}$:NID

Table 2.1: Parallel characterizations of Ow, SCB, and BO

$\mathcal{GU}$:QGP/$\mathcal{GU}$:1-QGP, and $\mathcal{GU}$:BCU/$\mathcal{GU}$:NID, the latter two pairs being logically related. Finally, the differences between BO and SCB are limited to $\mathcal{GU}$:QGP/$\mathcal{GU}$:1-QGP and $\mathcal{GU}$:BCU/$\mathcal{GU}$:NID. Therefore, SCB can be seen as a compromise between Ow and BO.

2.2 Shapley and Banzhaf levels values

In this section the games with levels structure of cooperation are introduced. This kind of games were suggested by Owen (1977) as a possible extension to his work. Indeed, the model generalizes the games with a priori unions introduced in Section 2.1 by considering a finite sequence of partitions of the players' set. The model is first studied by Winter (1989). In this first work, the author proposes and characterizes a generalization of the Owen value for this context. In a similar way a generalization of the Banzhaf-Owen value is proposed in Álvarez-Mozos & Tejada (2011) together with parallel characterizations of these two values.

A *levels structure of cooperation* is a pair $(N, \underline{B})$, where N is the set of players and $\underline{B} = \{B_0, \dots, B_{k+1}\}$ is a sequence of partitions of N such that $B_0 = \{\{i\} : i \in N\}$, $B_{k+1} = \{N\}$, and for each $r \in \{0, \dots, k+1\}$, B_{r+1} is coarser than B_r. That is to say, for each $r \in \{1, \dots, k+1\}$ and each $S \in B_r$, there is $B \subseteq B_{r-1}$ such that $S = \cup_{U \in B} U$. Each $U \in B_r$ is called a *union* and B_r is called the *r-th level of $\underline{B}$*. Without loss of generality, we also assume (when $|N| > 1$) that no partition is repeated, i.e., for every $r \in \{0, \dots, k\}$, $B_r \neq B_{r+1}$. Note that B_0 and B_{k+1} are fictitious levels which are introduced for notational convenience. Hence, we say that $(N, \underline{B})$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ is a levels structure of cooperation with k levels. The levels structure of cooperation with 0 levels is called *trivial* and is denoted by $(N, \underline{B_0})$, i.e., $\underline{B_0} = \{\{i : i \in N\}, \{N\}\}$. Note that such a levels structure of cooperation is actually not restricting the cooperation at all and that is why we call it trivial. The set of all levels structures of cooperation over the set N is denoted by $\mathcal{L}(N)$. The following example illustrates the above definitions.

Example 2.2.1. Let $N = \{1, 2, 3, 4, 5, 6\}$ and $\underline{B} = \{B_0, B_1, B_2, B_3\}$ be given by

$$\begin{aligned} B_3 &= \{\{1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6\}\}, \\ B_2 &= \{\{1, \quad 2, \quad 3\}, \{4, \quad 5, \quad 6\}\}, \\ B_1 &= \{\{1, \quad 2\}, \{3\}, \{4\}, \{5, \quad 6\}\}, \text{ and} \\ B_0 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}. \end{aligned}$$

Then, $(N, \underline{B}) \in \mathcal{L}(N)$ is a levels structure of cooperation with two levels. $\Diamond$

A *game with levels structure of cooperation* is a triple $(N, v, \underline{B})$, where $(N, v) \in \mathcal{G}$ and $(N, \underline{B}) \in \mathcal{L}(N)$. We denote by $\mathcal{GL}$ the set of all games with levels structure of cooperation.

In the framework of games with levels structure of cooperation we assume that players are initially organized into the coalition structure B_k as groups that bargain for the division of $v(N)$. Then, each union of the highest level is divided again according to the coalition structure B_{k-1} in order to divide the amount that the unions of the higher level have obtained, and so on and so forth until the lowest level, B_0, is reached.

Given $(N, \underline{B}) \in \mathcal{L}(N)$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ and $i \in N$, we denote by $(N, \underline{B^{-i}}) \in \mathcal{L}(N)$ the levels structure of cooperation obtained from $(N, \underline{B})$ by isolating player i from the union she belongs to at each level, i.e., $\underline{B^{-i}} = \{B_0, B_1^{-i}, \dots, B_k^{-i}, B_{k+1}\}$, where, for every $r \in \{1, \dots, k\}$, $B_r^{-i} = \{U \in B_r : i \notin U\} \cup \{U_r \setminus i, \{i\}\}$ given that $i \in U_r \in B_r$. For each level $r \in \{1, \dots, k\}$, the partition B_r can be seen as a set of players, i.e., each union $U \in B_r$ can be seen as a player. Then the levels structure of cooperation obtained from $(N, \underline{B}) \in \mathcal{L}(N)$ considering the unions of the r^{th} level as players is $(B_r, \underline{B_r}) \in \mathcal{L}(B_r)$, where $\underline{B_r} = \{B_r, \dots, B_{k+1}\}$.

Definition 2.2.2. Given $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$, for each $r \in \{1, \dots, k\}$ we define the r^{th} *level union game* $(B_r, v^r, \underline{B_r}) \in \mathcal{GL}$ as the game with levels structure of cooperation induced from $(N, v, \underline{B})$ by considering the coalitions of B_r as players, i.e., for each $r \in \{1, \dots, k\}$ and each $S \subseteq B_r$,

$$v^r(S) = v(\{i \in N : \text{there is } U \in S \text{ such that } i \in U\}).$$

Note that the r^{th} level union game generalizes the quotient game (Definition 2.1.1) from one level to an arbitrary levels structure of cooperation.

A *value on* $\mathcal{GL}$ is a map f that assigns a vector $\mathsf{f}(N, v, \underline{B}) \in \mathbb{R}^N$ to every game with levels structure of cooperation $(N, v, \underline{B}) \in \mathcal{GL}$.

Recall that Sh is the expected marginal contribution of a player to her predecessors given that all orderings of players are equally likely. To generalize the Shapley value to games with levels structure of cooperation Winter (1989) applied this idea taking into account only those permutations that are consistent with the levels structure of cooperation. By consistent we mean permutations that keep players in the same union consecutive. To formalize this idea, given a levels structure of cooperation $(N, \underline{B}) \in \mathcal{L}(N)$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ we define the following sets of permutations: $\Omega(\underline{B}) = \Omega_1(\underline{B}) \subseteq \Omega_2(\underline{B}) \subseteq \cdots \subseteq \Omega_k(\underline{B}) \subseteq \Pi(N)$. First of all,

$$\Omega_k(\underline{B}) = \{\sigma \in \Pi(N) : \forall i, j \in S \in B_k \text{ and } l \in N, \text{if } \sigma(i) < \sigma(l) < \sigma(j) \text{ then } l \in S\}.$$

Then, for $r \in \{k-1, \ldots, 1\}$ we recursively define

$$\Omega_r(\underline{B}) = \{\sigma \in \Omega_{r+1}(\underline{B}) : \forall i, j \in S \in B_r \text{ and } l \in N, \text{ if } \sigma(i) < \sigma(l) < \sigma(j) \text{ then } l \in S\}.$$

Observe that $\Omega_r(\underline{B})$ denotes the permutations of $\Omega_{r+1}(\underline{B})$ such that the elements of each union of B_r are consecutive. Let us see an example to illustrate the above definitions.

Example 2.2.3. Let $(N, \underline{B}) \in \mathcal{L}(N)$ be the levels structure of cooperation considered in Example 2.2.1. On the one hand,

$$(1,2,4,3,5,6) \notin \Omega_2(\underline{B}),$$
$$(1,3,2,4,5,6) \in \Omega_2(\underline{B}) \setminus \Omega_1(\underline{B}), \text{ and}$$
$$(3,2,1,5,6,4) \in \Omega_1(\underline{B}).$$

On the other hand it is easy to count the number of permutations of the sets defined above,

$$|\Omega_2(\underline{B})| = 3! \cdot 3! \cdot 2! = 72, \text{ and}$$
$$|\Omega_1(\underline{B})| = 72 \cdot \frac{2! \cdot 1! \cdot 2!}{3!} \cdot \frac{2! \cdot 1! \cdot 2!}{3!} = 32. \qquad \Diamond$$

Now we are in the position to recall the definition of the already known solution concept for games with levels structure of cooperation.

Definition 2.2.4. (Winter 1989). The *Shapley levels value*, Sh^{L}, is the value on $\mathcal{GL}$ defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ and $i \in N$ by

$$\mathsf{Sh}_i^{\mathsf{L}}(N, v, \underline{B}) = \frac{1}{|\Omega(\underline{B})|} \sum_{\sigma \in \Omega(\underline{B})} \left[v(P_i^\sigma \cup i) - v(P_i^\sigma) \right].$$

Winter (1989) proves that the Shapley levels value is the unique value on $\mathcal{GL}$ satisfying efficiency, additivity, anonymity[2], the null player property and coalitional symmetry. The first four properties are extensions of standard properties in the literature, whereas coalitional symmetry demands that the sum of the payoffs to the players belonging to two unions S and U of some level r be the same whenever S and U are symmetric players in the r^{th} level union game and they belong to the same union in the next level. It is worth to mention that the five properties are natural extensions of the properties used in Owen (1977) to characterize Ow, which is simply the restriction of Sh^{L} to games with levels structure of cooperation with a single level.

In this section we introduce a new value on $\mathcal{GL}$ that coincides with the Banzhaf-Owen value (Definition 2.1.3) when the levels structure of cooperation has just one level, i.e., when $\underline{B} = \{B_0, B_1, B_2\}$. The idea for defining this new value is to induce, for each player, a partition of the set of players that respects the restrictions of the levels structure of cooperation. In other words, instead of looking at which permutations are feasible for the given levels structure, as in Winter (1989), for each player we look at which coalitions are feasible for the given levels structure of cooperation. For doing so, for each player, we define a partition of the rest of the players such that any coalition of unions is consistent with the levels structure of cooperation from this player's point of view.

Given a levels structure of cooperation $(N, \underline{B}) \in \mathcal{L}(N)$, and a player $i \in N$, for every $r \in \{0, \dots, k+1\}$ let $U_r \in B_r$ be such that $\{i\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{k+1} = N$. Then, the *partition induced by* $\underline{B}$ *on* i is defined as follows:

$$P(i, \underline{B}) = \bigcup_{r=0}^{k} (B_r)_{|U_{r+1} \setminus U_r}.$$

Observe that $P(i, \underline{B}) \in P(N \setminus i)$. We denote $|P(i, \underline{B})|$ by m_i, and the unions of the partition induced by $\underline{B}$ on i, by $P(i, \underline{B}) = \{T_1, \dots, T_{m_i}\}$. Finally, the set of indices of the partition induced by $\underline{B}$ on i is denoted by $M_i = \{1, \dots, m_i\}$ which can be seen as the set of representatives of the unions in $P(i, \underline{B})$.

[2] In Winter (1989) it is called *individual symmetry*.

Example 2.2.5. For the levels structure of cooperation of Example 2.2.1 we have, for instance,

$$P(1,\underline{B}) = \{\{2\},\{3\},\{4,5,6\}\} \qquad \text{and}$$
$$P(3,\underline{B}) = \{\{1,2\},\{4,5,6\}\}. \qquad \Diamond$$

Using the partition induced by the levels structure of cooperation for each player, we define a new value on $\mathcal{GL}$, namely the Banzhaf levels value, which is built based on the Banzhaf-Owen value for games with a priori unions.

Definition 2.2.6. (Álvarez-Mozos & Tejada 2011). The *Banzhaf levels value*, Ba^{L}, is the value on $\mathcal{GL}$ defined for every $(N,v,\underline{B}) \in \mathcal{GL}$ and $i \in N$ by

$$\mathsf{Ba}_i^{\mathsf{L}}(N,v,\underline{B}) = \sum_{R \subseteq M_i} \frac{1}{2^{m_i}} \left[v(T_R \cup i) - v(T_R) \right],$$

where $T_R = \cup_{r\in R} T_r$.[3]

One can easily check that the coalitions considered in each marginal contribution, T_R, are the coalitions for which there exists $\sigma \in \Omega(\underline{B})$ such that $T_R = P_i^\sigma$. Therefore, exploiting for each $i \in N$ the link between coalitions of elements of $P(i,\underline{B})$ and permutations of $\Omega(\underline{B})$ the Shapley levels value, Sh^{L}, can be written in an alternative way.

Remark 2.2.7. Let $(N,v,\underline{B}) \in \mathcal{GL}$ and $i \in N$, then

$$\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B}) = \sum_{R \subseteq M_i} \frac{c_R^i}{|\Omega(\underline{B})|} \left[v(T_R \cup i) - v(T_R) \right],$$

where $c_R^i = |\{\sigma \in \Omega(\underline{B}) : P_i^\sigma = T_R\}|$.

Expressions of Sh^{L} and Ba^{L} above lead to the Owen, Ow, and Banzhaf-Owen, BO, values respectively for levels structure of cooperation with a single level. Hence, for games with the trivial levels structure of cooperation Sh^{L} and Ba^{L} lead to the Shapley, Sh, and Banzhaf, Ba, values respectively.

Remark 2.2.8. Let $(N,v) \in \mathcal{G}$, $P \in P(N)$, and $(N,\underline{B}) \in \mathcal{L}(N)$ with $\underline{B} = \{B_0, B_1, B_2\}$ and $B_1 = P$. Then,

$$\mathsf{Sh}^{\mathsf{L}}(N,v,\underline{B}) = \mathsf{Ow}(N,v,P), \qquad \mathsf{Sh}^{\mathsf{L}}(N,v,\underline{B_0}) = \mathsf{Sh}(N,v),$$
$$\mathsf{Ba}^{\mathsf{L}}(N,v,\underline{B}) = \mathsf{BO}(N,v,P), \text{ and} \qquad \mathsf{Ba}^{\mathsf{L}}(N,v,\underline{B_0}) = \mathsf{Ba}(N,v). \qquad \triangleleft$$

[3]Note that we abuse notation by omitting the dependence of T_R on i.

2.2.1 Parallel characterizations

In this section we characterize both Sh^{L} and Ba^{L} based on two different groups of properties. The first group applies only to games with the trivial levels structure of cooperation and points out which value on $\mathcal{G}$ does the value on $\mathcal{GL}$ generalize, either the Shapley value or the Banzhaf value. The second group of properties describes the performance of the values in $\mathcal{GL}$ with respect to the levels structure of cooperation and contains logically related properties.

We start with the first group of properties that only applies to games with the trivial levels structure of cooperation.

$\mathcal{GL}$:EFF A value on $\mathcal{GL}$, f, satisfies *efficiency* if for every $(N, v) \in \mathcal{G}$,

$$\sum_{i\in N} \mathsf{f}_i(N, v, \underline{B_0}) = v(N).$$

$\mathcal{GL}$:2-EFF A value on $\mathcal{GL}$, f, satisfies *2-efficiency* if for every $(N, v) \in \mathcal{G}$ and each pair of players $i, j \in N$,

$$\mathsf{f}_i(N, v, \underline{B_0}) + \mathsf{f}_j(N, v, \underline{B_0}) = \mathsf{f}_p(N^{ij}, v^{ij}, \underline{B_0}).$$

$\mathcal{GL}$:DPP A value on $\mathcal{GL}$, f, satisfies *the dummy player property* if for every $(N, v) \in \mathcal{G}$ and each dummy player $i \in N$ in (N, v),

$$\mathsf{f}_i(N, v, \underline{B_0}) = v(i).$$

$\mathcal{GL}$:SYM A value on $\mathcal{GL}$, f, satisfies *symmetry* if for every $(N, v) \in \mathcal{G}$ and each pair of symmetric players $i, j \in N$ in (N, v),

$$\mathsf{f}_i(N, v, \underline{B_0}) = \mathsf{f}_j(N, v, \underline{B_0}).$$

$\mathcal{GL}$:EMC A value on $\mathcal{GL}$, f, satisfies *equal marginal contributions* if for every $(N, v), (N, w) \in \mathcal{G}$ and $i \in N$ such that for every $S \subseteq N \setminus i$, $v(S \cup i) - v(S) = w(S \cup i) - w(S)$,

$$\mathsf{f}_i(N, v, \underline{B_0}) = \mathsf{f}_i(N, w, \underline{B_0}).$$

The above properties are standard in the literature. Indeed, they are based on $\mathcal{GU}$:EFF, $\mathcal{GU}$:2-EFF, $\mathcal{GU}$:DPP, $\mathcal{GU}$:SYM, and $\mathcal{GU}$:EMC presented in Section 2.1. Moreover, note that a value f on $\mathcal{GL}$ satisfies one of the properties above if and only if the value on $\mathcal{G}$ that f generalizes satisfies the corresponding property on

$\mathcal{G}$, for instance Sh^{L} satisfies $\mathcal{GL}$:EFF since Sh satisfies $\mathcal{G}$:EFF. In Theorem 2.2.9 (resp. Theorem 2.2.10) we use, together with other properties that are presented below, $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM and $\mathcal{GL}$:EMC to characterize the Shapley levels value Sh^{L} (resp. $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM and $\mathcal{GL}$:EMC to characterize the Banzhaf levels value Ba^{L}). Although all these properties are presented in a weak form, in the sense that they only concern the trivial levels structure, both Sh^{L} and Ba^{L} satisfy stronger versions of the corresponding properties as we later show in Propositions 2.2.13 and 2.2.14.

Next, we consider the second set of properties that applies to games with arbitrary levels structure of cooperation.

$\mathcal{GL}$:LGP A value on $\mathcal{GL}$, f, satisfies the *level game property* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ and every $U \in B_r$ with $r \in \{1, \cdots, k\}$,

$$\sum_{i \in U} \mathsf{f}_i(N, v, \underline{B}) = \mathsf{f}_U(B_r, v^r, \underline{B_r}).$$

$\mathcal{GL}$:SLGP A value on $\mathcal{GL}$, f, satisfies the *singleton level game property* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ and every $U \in B_r$ with $r \in \{1, \dots, k\}$, such that $U = \{i\}$ for some $i \in N$,

$$\mathsf{f}_i(N, v, \underline{B}) = \mathsf{f}_U(B_r, v^r, \underline{B_r}).$$

$\mathcal{GL}$:LBC A value on $\mathcal{GL}$, f, satisfies *level balanced contributions* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ and $i, j \in U \in B_1$,

$$\mathsf{f}_i(N, v, \underline{B}) - \mathsf{f}_i(N, v, \underline{B^{-j}}) = \mathsf{f}_j(N, v, \underline{B}) - \mathsf{f}_j(N, v, \underline{B^{-i}}).$$

$\mathcal{GL}$:LNID A value on $\mathcal{GL}$, f, satisfies *level neutrality under individual desertion* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ and $i, j \in U \in B_1$,

$$\mathsf{f}_i(N, v, \underline{B}) = \mathsf{f}_i(N, v, \underline{B^{-j}}).$$

The $\mathcal{GL}$:LGP is the natural extension of the $\mathcal{GU}$:QGP (see Section 2.1) to games with many levels of cooperation. It states that the total payoff obtained by the members of a union in a given level equals the payoff obtained by the union when considering it as a player in the corresponding level union game. The $\mathcal{GL}$:SLGP is a weaker version of $\mathcal{GL}$:LGP, which states that any union which is composed of a single player gets the same payoff in the original game and in the corresponding level game when considering the union as a player. The $\mathcal{GL}$:SLGP is also the natural extension of $\mathcal{GU}$:1-QGP (see Section 2.1) to games with many levels of cooperation. The $\mathcal{GL}$:LBC property is a reciprocity property that states that the isolation of a player from the levels structure affects the players in her same union of the first level in the same amount as if it happens the other way around. If we consider games with levels structure of cooperation of only one level, the property is equivalent to $\mathcal{GU}$:BCU (see Section 2.1). The $\mathcal{GL}$:LNID property is a stronger version of $\mathcal{GL}$:LBC and states that the isolation of a player from the levels structure does not affect the payoffs of the players which are in her same union in all the levels. As before, it coincides with $\mathcal{GU}$:NID when we consider games with a single level of cooperation.

Next, we state and prove the two characterization results, one for Sh^{L} (Theorem 2.2.9) and one for Ba^{L} (Theorem 2.2.10). We start characterizing the Shapley levels value.

Theorem 2.2.9. (Álvarez-Mozos & Tejada 2011). *The Shapley levels value,* Sh^{L}, *is the unique value on* $\mathcal{GL}$ *satisfying* $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:LGP, *and* $\mathcal{GL}$:LBC.

Proof. First we show that Sh^{L} satisfies the properties and then we prove that it is the only value on $\mathcal{GL}$ satisfying them.

(1) *Existence.* Note that, by Remark 2.2.8, we know that for every $(N, v) \in \mathcal{G}$, $\mathsf{Sh}^{\mathsf{L}}(N, v, \underline{B_0}) = \mathsf{Sh}(N, v)$. Hence, from Theorem 1.1.8 we have that Sh^{L} satisfies $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM, and $\mathcal{GL}$:EMC.

In the case of $\mathcal{GL}$:LGP, let $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$, and $U \in B_r$ for some $r \in \{1, \dots, k\}$. We prove that Sh^{L} satisfies $\mathcal{GL}$:LGP by induction over r. If $r = 1$, from the definition of the induced partition, $P(i, \underline{B}) \setminus \{\{j\} : j \in U \setminus i\}$ is the same partition for each $i \in U$. Moreover, for every $i \in U$ it holds that $P(U, \underline{B_1}) = P(i, \underline{B}) \setminus \{\{j\} : j \in U \setminus i\}$. For each $i \in U$, $R \subseteq M_U$, and $S \subseteq U \setminus i$, let $c^i_{R+S} = |\{\sigma \in \Omega(\underline{B}) : P^\sigma_i = T_R \cup S\}|$. By the way in which $\Omega(\underline{B})$ is constructed, given $R \subseteq M_U$ and $i \in U$, c^i_{R+S} is the same for any $S \subseteq U \setminus i$ of a given cardinality s, and hence it can be denoted by c^i_{R+s}. Moreover, for every $i, j \in U$, $R \subseteq M_U$, and $S \subseteq U \setminus i$,$c^i_{R+s} = c^j_{R+s}$ and thus c^i_{R+s} can be further denoted simply by c_{R+s}. Note that for every $s \in \{1, \dots, u-1\}$,

$$\frac{c_{R+s}}{c_{R+(s-1)}} = \frac{s}{u-s}. \tag{2.1}$$

Recall that $c^U_R = |\{\sigma \in \Omega(\underline{B_1}) : P^\sigma_U = T_R\}|$, then we can relate $\Omega(\underline{B})$ to $\Omega(\underline{B})$ as follows

$$c_{R+(u-1)} = c_{R+0} = \frac{c^U_R}{u} \cdot \frac{\Omega(\underline{B})}{\Omega(\underline{B_1})}. \tag{2.2}$$

Then, by Remark 2.2.7,

$$\begin{aligned}
\sum_{i\in U}\mathsf{Sh}^{\mathsf{L}}_i(N,v,\underline{B}) &= \frac{1}{|\Omega(\underline{B})|}\sum_{i\in U}\sum_{R\subseteq M_U}\sum_{S\subseteq U\setminus i} c_{R+s}\left[v(T_R\cup S\cup i)-v(T_R\cup S)\right]\\
&= \frac{1}{|\Omega(\underline{B})|}\sum_{R\subseteq M_U}\sum_{i\in U}\sum_{S\subseteq U\setminus i} c_{R+s}\left[v(T_R\cup S\cup i)-v(T_R\cup S)\right]\\
&= \frac{1}{|\Omega(\underline{B})|}\sum_{R\subseteq M_U}\left[\begin{array}{c} u\cdot c_{R+(u-1)}v(T_R\cup U)-u\cdot c_{R+0}v(T_R)\\ +\sum_{\emptyset\neq S\subsetneq U}\left[s\cdot c_{R+(s-1)}-(u-s)\cdot c_{R+s}\right]v(T_R\cup S)\end{array}\right]\\
&= \sum_{R\subseteq M_U}\left[u\frac{c_{R+(u-1)}}{|\Omega(\underline{B})|}v(T_R\cup U)-u\frac{c_{R+0}}{|\Omega(\underline{B})|}v(T_R)\right]\\
&= \frac{1}{|\Omega(\underline{B_1})|}\sum_{R\subseteq M_U} c^U_R\left[v^1(T_R\cup U)-v^1(T_R)\right]=\mathsf{Sh}^{\mathsf{L}}_U(B_1,v^1,\underline{B_1}),
\end{aligned} \tag{2.3}$$

where the third equality is obtained by rearranging the terms of the summation, the fourth equality holds by eq. (2.1), and the fifth equality holds by eq. (2.2) which completes the first step of the induction.

Now, take $r\in\{2,\dots,k\}$ and suppose that for every $S\in B_{r-1}$ the following equality holds $\sum_{i\in S}\mathsf{Sh}^{\mathsf{L}}_i(N,v,\underline{B}) = \mathsf{Sh}^{\mathsf{L}}_S(B_{r-1},v^{r-1},\underline{B_{r-1}})$ (induction hypothesis). Let $U\in B_r$. Then,

$$\sum_{i\in U}\mathsf{Sh}^{\mathsf{L}}_i(N,v,\underline{B}) = \sum_{\substack{S\in B_{r-1}\\ S\subseteq U}}\sum_{i\in S}\mathsf{Sh}^{\mathsf{L}}_i(N,v,\underline{B}) = \sum_{\substack{S\in B_{r-1}\\ S\subseteq U}}\mathsf{Sh}^{\mathsf{L}}_S(B_{r-1},v^{r-1},\underline{B_{r-1}})$$

by the induction hypothesis. Finally, we can follow the argument from Eq. (2.3) with $\left(B_{r-1},v^{r-1},\underline{B_{r-1}}\right)$ instead of $(N,v,\underline{B})$ and $U\in B_{r-1}$ instead of $i\in N$ to obtain

$$\sum_{\substack{S\in B_{r-1}\\ S\subseteq U}}\mathsf{Sh}^{\mathsf{L}}_S(B_{r-1},v^{r-1},\underline{B_{r-1}}) = \mathsf{Sh}^{\mathsf{L}}_U(B_r,v^r,\underline{B_r}),$$

which completes the induction procedure.

In order to show that Sh^{L} satisfies $\mathcal{GL}$:LBC, let $(N,v,\underline{B})\in\mathcal{GL}$ with $\underline{B}=\{B_0,\dots,B_{k+1}\}$ and $i,j\in U\in B_1$. Then, it is easy to check that $P(i,\underline{B})\setminus\{j\}=P(j,\underline{B})\setminus\{i\}$. Hence, we may define $P(ij,\underline{B})=P(i,\underline{B})\setminus\{j\}=P(j,\underline{B})\setminus\{i\}$, $m_{ij}=|P(ij,\underline{B})|$, and $M_{ij}=\{1,\dots,m_{ij}\}$. Moreover, for every $R\subseteq M_{ij}$, define

$$\begin{aligned}
c^{i,-j}_R &= |\{\sigma\in\Omega(\underline{B^{-j}}) : P^\sigma_i = T_R\}| \quad\text{and}\\
c^{i,-j}_{R+j} &= |\{\sigma\in\Omega(\underline{B^{-j}}) : P^\sigma_i = T_R\cup j\}|
\end{aligned}$$

Note that, by definition, $c^i_R = c^j_R$, $c^i_{R+j} = c^j_{R+i}$, $c^{i,-j}_R = c^{j,-i}_R$, and $c^{i,-j}_{R+j} = c^{j,-i}_{R+i}$. Then,

$$\begin{aligned}
&\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B}) - \mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B^{-j}}) \\
&\quad = \sum_{R \subseteq M_{ij}} \left\{ \frac{c^i_{R+j}}{|\Omega(\underline{B})|} [v(T_R \cup j \cup i) - v(T_R \cup j)] + \frac{c^i_R}{|\Omega(\underline{B})|} [v(T_R \cup i) - v(T_R])\right\} \\
&\quad - \sum_{R \subseteq M_{ij}} \left\{ \frac{c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|} [v(T_R \cup j \cup i) - v(T_R \cup j)] + \frac{c^{i,-j}_R}{|\Omega(\underline{B^{-j}})|} [v(T_R \cup i) - v(T_R)]\right\} \\
&\quad = \sum_{R \subseteq M_{ij}} \left\{ \left[\frac{c^i_{R+j}}{|\Omega(\underline{B})|} - \frac{c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|} \right] [v(T_R \cup j \cup i) - v(T_R \cup j)] \right. \\
&\qquad \left. + \left[\frac{c^i_R}{|\Omega(\underline{B})|} - \frac{c^{i,-j}_R}{|\Omega(\underline{B^{-j}})|} \right] [v(T_R \cup i) - v(T_R)] \right\}, \qquad (2.4)
\end{aligned}$$

Observe that, if the above equation depends on i in the same way it depends on j, Sh^{L} satisfies $\mathcal{GL}$:LGP. To show this fact we claim that

$$\frac{c^i_R + c^i_{R+j}}{|\Omega(\underline{B})|} = \frac{c^{i,-j}_R + c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|}. \qquad (2.5)$$

Indeed, let us define, for $r \in \{1, \ldots, k\}$,

$$\begin{aligned}
\lambda^r_R &= |\{\sigma \in \Omega_r(\underline{B}) : P^\sigma_i = T_R\}| + |\{\sigma \in \Omega_r(\underline{B}) : P^\sigma_i = T_R \cup j\}| \quad \text{and} \\
\lambda^{-r}_R &= |\{\sigma \in \Omega_r(\underline{B^{-j}}) : P^\sigma_i = T_R\}| + |\{\sigma \in \Omega_r(\underline{B^{-j}}) : P^\sigma_i = T_R \cup j\}|.
\end{aligned}$$

Observe that $\lambda^1_R = c^i_R + c^i_{R+j}$ and $\lambda^{-1}_R = c^{i,-j}_R + c^{i,-j}_{R+j}$. Next, we prove that $\frac{\lambda^r_R}{|\Omega_r(\underline{B})|} = \frac{\lambda^{-r}_R}{|\Omega_r(\underline{B^{-j}})|}$ by backward induction on r. Recall that $U_{k+1} = N$ and define for every $r \in \{1, \ldots, k\}$

$$\begin{aligned}
A^r &= |\{U \in B_r \setminus U_r : U \subseteq U_{r+1} \text{ and } U \cap T_R = \emptyset\}| \quad \text{and} \\
B^r &= |\{U \in B_r \setminus U_r : U \subseteq U_{r+1} \text{ and } U \subseteq T_R\}|.
\end{aligned}$$

Observe that $A^k + B^k + 1 = b_k$ and that, for each $r \in \{1, \cdots, k\}$, $|U_r \cap T_R| + |U_r \setminus T_R| = u_r$.

We start proving the case $r = k$. Recall that $U_k \in B_k$ is such that $i, j \in U_k$. In particular, $i, j \in U_k \setminus T_R$ and thus $|U_k \setminus T_R| \geq 2$. By definition of λ_R^r,

$$\begin{aligned}
\lambda_R^k = &\prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 1)! \\
&+ \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R| + 1)! \cdot (|U_k \setminus T_R| - 2)! \\
&= \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \cdot u_k.
\end{aligned}$$

Similarly, by definition of λ_R^{-k},

$$\begin{aligned}
\lambda_R^{-k} = &\prod_{S \in B_k \setminus U_k} |S|! \cdot (A^k + 1)! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \\
&+ \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot (B^k + 1)! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \\
&= \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \cdot (b_k + 1).
\end{aligned}$$

Hence, for every $R \subseteq M_{ij}$, $\frac{\lambda_R^k}{\lambda_R^{-k}} = \frac{u_k}{b_k+1}$. To conclude with the first step of the induction one can easily check that $\frac{\Omega_k(\underline{B})}{\Omega_k(\underline{B^{-j}})} = \frac{u_k}{b_k+1}$.

Now suppose that for every $R \subseteq M_{ij}$, $\frac{|\Omega_{r+1}(\underline{B})|}{|\Omega_{r+1}(\underline{B^{-j}})|} = \frac{\lambda_R^{r+1}}{\lambda_R^{-(r+1)}}$, for some $r \in \{2, \dots, k\}$. By definition of λ_R^k,

$$\begin{aligned}
\frac{\lambda_R^r}{\lambda_R^{r+1}} = &\prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|! \\
&\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 1)! + (|U_r \cap T_R| + 1)! \cdot (|U_r \setminus T_R| - 1)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 1)! + (|U_{r+1} \cap T_R| + 1)! \cdot (|U_{r+1} \setminus T_R| - 2)!} \\
&= \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|! \\
&\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 2)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 2)!} \cdot \frac{u_{r+1}}{u_r},
\end{aligned}$$

where $h(S) = |\{S' \in B_r : S' \subseteq S\}|$ for each $S \in B_{r+1}$. Similarly, by definition of λ_R^{-k},

$$\frac{\lambda_R^{-r}}{\lambda_R^{-(r+1)}} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|!$$
$$\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 2)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 2)!}.$$

Combining the two expressions above we obtain

$$\frac{\lambda_R^r}{\lambda_R^{-r}} = \frac{\lambda_R^{r+1}}{\lambda_R^{-(r+1)}} \cdot \frac{u_r}{u_{r+1}}. \tag{2.6}$$

Furthermore,

$$\frac{|\Omega_r(\underline{B})|}{|\Omega_{r+1}(\underline{B})|} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot \frac{h(U_{r+1})!}{u_{r+1}!} \cdot \left(\prod_{\substack{S' \in B_r \setminus U_r \\ S' \subseteq U_{r+1}}} |S'|! \right) \cdot u_r!,$$

and

$$\frac{|\Omega_r(\underline{B^{-j}})|}{|\Omega_{r+1}(\underline{B^{-j}})|} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot \frac{h(U_{r+1})!}{(u_{r+1}-1)!} \cdot \left(\prod_{\substack{S' \in B_r \setminus U_r \\ S' \subseteq U_{r+1}}} |S'|! \right) \cdot (u_r - 1)!.$$

Thus

$$\frac{|\Omega_r(\underline{B})|}{|\Omega_r(\underline{B^{-j}})|} = \frac{|\Omega_{r+1}(\underline{B})|}{|\Omega_{r+1}(\underline{B^{-j}})|} \cdot \frac{u_r}{u_{r+1}}. \tag{2.7}$$

Hence, from Eq. (2.6) and (2.7), using the induction hypothesis we obtain,

$$\frac{\lambda_R^r}{|\Omega_r(\underline{B})|} = \frac{\lambda_R^{-,r}}{|\Omega_r(\underline{B^{-j}})|},$$

Thus, the claim in Eq. (2.5) holds and the proof concludes.

(2) *Uniqueness.* In Theorem 1.1.8 it is proved that any value on $\mathcal{GL}$ that satisfies $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM, and $\mathcal{GL}$:EMC is unique for games with the trivial levels structure of cooperation. In other words, let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM, and $\mathcal{GL}$:EMC, then for every $(N, v) \in \mathcal{G}$

$$\mathsf{f}^1(N, v, \underline{B_0}) = \mathsf{f}^2(N, v, \underline{B_0}) = \mathsf{Sh}(N, v).$$

Hence, let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying $\mathcal{GL}$:LGP and $\mathcal{GL}$:LBC and such that for every $(N, v) \in \mathcal{G}$, $\mathsf{f}^1(N, v, \underline{B_0}) = \mathsf{f}^2(N, v, \underline{B_0})$. We prove that for every $(N, v, \underline{B}) \in \mathcal{GL}$, with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $k \geq 1$, $\mathsf{f}^1(N, v, \underline{B}) = \mathsf{f}^2(N, v, \underline{B})$ by induction on the number of levels k. The case $k = 1$ holds by Theorem 2.1.5. Now, suppose that $\mathsf{f}^1(N', v', \underline{B'}) = \mathsf{f}^2(N', v', \underline{B'})$ for every $(N', v', \underline{B'}) \in \mathcal{GL}$ such that $(N', \underline{B'})$ has at most k levels and let $(N, v, \underline{B}) \in \mathcal{GL}$ be such that $(N, \underline{B})$ has $k+1$ levels, i.e., $\underline{B} = \{B_0, \ldots, B_{k+2}\}$. Let also $i \in N$. We prove that $\mathsf{f}^1_i(N, v, \underline{B}) = \mathsf{f}^2_i(N, v, \underline{B})$ by a second induction on $u = |U|$, where $i \in U \in B_1$. If $u = 1$, i.e., $U = \{i\}$, since both f^1 and f^2 satisfy $\mathcal{GL}$:LGP,

$$\mathsf{f}^1_i(N, v, \underline{B}) = \mathsf{f}^1_U(B_1, v^1, \underline{B_1}) = \mathsf{f}^2_U(B_1, v^1, \underline{B_1}) = \mathsf{f}^2_i(N, v, \underline{B}),$$

where the second equality holds by the first induction hypothesis. Hence, suppose that $\mathsf{f}^1_l(N, v, \underline{B}) = \mathsf{f}^2_l(N, v, \underline{B})$ for every $(N, v, \underline{B}) \in \mathcal{GL}$ such that $(N, \underline{B})$ has $k+1$ levels and every $l \in U \in B_1$ such that $|U| \leq u$. Now suppose that $|U| = u + 1$ and let $j \in U \setminus i$. Since f^1 and f^2 satisfy $\mathcal{GL}$:LBC,

$$\begin{aligned} \mathsf{f}^1_i(N, v, \underline{B}) - \mathsf{f}^1_j(N, v, \underline{B}) &= \mathsf{f}^1_i(N, v, \underline{B^{-j}}) - \mathsf{f}^1_j(N, v, \underline{B^{-i}}) \\ &= \mathsf{f}^2_i(N, v, \underline{B^{-j}}) - \mathsf{f}^2_j(N, v, \underline{B^{-i}}) = \mathsf{f}^2_i(N, v, \underline{B}) - \mathsf{f}^2_j(N, v, \underline{B}), \end{aligned} \tag{2.8}$$

where the second equality follows from the second induction hypothesis, since $i \in U \setminus j \in B_1^{-j}$, $j \in U \setminus i \in B_1^{-i}$, $|U \setminus j| = |U \setminus i| = u$, and $(N, \underline{B^{-j}})$ and $(N, \underline{B^{-i}})$ have $k+1$ levels. Now, adding up Eq. (2.8) for all $j \in U \setminus i$,

$$(u+1)\mathsf{f}^1_i(N, v, \underline{B}) - \sum_{j \in U} \mathsf{f}^1_j(N, v, \underline{B}) = (u+1)\mathsf{f}^2_i(N, v, \underline{B}) - \sum_{j \in U} \mathsf{f}^2_j(N, v, \underline{B}). \tag{2.9}$$

Finally, since f^1 and f^2 satisfy $\mathcal{GL}$:LGP,

$$\sum_{j \in U} \mathsf{f}^1_j(N, v, \underline{B}) = \mathsf{f}^1_U(B_1, v^r, \underline{B_1}) = \mathsf{f}^2_U(B_1, v^r, \underline{B_1}) = \sum_{j \in U} \mathsf{f}^2_j(N, v, \underline{B}), \tag{2.10}$$

where the second equality holds by the first induction hypothesis since $(B_1, \underline{B_1})$ has k levels. Combining Eq. (2.9) and (2.10) we obtain $\mathsf{f}^1_i(N, v, \underline{B}) = \mathsf{f}^2_i(N, v, \underline{B})$, which completes the proof. □

In the next result we characterize the Banzhaf levels value by means of six properties, four properties that characterize the Banzhaf value, Ba, and two additional properties that describe the way in which the Banzhaf levels value, Ba^{L}, deals with the levels structure of cooperation. Recall that the last two properties are logically related to those used to characterize the Shapley levels value.

Theorem 2.2.10. (Álvarez-Mozos & Tejada 2011). *The Banzhaf levels value,* Ba^{L}*, is the unique value on* $\mathcal{GL}$ *satisfying* $\mathcal{GL}$:2-EFF*,* $\mathcal{GL}$:DPP*,* $\mathcal{GL}$:SYM*,* $\mathcal{GL}$:EMC*,* $\mathcal{GL}$:SLGP*, and* $\mathcal{GL}$:LNID*.*

Proof. As in the previous theorem, we first show that Ba^{L} satisfies the properties and then prove that it is the only value on $\mathcal{GL}$ satisfying them.

(1) *Existence.* Note that, by Remark 2.2.8, we know that for every $(N, v) \in \mathcal{G}$, $\mathsf{Ba}^{\mathsf{L}}(N, v, \underline{B_0}) = \mathsf{Ba}(N, v)$. Hence, from Theorem 1.1.12 we have that Ba^{L} satisfies $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, and $\mathcal{GL}$:EMC.

In the case of $\mathcal{GL}$:SLGP, the proof follows immediately taking into account the fact that, for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ and $U = \{i\} \in B_r$ for some $r \in \{1, \dots, k\}$, $P(i, \underline{B}) = P(U, \underline{B_r})$.

In the case of $\mathcal{GL}$:LNID, we only need to check that for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \dots, B_{k+1}\}$, and every $i, j \in U \in B_1$, $P(i, \underline{B}) = P(i, \underline{B^{-j}})$, which follows from the definition of the partition induced by $\underline{B}$ on i.

(2) *Uniqueness.* From the characterization in Theorem 1.1.12, we have that any value on $\mathcal{GL}$ that satisfies $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, and $\mathcal{GL}$:EMC is unique for games with the trivial levels structure of cooperation. In other words, let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, and $\mathcal{GL}$:EMC, then for every $(N, v) \in \mathcal{G}$,

$$\mathsf{f}^1(N, v, \underline{B_0}) = \mathsf{f}^2(N, v, \underline{B_0}) = \mathsf{Ba}(N, v).$$

Hence, let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying $\mathcal{GL}$:SLGP, $\mathcal{GL}$:LNID, and such that for every $(N, v) \in \mathcal{G}$, $\mathsf{f}^1(N, v, \underline{B_0}) = \mathsf{f}^2(N, v, \underline{B_0})$. We prove that for every $(N, v, \underline{B}) \in \mathcal{GL}$, with $\underline{B} = \{B_0, \dots, B_{k+1}\}$ and $k \geq 1$, $\mathsf{f}^1(N, v, \underline{B}) = \mathsf{f}^2(N, v, \underline{B})$ by induction on the number k of levels of $(N, \underline{B})$. The case $k = 1$ holds by Theorem 2.1.7. Now, suppose that $\mathsf{f}^1(N', v', \underline{B'}) = \mathsf{f}^2(N', v', \underline{B'})$ for every $(N', v', \underline{B'}) \in \mathcal{GL}$ such that $(N', \underline{B'})$ has at most k levels and let $(N, v, \underline{B}) \in \mathcal{GL}$ be such that $(N, \underline{B})$ has $k+1$ levels, i.e., $\underline{B} = \{B_0, \dots, B_{k+2}\}$. Let also $i \in N$. We prove that $\mathsf{f}^1_i(N, v, \underline{B}) = \mathsf{f}^2_i(N, v, \underline{B})$ by a second induction on $u = |U|$, where $i \in U \in B_1$. If $u = 1$, i.e. $U = \{i\}$, since both f^1 and f^2 satisfy $\mathcal{GL}$:SLGP,

$$\mathsf{f}^1_i(N, v, \underline{B}) = \mathsf{f}^1_U(B_1, v^1, \underline{B_1}) = \mathsf{f}^2_U(B_1, v^1, \underline{B_1}) = \mathsf{f}^2_i(N, v, \underline{B}),$$

where the second equality follows from the first induction hypothesis since $(B_1, \underline{B_1}) \in \mathcal{L}(B_1)$ is a levels structure of cooperation with k levels. Now suppose that $\mathsf{f}^1_l(N, v, \underline{B}) = \mathsf{f}^2_l(N, v, \underline{B})$ for every $(N, v, \underline{B})$ such that $\underline{B} = \{B_0, \dots, B_{k+2}\}$ and every $l \in U \in B_1$ where $|U| \leq u$. Next, suppose that $|U| = u+1$ and let $j \in U \setminus i$.

Since f^1 and f^2 satisfy $\mathcal{GL}$:LNID,

$$\mathsf{f}_i^1(N, v, \underline{B}) = \mathsf{f}_i^1(N, v, \underline{B^{-j}}) = \mathsf{f}_i^2(N, v, \underline{B^{-j}}) = \mathsf{f}_i^2(N, v, \underline{B}),$$

where the second equality holds by the second induction hypothesis since $i \in U \setminus j \in B_1^{-j} \in \underline{B^{-j}}$, $(N, \underline{B^{-j}})$ has $k+1$ levels, and $|U \setminus j| = u$. This concludes the proof. □

Table 2.2 summarizes the parallel characterizations of Sh^{L} and Ba^{L} presented above.

Sh^{L}	Ba^{L}
$\mathcal{GL}$:EFF	$\mathcal{GL}$:2-EFF $\mathcal{GL}$:DPP
$\mathcal{GL}$:SYM $\mathcal{GL}$:EMC	$\mathcal{GL}$:SYM $\mathcal{GL}$:EMC
$\mathcal{GL}$:LGP $\mathcal{GL}$:BCU	$\mathcal{GL}$:SLGP $\mathcal{GL}$:LNID

Table 2.2: Parallel characterizations of Sh^{L} and Ba^{L}

In the current section we have proposed a new value for games with levels structure of cooperation, the Banzhaf levels value, and we have provided characterizations of this new value and the Shapley levels value. It should be pointed out that, in both theorems, the group of properties that apply only to games with trivial levels structures of cooperation can be replaced by any other group of properties that characterizes either the Shapley or the Banzhaf value (see Table 1.1). The remaining properties, $\mathcal{GL}$:LGP and $\mathcal{GL}$:LBC in Theorem 2.2.9 and $\mathcal{GL}$:SLGP and $\mathcal{GL}$:LNID in Theorem 2.2.10, describe the behavior of the values with respect to the non trivial levels structure of cooperation. Moreover, since these latter properties are logically comparable, the results help on deciding which value to use in a particular situation.

Next, we check that the proposed properties are independent, and hence we cannot drop any of them from the characterizations. We start examining the properties used for the characterization of the Shapley levels value, Sh^{L}.

Remark 2.2.11. The properties considered in Theorem 2.2.9 are independent as the following examples show:

(i) The value on $\mathcal{GL}$, g^1, defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ by

$$\mathsf{g}^1(N, v, \underline{B}) = 0,$$

satisfies $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:LGP, $\mathcal{GL}$:LBC, but not $\mathcal{GL}$:EFF.

(ii) Let a and b be two distinct, fixed, and indivisible players. In this context, by indivisible we mean that there are no players $i_1, \ldots, i_l$ such that $\{i_1, \ldots, i_l\} = a$ or $\{i_1, \ldots, i_l\} = b$. In other words, a and b are not unions of any levels structure of cooperation. Let g^2 be the value on $\mathcal{GL}$ defined as follows:

- If $N = \{a, b\}$ and $\underline{B} = \underline{B_0}$,

$$\mathrm{g}^2_a(N, v, \underline{B}) = \frac{3}{4}\left[v(N) - v(b)\right] + \frac{1}{4}v(a) \quad \text{and}$$
$$\mathrm{g}^2_b(N, v, \underline{B}) = \frac{1}{4}\left[v(N) - v(a)\right] + \frac{3}{4}v(b).$$

- Otherwise, $\mathrm{g}^2(N, v, \underline{B}) = \mathsf{Sh}^{\mathsf{L}}(N, v, \underline{B})$.

Thus, g^2 satisfies $\mathcal{GL}$:EFF, $\mathcal{GL}$:EMC, $\mathcal{GL}$:LGP, $\mathcal{GL}$:LBC, but not $\mathcal{GL}$:SYM.

(iii) Consider the value on $\mathcal{GL}$, g^3, defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ by

$$\mathrm{g}^3(N, v, \underline{B}) = \begin{cases} \mathsf{Sh}^{\mathsf{L}}(N, v, \underline{B}) & \text{if } (N, v, \underline{B}) \notin \mathcal{C} \\ \alpha_{i(N,v)}\mathbf{1}_{i(N,v)} & \text{if } (N, v, \underline{B}) \in \mathcal{C} \end{cases}$$

where $\mathbf{1}_k \in \mathbb{R}^N$ is such that $\mathbf{1}_k(l) = 1$ if $k = l$ and $\mathbf{1}_k(l) = 0$ if $k \neq l$ and

$$\mathcal{C} = \{(N, v, \underline{B}) \in \mathcal{GL} : v = \beta_i \tau_i + (\alpha_i - \beta_i)\delta_N, \text{for some } i = i(N, v) \in N \text{ and } 0 \leq \beta_i < \alpha_i\},$$

where τ and δ are defined for every $S \subseteq N$ by

$$\tau_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_N(S) = \begin{cases} 1 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases}$$

Then, g^3 satisfies $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM, $\mathcal{GL}$:LGP, $\mathcal{GL}$:LBC, but not $\mathcal{GL}$:EMC.

(iv) The value on $\mathcal{GL}$, g^4, defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ by

$$\mathrm{g}^4(N, v, \underline{B}) = \mathsf{Sh}(N, v),$$

satisfies $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:LBC, but not $\mathcal{GL}$:LGP.

(v) Let a and b be two distinct, fixed, and indivisible players. Let g^5 be the value on $\mathcal{GL}$ defined as follows:

- If $N = \{a, b\}$ and $\underline{B} = \underline{B_0}$, $\mathrm{g}^5(N, v, \underline{B}) = \left(\frac{v(N)}{2}, \frac{v(N)}{2}\right)$.

- Otherwise, $\mathrm{g}^5(N, v, \underline{B}) = \mathsf{Sh}^{\mathsf{L}}(N, v, \underline{B})$.

Thus, g^5 satisfies $\mathcal{GL}$:EFF, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:LGP, but not $\mathcal{GL}$:LBC. $\triangleleft$

Lastly, we examine the properties used for the characterization of the Banzhaf levels value, Ba^{L}.

Remark 2.2.12. The properties considered in Theorem 2.2.10 are independent as the following examples show:

(i) The value on $\mathcal{GL}$, g^6, defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ by

$$\mathrm{g}_i^6(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{|R|!(m_i - |R| - 1)!}{m_i!} \left[v(T_R \cup i) - v(T_R)\right],$$

satisfies $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:SLGP, $\mathcal{GL}$:LNID, but not $\mathcal{GL}$:2-EFF.

(ii) The value on $\mathcal{GL}$, g^1, defined above satisfies $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:SLGP, $\mathcal{GL}$:LNID, but not $\mathcal{GL}$:DPP.

(iii) Let a, b be two distinct, fixed, and indivisible players. Recall that by indivisible we mean that a and b are individual players and not unions, moreover a and b are not obtained after a merging of any other pair of players. Let g^7 be the value on $\mathcal{GL}$ defined as follows:

- If $N = \{a, b\}$ and $\underline{B} = \underline{B_0}$,

$$\mathrm{g}_a^7(N, v, \underline{B}) = \frac{3}{4}\left[v(N) - v(b)\right] + \frac{1}{4}v(a) \quad \text{and}$$
$$\mathrm{g}_b^7(N, v, \underline{B}) = \frac{1}{4}\left[v(N) - v(a)\right] + \frac{3}{4}v(b).$$

- Otherwise, $\mathrm{g}^7(N, v, \underline{B}) = \mathsf{Ba}^{\mathsf{L}}(N, v, \underline{B})$.

Thus, g^7 satisfies $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:EMC, $\mathcal{GL}$:SLGP, $\mathcal{GL}$:LNID, but not $\mathcal{GL}$:SYM.

(iv) Again, let a, b be two distinct, fixed, and indivisible players. The value on $\mathcal{GL}$, g^8, defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ by

$$\mathrm{g}^8(N, v, \underline{B}) = \begin{cases} (0,0) & \text{if } (N, v, \underline{B}) = (\{a, b\}, \delta_{\{a\}}, \underline{B_0}) \\ \mathsf{Ba}^{\mathsf{L}}(N, v, \underline{B}) & \text{otherwise} \end{cases}$$

satisfies $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, $\mathcal{GL}$:SLGP, $\mathcal{GL}$:LNID, but not $\mathcal{GL}$:EMC.

(v) The value on $\mathcal{GL}$, g^9, defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ by

$$\mathrm{g}^9(N, v, \underline{B}) = \mathsf{Ba}(N, v)$$

satisfies $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:LNID, but not $\mathcal{GL}$:SLGP.

(vi) The value on $\mathcal{GL}$, g^{10}, defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ and $i \in N$ by

$$\mathsf{g}_i^{10}(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{1}{2^{m_i - |T_R \cap U_k|}} \cdot \frac{|T_R \cap U_k|!(|U_k \setminus T_R| - 1)!}{|U_k|!} \left[v(T_R \cup i) - v(T_R) \right],$$

satisfies $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, $\mathcal{GL}$:SLGP, but not $\mathcal{GL}$:LNID. Recall that U_k is the union of the k-th level that contains player i. $\lhd$

To end with, it is worth to discuss that the properties that apply only to the trivial levels structure, namely, $\mathcal{GL}$:EFF, $\mathcal{GL}$:2-EFF, $\mathcal{GL}$:DPP, $\mathcal{GL}$:SYM, $\mathcal{GL}$:EMC, can be required for games with arbitrary levels structure of cooperation. However we have considered the weak versions of them since they are enough to show the uniqueness. Formally, consider the following strong versions of the aforementioned properties.

$\mathcal{GL}$:EFF* A value on $\mathcal{GL}$, f, satisfies *strong efficiency* if for every $(N, v, \underline{B}) \in \mathcal{GL}$,

$$\sum_{i \in N} \mathsf{f}_i(N, v, \underline{B}) = v(N).$$

$\mathcal{GL}$:2-EFF* A value on $\mathcal{GL}$, f, satisfies *strong 2-efficiency* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $i, j \in U_1 \in B_1$,

$$\mathsf{f}_i(N, v, \underline{B}) + \mathsf{f}_j(N, v, \underline{B}) = \mathsf{f}_p(N^{ij}, v^{ij}, \underline{B^{ij}}),$$

where $(N^{ij}, v^{ij}, \underline{B^{ij}})$ is the game with levels structure of cooperation obtained from the original one such that players i and j have merged into the new player $p = \{i, j\}$, i.e., (N^{ij}, v^{ij}) is as defined in Section 1.1.2 (back in page 9) and $\underline{B^{ij}} = \{B_0^{ij}, \ldots, B_{k+1}^{ij}\}$ is such that $B_0^{ij} = \{\{l\} : l \in N^{ij}\}$ and for every $r \in \{1, \ldots, k+1\}$ if $U_r \in B_r$ is such that $i, j \in U_r$, then $B_r^{ij} = (B_r \setminus U_r) \cup ((U_r \setminus \{i, j\}) \cup p)$.

$\mathcal{GL}$:DPP* A value on $\mathcal{GL}$, f, satisfies the *strong dummy player property* if for every $(N, v, \underline{B}) \in \mathcal{GL}$, if $i \in N$ is a dummy player in (N, v),

$$\mathsf{f}_i(N, v, \underline{B}) = v(i).$$

$\mathcal{GL}$:SYM* A value on $\mathcal{GL}$, f, satisfies *strong symmetry* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and every pair $i, j \in U_1 \in B_1$ of symmetric players in (N, v),

$$\mathsf{f}_i(N, v, \underline{B}) = \mathsf{f}_j(N, v, \underline{B}).$$

$\mathcal{GL}$:EMC* A value on $\mathcal{GL}$, f, satisfies *strong equal marginal contributions* if for every $(N,v,\underline{B}),(N,w,\underline{B}) \in \mathcal{GL}$ and every $i \in N$ such that for every $S \subseteq N \setminus i$, $v(S \cup i) - v(S) = w(S \cup i) - w(S)$,

$$\mathsf{f}_i(N,v,\underline{B}) = \mathsf{f}_i(N,w,\underline{B}).$$

Then, we can prove the following two propositions.

Proposition 2.2.13. *The Shapley levels value* Sh^{L} *satisfies* $\mathcal{GL}$:EFF*, $\mathcal{GL}$:SYM*, *and* $\mathcal{GL}$:EMC*.

Proof. The $\mathcal{GL}$:EFF* property is proved in Winter (1989).

In the case of $\mathcal{GL}$:SYM*, let $(N,v,\underline{B}) \in \mathcal{GL}$ and $i,j \in N$ be symmetric players in $(N,v) \in \mathcal{G}$ that belong to the same union at any level $r \geq 1$. From Remark 2.2.7 it follows that

$$\begin{aligned}\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B}) &= \sum_{R \subseteq M_{ij}} \frac{c_R^i}{\Omega(\underline{B})}\left[v(T_R \cup i) - v(T_R)\right] + \frac{c_{R+j}^i}{\Omega(\underline{B})}\left[v(T_R \cup j \cup i) - v(T_R \cup j)\right] \\ &= \sum_{R \subseteq M_{ij}} \frac{c_R^j}{\Omega(\underline{B})}\left[v(T_R \cup j) - v(T_R)\right] + \frac{c_{R+i}^j}{\Omega(\underline{B})}\left[v(T_R \cup i \cup j) - v(T_R \cup i)\right] = \mathsf{Sh}_j^{\mathsf{L}}(N,v,\underline{B}),\end{aligned}$$

where M_{ij}, c_R^i, c_R^j, c_{R+j}^i, and c_{R+i}^j are as defined in the proof of Theorem 2.2.9. The expression above follows from the fact that for each $R \subseteq M_{ij}$, $c_R^i = c_R^j$ and $c_{R+j}^i = c_{R+i}^j$, and the symmetry of i and j.

Finally, the $\mathcal{GL}$:EMC* property can be easily proved taking into account that $\Omega(\underline{B})$ only depends on $(N,\underline{B})$. ☐

Proposition 2.2.14. *The Banzhaf levels value* Ba^{L} *satisfies* $\mathcal{GL}$*:2-*EFF*, $\mathcal{GL}$:DPP*, $\mathcal{GL}$:SYM*, *and* $\mathcal{GL}$:EMC*.

Proof. The $\mathcal{GL}$:EMC* property can be proved easily taking into account the fact that $P(i,\underline{B})$ does not depend on the game (N,v).

In the case of $\mathcal{GL}$:SYM*, let $(N,v,\underline{B}) \in \mathcal{GL}$ and $i,j \in N$ be symmetric players in $(N,v) \in \mathcal{G}$ that belong to the same union at any level $r \geq 1$. Then,

$$\begin{aligned}\mathsf{Ba}_i^{\mathsf{L}}(N,v,\underline{B}) &= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_i}}\left\{\left[v(T_R \cup i) - v(T_R)\right] + \left[v(T_R \cup j \cup i) - v(T_R \cup j)\right]\right\} \\ &= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_j}}\left\{\left[v(T_R \cup j) - v(T_R)\right] + \left[v(T_R \cup i \cup j) - v(T_R \cup i)\right]\right\} = \mathsf{Ba}_j^{\mathsf{L}}(N,v,\underline{B}),\end{aligned}$$

where $M_{ij} = \{1, \ldots, m_{ij}\}$, $m_{ij} = |P(ij, \underline{B})|$, $P(ij, \underline{B}) = P(i, \underline{B}) \setminus \{i\} = P(j, \underline{B}) \setminus \{j\}$, and $m_i = m_j$, and the second equality holds since i and j are symmetric players and therefore $v(S \cup i) = v(S \cup j)$ for all $S \subseteq M_{ij}$.

The $\mathcal{GL}$:DPP* property holds straightforwardly since, in this case, we have that

$$\mathsf{Ba}_i^{\mathsf{L}}(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{1}{2^{m_i}} \left[v(T_R \cup i) - v(T_R) \right] = \sum_{R \subseteq M_i} \frac{1}{2^{m_i}} v(i) = v(i).$$

Finally, we prove the $\mathcal{GL}$:2-EFF* property. Let $(N, v, \underline{B}) \in \mathcal{GL}$ and $i, j \in N$ be two players that belong to the same union at any level $r \geq 1$. Then,

$$\begin{aligned}
&\mathsf{Ba}_i^{\mathsf{L}}(N, v, \underline{B}) + \mathsf{Ba}_j^{\mathsf{L}}(N, v, \underline{B}) \\
&= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_{ij}}} \begin{bmatrix} v(T_R \cup i) - v(T_R) \\ +v(T_R \cup j \cup i) - v(T_R \cup j) \end{bmatrix} + \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_{ij}}} \begin{bmatrix} v(T_R \cup j) - v(T_R) \\ +v(T_R \cup i \cup j) - v(T_R \cup i) \end{bmatrix} \\
&= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_{ij}-1}} \left[v(T_R \cup i \cup j) - v(T_R) \right] = \sum_{R \subseteq M_p} \frac{1}{2^{m_p}} \left[v^{ij}(T_R \cup p) - v^{ij}(T_R) \right] \\
&= \mathsf{Ba}_p^{\mathsf{L}}(N^{ij}, v^{ij}, \underline{B^{ij}}),
\end{aligned}$$

where M_{ij} is as defined in the proof of Theorem 2.2.9 above. □

2.2.2 An example

We conclude the chapter by examining an example to illustrate the use of the two different values in a decision problem. Before doing so, we make a comment on the validity of the application of the Banzhaf levels value. Laruelle & Valenciano (2004) claim that, in the context of voting games with a single level structure of cooperation and the Banzhaf levels value, only comparisons between players that belong to the same union of the first level are meaningful. The reason why they state so is that the number that Ba^{L} assigns to player i can be interpreted as the mathematical expectation of the decisiveness of player i when considering the probability distribution defined on the set of permutations of players conditional to the partition induced by the levels structure on player i. Since players that belong to different unions give rise to different induced partitions, their corresponding probability distributions are different, and hence, Laruelle & Valenciano (2004) conclude that they cannot be compared. Nevertheless, when the levels structure of cooperation is pin down and the players cannot behave strategically and change their position in the structure, as it is the case in the example below, we can do compare the values of players belonging to different unions, even in the case of simple games. We argue that even though the probability distribution of each agent is different, all of them are obtained from the same fixed structure following the same rules, which can be seen as public knowledge. Therefore, we may interpret the Banzhaf levels value as the subjective expectation of any player about the outcome of the game, provided the following condition holds: all agents believe that, for any arbitrary given agent, all possible coalitions that may form before she takes a decision -which may be different depending on the player considered- are equally likely.

Example 2.2.15. Consider a grid computing network to which some departments of several universities contribute with resources, e.g., memory, databases or processing capacity. The whole network resources are used for purposes of calculations demanding massive resources such as climate predictions. The departments involved are willing to use the grid computing network for their investigations and the problem arises when more than one department simultaneously request access to the common resources, which can only be accessed by one department at a time.

Moreover, consider a numerical example where the amount of resources that each department contributes with can be measured, e.g., either TB or Ghz. The total amount of resources add up to 41 units that are provided by 10 departments namely A, B, C, D, E, F, G, H, I and J, which respectively contribute 3, 1, 2, 10, 3, 5, 2, 3, 2, and 10 units.

In order to measure the contribution of each department to the network we assume that a grid computing network needs a minimum of 21 units to operate. Hence, any group of departments whose resources add up to 21 units or more could form a smaller network. Even though, all departments prefer to be part of a network as big as possible, we consider this possibility in order to measure the bargaining strength of each department.

The situation described so far can be modeled by a simple game (N, v), where N is the set of departments and the characteristic function $v(S)$ equals 1 if the aggregate amount of resources of coalition S is at least equal to 21 and equals 0 otherwise. Therefore, the priority rule needed to decide which department will use the grid first can be based on either the Shapley or the Banzhaf value, Sh or Ba, respectively. More precisely, we first normalize the Banzhaf value and the payoff of each department is interpreted as the probability -henceforth just *priority*- that the corresponding department can make use of the common resources when all departments simultaneously request access. These values (Sh and $\overline{\mathsf{Ba}}$) are depicted in Table 2.3.

However, each department involved is part of a university which, in turn, is in a given country. It may happen that when bargaining for the priority the departments are not autonomous anymore and need the permission of the university or country they belong to. If we take into account these restrictions, a levels structure of cooperation emerges naturally, and hence, the Shapley and Banzhaf levels values, Sh^{L} and Ba^{L}, could be used as basis for a priority rule. Consider for instance, that the 10 departments are part of 6 different universities which, in turn, are in 4 countries. More precisely, suppose there is the following levels structure of cooperation, $\{\{A\},\{B,C\},\{D\},\{E,F\},\{G,H,I\},\{J\}\}$ and $\{\{A,B,C\},\{D,E,F\},\{G,H,I\},\{J\}\}$, i.e. for instance Dep. B and Dep. C belong to the same university, which at its turn it is located in the same country as the university which Dep. A belongs to.

Table 2.3 comprises the different values considered in this section[4].

Dep.	Res.	Sh	Sh^{L}	Ba	Ba^{L}	$\overline{\mathsf{Ba}}$	$\overline{\mathsf{Ba}^{\mathsf{L}}}$
A	3	.0690	.0833	.1523	.1250	.0736	.0800
B	1	.0341	.0417	.0724	.0625	.0358	.0400
C	2	.0405	.0417	.0898	.0625	.0434	.0400
D	10	.2579	.2500	.4961	.3750	.2396	.2400
E	3	.0690	.0417	.1523	.0625	.0736	.0400
F	5	.1214	.2083	.2773	.3125	.1340	.2000
G	2	.0405	.0278	.0898	.0625	.0434	.0400
H	3	.0690	.1110	.1523	.1875	.0736	.1200
I	2	.0405	.0278	.0898	.0625	.0434	.0400
J	10	.2579	.1667	.4961	.2500	.2396	.1600

Table 2.3: The different measures of priority.

From Table 2.3, it follows that when considering the restrictions given by the levels structure of cooperation the priorities change significantly. For instance, a relevant such a difference is the change in the priority assigned to Dep. J. When the departments are considered autonomous, it is given top priority together with Dep. D. However, when the universities and countries are taken into account it ranks third, having Dep. F priority over Dep. J. This is explained by the fact that even so Dep. J is one of the departments whose contribution is the highest, the aggregate resources of its country are not so high compared to the aggregate resources of the remaining countries. Finally, the difference between Sh^{L} and Ba^{L} reveals intensely on the values of Dep. E and Dep. I, since Ba^{L} gives equal priority to both of them, whereas Sh^{L} doubles the value of Dep. E. $\Diamond$

[4]By $\bar{f}$ we denote the normalized f value. The different values have been calculated using a MATLAB© routine.

3

Share functions for monotone games

The concept of share functions is first introduced in van der Laan & van den Brink (1998). Share functions represent an alternative approach to the problem of allocating payoffs to the agents participating in a game. As mentioned in Section 1.1.2, a share vector for a game (N, v) is an $|N|$-dimensional vector $x \in \mathbb{R}^N$ such that for every $i \in N$, $x_i \geq 0$ and $\sum_{i \in N} x_i = 1$. The amount x_i represents player i's share in the worth to be distributed.

Hence, a first singularity of share functions over values is that share functions avoid the "efficiency issue". In this chapter a family of share functions will be studied in each of the following frameworks: monotone games ($\mathcal{M}$), monotone games with a priori unions, and monotone games with levels structure of cooperation. Each of these families will be generated by real valued functions on $\mathcal{M}$ satisfying certain conditions. On the one hand, these families will include the share functions associated with many of the values studied in previous chapters. In particular, the share functions associated with Sh, Ba, Ow, SCB, and Sh^{L} lie within these families. On the other hand, Banzhaf-like share functions will be proposed for games with a priori unions and games with levels structure of cooperation that differ from the ones associated with BO and Ba^{L}. Besides these points, each of the families will allow us to build new share functions in each of the considered frameworks just by choosing a real valued function on $\mathcal{M}$ that satisfies the demanded properties.

The main contribution of this chapter is a joint work with Oriol Tejada from the ETH-Zürich on the one hand, and René van den Brink and Gerard van der Laan from the Free University of Amsterdam on the other hand. At the moment an article is under the peer-reviewing process of an international journal and the main results of it are published (Álvarez-Mozos et al. 2011) on the working paper series of the Department of Statistics and Operations Research of the University of Santiago de Compostela. The chapter is organized as follows. Section 3.1

deals with share functions on monotone games following van der Laan & van den Brink (1998). However, instead of being just a review of the results contained in the aforementioned paper we have tried to round off some of the results. Theorem 3.1.2 is the main contribution in this regard. Section 3.2 focuses on share functions for monotone games with a priori unions. Two papers constitute the basis of this section, van der Laan & van den Brink (2002) and van den Brink & van der Laan (2005). However, instead of just recalling the main results of these works we have tried to extend the results a little bit. In particular, we have extended the family of share functions defined in these papers so that the share function associated with the Symmetric coalitional Banzhaf value, SCB, lies within the family. Moreover, two characterizations of this family of share functions for monotone games with a priori unions are proposed. Finally, Section 3.3 focuses on share functions for monotone games with levels structure of cooperation. A family of share functions is introduced for this class of games using a multiplication property. Roughly speaking, the multiplication property builds a sharing for monotone games with levels structure of cooperation using a sharing of monotone games. First, the family is introduced and next, three characterization results are proposed.

3.1 Share functions on $\mathcal{M}$

In van der Laan & van den Brink (1998), a class of share functions is defined and characterized based on real valued functions on the set of monotone games[1], $\mu : \mathcal{M} \longrightarrow \mathbb{R}$. The Shapley and Banzhaf share functions (Definitions 1.1.16 and 1.1.17 back in page 15) are included in this class for appropriate choices of μ functions. In this section, we try to round off the results of van der Laan & van den Brink (1998). More precisely, in Section 3.1.1 we propose a set of properties for μ functions that will allow us to write them as a sum of weighted marginal contributions where the weights are of a certain type. Finally, in Section 3.1.2 we merge the aforesaid result with the characterization in van der Laan & van den Brink (1998).

3.1.1 Real valued functions on $\mathcal{M}$

We consider some properties that real valued functions on $\mathcal{M}$, $\mu : \mathcal{M} \to \mathbb{R}$, might satisfy. We say that

[1] Recall that the set $\mathcal{M}$ is introduced back on page 3, Definition 1.1.1

(a) μ is *positive* if $\mu(N, v_0) = 0$ and for every $(N, v) \in \mathcal{M}^+$,

$$\mu(N, v) > 0.$$

(b) μ is *additive* if for every pair of monotone games (N, v), $(N, w) \in \mathcal{M}$,

$$\mu(N, v + w) = \mu(N, v) + \mu(N, w).$$

(c) μ is *anonymous* if, for every monotone game $(N, v) \in \mathcal{M}$, every permutation $\pi \in \Pi(N)$, and every $\emptyset \neq S \subseteq N$,

$$\mu(S, v_{|S}) = \mu(\pi(S), (\pi^{-1}v)_{|\pi(S)}).$$

The positivity property states that the only monotone game that should be assigned zero is the null game and any other monotone game is to be assigned a positive amount. The additivity is the standard additivity property for mappings. Finally, observe that anonymity requires the function to be independent of players' names.

Remark 3.1.1. Note that additivity, positivity, and anonymity are independent properties. Indeed, in the first place, $\mu(N, v) = v(N)^2$ satisfies positivity, anonymity but not additivity. In the second place, $\mu(N, v) = -v(N)$ satisfies additivity, anonymity but not positivity. In the third place, take a fixed player a and let μ be defined by,

$$\mu(N, v) = \begin{cases} 2v(N) & \text{if } N = \{a\}, \\ v(N) & \text{otherwise.} \end{cases}$$

It is clear that μ is additive and positive. Consider the set of players $N = \{a, b\}$ for some other player b and let $\pi \in \Pi(N)$ be the permutation that exchanges a with b, i.e., $\pi(a) = b$ and $\pi(b) = a$. Let also $(N, v) \in \mathcal{M}$ be a monotone game such that $v(a) > 0$. Then,

$$\mu\left(\{a\}, v_{|\{a\}}\right) = 2v(a) \neq v(a) = \mu(\{b\}, (\pi^{-1}v)_{|\{b\}}) = \mu(\{\pi(a)\}, (\pi^{-1}v)_{|\{\pi(a)\}}).$$

That is, μ is not anonymous. $\triangleleft$

In the next result we show that any real valued function μ that is positive, additive, and anonymous can be written as a weighted sum of marginal contributions, the weights depending only on the sizes of the corresponding coalitions.

Theorem 3.1.2. *Let μ be a real valued function on $\mathcal{M}$. Then, the two following statements are equivalent:*

(a) μ is positive, additive, and anonymous.

(b) There is a unique family of weights, $\{\omega_\mu^{n,s} > 0 : n \in \mathbb{N},\, s \in \{0, \ldots, n-1\}\}$ such that, for every $(N, v) \in \mathcal{M}$,

$$\mu(N, v) = \sum_{i \in N} \sum_{S \subseteq N \setminus i} \omega_\mu^{|N|,|S|} \left[v(S \cup i) - v(S)\right].$$

Proof. We only prove that *(a)* implies *(b)*, the reverse implication is straightforward. Let $n \in \mathbb{N}$, N be a set of players with $|N| = n$, and $\mu : \mathcal{M} \longrightarrow \mathbb{R}$ be fixed. Recall that by $\mathcal{M}^N$ we denote the set of monotone games on N. It is well known that $\mathcal{M}^N$ is a cone in the euclidean space of dimension $2^n - 1$. As a consequence μ can be cast as the restriction of a map $\mu^* : \mathbb{R}^{2^n-1} \longrightarrow \mathbb{R}$ to $\mathcal{M}^N$. Moreover, observe that $\mathcal{M}^N$ has positive measure in $\mathbb{R}^{2^n-1}$ because $\{(N, u_S)\}_{\emptyset \neq S \subseteq N}$ is a basis of $\mathbb{R}^{2^n-1}$ and for every $\emptyset \neq S \subseteq N$, $(N, u_S) \in \mathcal{M}^+$.

Therefore, since μ is positive, μ^* is bounded from below in a set of positive measure. From the solution of the Cauchy equation for several variables applied to μ (see for instance, Aczél & Dhombres (1989) Proposition 1 on page 35), for every $\emptyset \neq S \subseteq N$ there is a scalar $a^S \in \mathbb{R}$, such that for every $(N, v) \in \mathcal{M}^N$,

$$\mu(N, v) = \mu^*(N, v) = \sum_{\emptyset \neq S \subseteq N} a^S v(S). \tag{3.1}$$

Note that the reasoning above applies for any given μ and N. Hence, for every positive, additive, and anonymous μ, every finite set of players N, and every coalition $\emptyset \neq S \subseteq N$ there are scalars $a_\mu^{N,S} \in \mathbb{R}$ such that for every $(N, v) \in \mathcal{M}$,

$$\mu(N, v) = \sum_{\emptyset \neq S \subseteq N} a_\mu^{N,S} v(S).$$

Next, let N' be a set of players with $|N'| = |N|$ and $N' \cap N = \emptyset$, $\emptyset \neq S \subseteq N$, and $S' \subseteq N'$ such that $|S| = |S'|$. Let $\pi \in \Pi(N \cup N')$ be a permutation that exchanges S with S' and N with N'. By anonymity,

$$\mu(N, u_{S|N}) = \mu(\pi(N), (\pi^{-1} u_S)_{|\Pi(N)}) = \mu(N', u_{S'|N'}). \tag{3.2}$$

We prove that $a_\mu^{N,S} = a_\mu^{N',S'}$ by backward induction on the cardinality of S. If $S = N$ and $S' = N'$ then from Eq. (3.1) and (3.2) we have that $a_\mu^{N',N'} = a_\mu^{N,N}$. Next, assume that $a_\mu^{N,S} = a_\mu^{N',S'}$ whenever $|S| = |S'| > s > 0$. Now, let $|S'| = |S| = s > 0$. By Eq. (3.1) and (3.2),

$$\sum_{\substack{T' \in 2^{N'} \\ S' \subseteq T'}} a_\mu^{N',T'} = \sum_{\substack{T \in 2^N \\ S \subseteq T}} a_\mu^{N,T}$$

and, therefore

$$a_\mu^{N',S'} + \sum_{\substack{T' \in 2^{N'} \\ S' \subsetneq T'}} a_\mu^{N',T'} = a_\mu^{N,S} + \sum_{\substack{T \in 2^N \\ S \subsetneq T}} a_\mu^{N,T}.$$

Then, by the induction hypothesis we have $a_\mu^{N,S} = a_\mu^{N',S'}$. That is, for every $(N, v) \in \mathcal{M}$

$$\mu(N, v) = \sum_{\emptyset \neq S \subseteq N} a_\mu^{|N|,|S|} v(S). \tag{3.3}$$

Analogously to Corollary 3.8 of van der Laan & van den Brink (1998), for every $n \in \mathbb{N}$ we define first

$$\omega_\mu^{n,n-1} = \frac{1}{n} a_\mu^{n,n}, \tag{3.4}$$

and recursively, for every $s \in \{0, \dots, n-2\}$

$$\omega_\mu^{n,s} = \frac{1}{s+1} \left(a_\mu^{n,s+1} + (n-s-1)\omega_\mu^{n,s+1} \right). \tag{3.5}$$

Using Eq. (3.4) and (3.5), we can rewrite Eq. (3.3) as

$$\sum_{\emptyset \neq S \subseteq N} a_\mu^{|N|,|S|} v(S) = \sum_{i \in N} \sum_{S \subseteq N \setminus i} \omega_\mu^{n,s} \left[v(S \cup i) - v(S) \right],$$

where $n = |N|$ and $s = |S|$.

That is, there is a family of weights, $\{\omega_\mu^{n,s} : n \in \mathbb{N} \text{ and } s \in \{0, \dots, n-1\}\}$ such that, for every $(N, v) \in \mathcal{M}$,

$$\mu(N, v) = \sum_{i \in N} \sum_{S \subseteq N \setminus i} \omega_\mu^{n,s} \left[v(S \cup i) - v(S) \right]. \tag{3.6}$$

Next, let $\{\omega_\mu^{n,s} : n \in \mathbb{N} \text{ and } s \in \{0, \dots, n-1\}\}$ and $\{\delta_\mu^{n,s} : n \in \mathbb{N} \text{ and } s \in \{0, \dots, n-1\}\}$ be two systems of weights that satisfy Eq. (3.6). For each $n \in \mathbb{N}$ and $s \in \{0, \dots, n-1\}$, let $(N, v^s) \in \mathcal{M}$ with $|N| = n$ and the characteristic function, defined for every $T \subseteq N$ by $v^s(T) = 1$ if $|T| > s$ and $v^s(T) = 0$ otherwise. Then,

$$\mu(N,v^s)-\mu(N,v^s)=\sum_{i\in N}\sum_{S\subseteq N\setminus i}(\omega_\mu^{n,|S|}-\delta_\mu^{n,|S|})\,[v^s(S\cup i)-v^s(S)]\;=(\omega_\mu^{n,s}-\delta_\mu^{n,s})\binom{n-1}{s}n,$$

which implies that for every $n\in\mathbb{N}$ and $s\in\{0,\dots,n-1\}$, $\omega_\mu^{n,s}=\delta_\mu^{n,s}$. Hence, the system of weights is unique.

Lastly, since the game $(N,v^s)\in\mathcal{M}$ is not null and μ is positive,

$$0<\mu(N,v^s)=\sum_{i\in N}\sum_{T\subseteq N\setminus i}\omega_\mu^{n,|T|}\,[v^s(T\cup i)-v^s(T)]=\omega_\mu^{n,s}\binom{n-1}{s}n,$$

which implies that for every $n\in\mathbb{N}$ and $s\in\{0,\dots,n-1\}$, $\omega_\mu^{n,s}>0$. $\square$

Two important examples of additive, positive and anonymous real valued functions are

$$\mu^{\mathsf{Sh}}(N,v)=v(N)$$

and

$$\mu^{\mathsf{Ba}}(N,v)=\frac{1}{2^{n-1}}\sum_{i\in N}\sum_{S\subseteq N\setminus i}\left[v(S\cup i)-v(S)\right],$$

which respectively induce two families of weights defined by

$$\omega_{\mathsf{Sh}}^{n,s}=\omega_{\mu^{\mathsf{Sh}}}^{n,s}=\frac{s!(n-s-1)!}{n!}\tag{3.7}$$

and

$$\omega_{\mathsf{Ba}}^{n,s}=\omega_{\mu^{\mathsf{Ba}}}^{n,s}=\frac{1}{2^{n-1}}.\tag{3.8}$$

3.1.2 A family of share functions on $\mathcal{M}$

In the last part of this section we focus on share functions on $\mathcal{M}$. We start by considering three properties that a share function on $\mathcal{M}$ might satisfy.

$\mathcal{M}$:NPP A share function on $\mathcal{M}$, ρ, satisfies the *null player property* if for every $(N,v)\in\mathcal{M}^+$ and every null player $i\in N$ in (N,v),

$$\rho_i(N,v)=0.$$

$\mathcal{M}$:SYM A share function on $\mathcal{M}$, ρ, satisfies *symmetry* if for every $(N,v)\in\mathcal{M}$ and every pair $i,j\in N$ of symmetric players in (N,v),

$$\rho_i(N,v)=\rho_j(N,v).$$

$\mathcal{M}$:μ-ADD Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{M}$, ρ, satisfies *μ-additivity* if for every pair of monotone games $(N, v), (N, w) \in \mathcal{M}$,

$$\mu(N, v + w)\rho(N, v + w) = \mu(N, v)\rho(N, v) + \mu(N, w)\rho(N, w).$$

The properties $\mathcal{M}$:NPP and $\mathcal{M}$:SYM are standard, whereas $\mathcal{M}$:μ-ADD generalizes $\mathcal{G}$:ADD. Note that since efficiency is somehow included in the definition of a share function on $\mathcal{M}$, the three properties above are essentially the properties used in the first characterization of the Shapley value (see Theorem 1.1.7). The only difference is that the additivity ($\mathcal{G}$:ADD) is replaced by the more general μ-additivity ($\mathcal{M}$:μ-ADD), and hence, it allows for a characterization of a wider class of solutions.

The next result shows that the above three properties determine a unique share function on $\mathcal{M}$ and that none of the properties can be left out.

Proposition 3.1.3. *Let $\mu : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous. Then, the only share function on $\mathcal{M}$ satisfying $\mathcal{M}$:NPP, $\mathcal{M}$:SYM, and $\mathcal{M}$:μ-ADD is the μ-share function on $\mathcal{M}$ defined for every $(N, v) \in \mathcal{M}$ and $i \in N$ by*

$$\rho_i^\mu(N, v) = \frac{\sum_{S \subseteq N \setminus i} \omega_\mu^{n,s} \left[v(S \cup i) - v(S)\right]}{\mu(N, v)} \qquad \text{if } (N, v) \in \mathcal{M}^+$$

and $\rho_i^\mu(N, v_0) = \frac{1}{n}$. Moreover, the three properties are independent.

Proof. We only prove the independence of the properties since the characterization result holds as a direct consequence of Theorem 3.5 in van der Laan & van den Brink (1998).

In the first place, let a and b be two fixed and different players, $N = \{a, b\}$ and $(N, v) \in \mathcal{M}$. Since μ is additive, positive, and anonymous, we can write

$$\mu(N, v) = \lambda_1 v(a) + \lambda_1 v(b) + \lambda_2 v(a, b),$$

for some $\lambda_1, \lambda_2 > 0$. Then, we define ρ as follows

- If $N = \{a, b\}$ and $(N, v) \in \mathcal{M}^+$,

$$\begin{cases} \rho_a(N, v) &= \frac{1}{\mu(N,v)} \left[\lambda_1 v(a) + \lambda_2 (v(a, b) - v(b))\right] \\ \rho_b(N, v) &= \frac{1}{\mu(N,v)} \left[(\lambda_1 + \lambda_2) v(b)\right] \end{cases}$$

- Otherwise,

$$\rho(N, v) = \rho^\mu(N, v).$$

It is straightforward to check that ρ satisfies $\mathcal{M}$:NPP and $\mathcal{M}$:μ-ADD but not $\mathcal{M}$:SYM. To prove this last assertion, consider the game $(N, v) \in \mathcal{M}$ where $N = \{a, b\}$ and $v(a) = v(b) = v(a, b) = 1$. It is clear that a and b are symmetric players in (N, v) but

$$\mu(N, v)\rho_a(N, v) = \lambda_1 \neq \lambda_1 + \lambda_2 = \mu(N, v)\rho_b(N, v).$$

In the second place, let $\mu : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous. Take $\mu' : \mathcal{M} \to \mathbb{R}$ a positive, additive, and anonymous function different from μ, i.e., $\mu' \neq \mu$. Then, from the uniqueness of the characterization result, $\rho^{\mu'}$ satisfies $\mathcal{M}$:NPP and $\mathcal{M}$:SYM but not $\mathcal{M}$:μ-ADD.

Lastly, let ρ be defined, for every $(N, v) \in \mathcal{M}$ and every $i \in N$ by $\rho_i(N, v) = 1/n$. Then, ρ satisfies $\mathcal{M}$:μ-ADD for any additive μ and $\mathcal{M}$:SYM, but not $\mathcal{M}$:NPP. $\square$

In the proof above we made use of Theorem 3.5 in van der Laan & van den Brink (1998). However, the original formulation of this result considers symmetry instead of anonymity. We say that μ is *symmetric* if for every pair of symmetric players $i, j \in N$ in $(N, v) \in \mathcal{M}$ and every $S \subseteq N \setminus \{i, j\}$, $\mu(S \cup i, v_{|S\cup i}) = \mu(S \cup j, v_{|S\cup j})$. We may do so since in this framework anonymity and symmetry are actually equivalent properties as it is shown next.

Proposition 3.1.4. *A mapping $\mu : \mathcal{M} \to \mathbb{R}$ is symmetric if and only if it is anonymous.*

Proof. Let $\mu : \mathcal{M} \to \mathbb{R}$ be anonymous. On the one hand, let $i, j \in N$ be two symmetric players in $(N, v) \in \mathcal{M}$ and let $S \subseteq N \setminus \{i, j\}$. Consider the permutation $\pi \in \Pi(N)$ that leaves any player in $N \setminus \{i, j\}$ invariant and exchanges player i with player j. This type of permutations are known as transpositions. We denote the set of all transpositions over N by $\Pi^*(N)$. Then, since μ is anonymous

$$\mu(S \cup i, v_{|S\cup i}) = \mu(\pi(S \cup i), (\pi^{-1}v)_{|\pi(S\cup i)}). \tag{3.9}$$

Next, by definition for every $T \subseteq \pi(S \cup i) = S \cup j$,

$$(\pi^{-1}v)(T) = v(\pi^{-1}(T)) = \begin{cases} v((T \setminus j) \cup i) & \text{if } j \in T \\ v(T) & \text{if } j \notin T, \end{cases}$$

which, means that $(\pi(S \cup i), (\pi^{-1}v)_{|\pi(S\cup i)}) = (S \cup j, v_{|S\cup j})$ because i and j are symmetric players in (N, v). Hence, by Eq. (3.9) μ is symmetric.

Let now $\mu : \mathcal{M} \to \mathbb{R}$ be symmetric. Let N be a set of players with $|N| \geq 2$ (otherwise the result is straightforward) and $\pi \in \Pi(N)$. It is well known that every permutation can be written as a finite composition of transpositions, i.e., there are $\pi_1, \ldots, \pi_r \in \Pi^*(N)$ such that,

$$\pi = \pi_1 \circ \cdots \circ \pi_r.$$

Next, we claim that for every $(N, v) \in \mathcal{M}$, $\pi \in \Pi^*(N)$, and $\emptyset \neq S \subseteq N$,

$$\mu(S, v_{|S}) = \mu(\pi(S), (\pi^{-1}v)_{|\pi(S)}). \tag{3.10}$$

Observe that if the claim above holds then μ is anonymous. Indeed, let $(N, v) \in \mathcal{M}$, $\pi \in \Pi(N)$, and $\emptyset \neq S \subseteq N$. Let also $\pi = \pi_1 \circ \cdots \circ \pi_r$ be a decomposition of π in transpositions. Then,

$$\begin{aligned}
\mu(S, v_{|S}) &= \mu(\pi_r(S), (\pi_r^{-1}v)_{|\pi_r(S)}) = \mu(\pi_{r-1}(\pi_r(S)), (\pi_{r-1}^{-1}(\pi_r^{-1}v))_{|\pi_{r-1}(\pi_r(S))}) \\
&= \cdots = \mu(\pi_1(\cdots(\pi_r(S))\cdots), (\pi_1^{-1}(\cdots(\pi_r^{-1}v)\cdots))_{|\pi_1(\cdots(\pi_r(S))\cdots)}) \\
&= \mu\left((\pi_1 \circ \cdots \circ \pi_r)(S), ((\pi_1 \circ \cdots \circ \pi_r)^{-1}v)_{|\pi_1 \circ \cdots \circ \pi_r(S)}\right) = \mu(\pi(S), (\pi^{-1}v)_{|\pi(S)}),
\end{aligned}$$

because, for every $T \subseteq (\pi_1 \circ \cdots \circ \pi_r)(S)$,

$$(\pi_1^{-1}(\cdots(\pi_r^{-1}v)\cdots))(T) = v((\pi_r^{-1} \circ \cdots \circ \pi_1^{-1})(T)) = v((\pi_1 \circ \cdots \circ \pi_r)^{-1}(T)).$$

Hence, it only remains to prove the claim in Eq. (3.10). Let $\pi \in \Pi^*(N)$ be a transposition, then there are $i, j \in N$ such that $\pi(i) = j$, $\pi(j) = i$, and for every $l \in N \setminus \{i, j\}$, $\pi(l) = l$. Let $\emptyset \neq S \subseteq N$, we distinguish three cases.

Case 1: $S \cap \{i, j\} = \emptyset$.

Since $\pi_{|S}$ is the identity permutation, that is, the permutation which leaves every player invariant. Eq. (3.10) is trivially satisfied.

Case 2: $S \cap \{i, j\} \neq \emptyset$ and $\pi(S) \cap \{i, j\} \neq \emptyset$.

We can assume without loss of generality that $i \in S \cap \{i, j\}$, and thus $j \in \pi(S) \cap \{i, j\}$. Then, consider the game (N', v'), defined by $N' = S \cup j$ and for every $T \subseteq N'$, by

$$v'(T) = \begin{cases} v(T) & \text{if } j \notin T, \\ v((T \setminus j) \cup i) & \text{if } j \in T. \end{cases}$$

By construction, i and j are symmetric players in (N', v'). Let $S' = S \setminus i$, then, by symmetry,

$$\mu(S, v_{|S}) = \mu(S' \cup i, v'_{|S' \cup i}) = \mu(S' \cup j, v'_{|S' \cup j}) = \mu(\pi(S), (\pi^{-1}v)_{|\pi(S)}).$$

Case 3: $\{i, j\} \subseteq S$.

Observe that in this case we have $\pi(S) = S$. Let $k \notin N$ be an extra (fictitious) player. Then, define the game (N', v') by $N' = N \cup k$ and for every $T \subseteq N'$ by $v'(T) = v(T \setminus k)$. Let $\pi_{i,k} \in \Pi^*(N')$ be the transposition that exchanges i with k and leaves the remaining players invariant. Similarly, let $\pi_{j,i}, \pi_{j,k} \in \Pi^*(N')$ be the transpositions that exchange j with i and j with k, respectively. It is an easy exercise to check that for every $l \in N$, $\pi(l) = (\pi_{j,k} \circ \pi_{j,i} \circ \pi_{i,k})(l)$. Let $T = S \setminus \{i, j\}$ and define,

$$\begin{aligned}
S^0 &= T \cup \{i, j\} = S, \\
S^1 &= \pi_{i,k}(S^0) = \pi_{i,k}(T \cup \{i, j\}) = T \cup \{k, j\}, \\
S^2 &= \pi_{j,i}(S^1) = (\pi_{j,i} \circ \pi_{i,k})(T \cup \{i, j\}) = T \cup \{k, i\}, \text{ and} \\
S^3 &= \pi_{j,k}(S^2) = (\pi_{j,k} \circ \pi_{j,i} \circ \pi_{i,k})(T \cup \{i, j\}) = T \cup \{i, j\} = S.
\end{aligned}$$

Observe that $S \cap \pi_{i,k}(S) \neq \emptyset$ and $\pi_{i,k}(S) \setminus S \neq \emptyset$, hence, by Case 2,

$$\mu(S, v_{|S}) = \mu(\pi_{i,k}(S), (\pi_{i,k}^{-1} v)_{|\pi_{i,k}(S)}) = \mu(S^1, (\pi_{i,k} v)_{|S^1}), \tag{3.11}$$

where the second equality holds because the inverse of every transposition is the transposition itself. Next, note that $S^1 \cap \pi_{j,i}(S^1) \neq \emptyset$ and $\pi_{j,i}(S^1) \setminus S^1 \neq \emptyset$, hence, we can apply Case 2 to obtain,

$$\mu(S^1, (\pi_{i,k} v)_{|S^1}) = \mu(\pi_{j,i}(S^1), (\pi_{j,i}^{-1}(\pi_{i,k} v))_{|\pi_{j,i}(S^1)}) = \mu(S^2, ((\pi_{j,i} \circ \pi_{i,k}) v)_{|S^2}). \tag{3.12}$$

Finally, observe again that $S^2 \cap \pi_{k,j}(S^2) \neq \emptyset$ and $\pi_{k,j}(S^2) \setminus S^2 \neq \emptyset$, hence, we can again apply case 2 to obtain,

$$\begin{aligned}
\mu(S^2, ((\pi_{j,i} \circ \pi_{i,k}) v)_{|S^2}) &= \mu(\pi_{k,j}(S^2), \pi_{k,j}^{-1}((\pi_{j,i} \circ \pi_{i,k}) v)_{|\pi_{k,j}(S^2)}) \\
&= \mu(S^3, ((\pi_{k,j} \circ \pi_{j,i} \circ \pi_{i,k}) v)_{|S^3}) = \mu(S, (\pi v)_{|S}) = \mu(\pi(S), (\pi^{-1} v)_{|\pi(S)}),
\end{aligned} \tag{3.13}$$

where the last equality follows from the fact that $\pi^{-1} = \pi$. Taking into account Eq. (3.11), (3.12), and (3.13) the desired result follows. □

Note that, when $\mu = \mu^{\mathsf{Sh}}$, the unique share function satisfying the corresponding properties is ρ^{Sh}, whereas when $\mu = \mu^{\mathsf{Ba}}$, it is ρ^{Ba}. Observe that when $\mu = \mu^{\mathsf{Sh}}$, the property of $\mathcal{M}$:μ-ADD is equivalent to $\mathcal{G}$:ADD, and hence, the characterization result above coincides with the characterization of Sh presented in Theorem 1.1.7.

3.2 Share functions on $\mathcal{MU}$

In this section, we extend the family of share functions defined in Section 3.1 to monotone games with a priori unions based on the so-called "multiplication" property. This property is first suggested by Owen (1977) when he introduces the Owen value. According to this property, the fraction of the total payment $v(N)$, received by a player in a game with a priori unions should be equal to the product of the share of the coalition she belongs to in the quotient game, and her share in some internal game played among the members of her union. In van der Laan & van den Brink (2002) and van den Brink & van der Laan (2005) the idea is formalized and applied to Ow and BO. As a result, the unified approach to ρ^{Sh} and ρ^{Ba} presented in Section 3.1 is extended to monotone games with a priori unions. In this section this approach is generalized to cover the share function associated with SCB.

A *monotone game with a priori unions* is a triple (N, v, P) where $(N, v) \in \mathcal{M}$ and $P \in P(N)$. The set of monotone games with a priori unions and fixed player set N is denoted by $\mathcal{MU}^N$ and the set of monotone games with a priori unions and any finite set of players by $\mathcal{MU}$. The set of monotone non null games with a priori unions is denoted by $\mathcal{MU}^+$, i.e., $\mathcal{MU}^+ = \{(N, v, P) \in \mathcal{MU} : (N, v) \in \mathcal{M}^+\}$. A *share function on $\mathcal{MU}$* is a map, ρ, that assigns a share vector, $\rho(N, v, P)$, to every monotone game with a priori unions, $(N, v, P) \in \mathcal{MU}$.

The only value on $\mathcal{GU}$ which is efficient among the three presented in Section 2.1 is the Owen value. Therefore, we can define a share function on $\mathcal{MU}$ associated to each value on $\mathcal{GU}$ dividing the payoffs by a proper amount.

Definition 3.2.1. The *Owen share function*, ρ^{Ow}, is the share function on $\mathcal{MU}$ defined for every $(N, v, P) \in \mathcal{MU}$ and $i \in N$ by

$$\rho_i^{\mathsf{Ow}}(N, v, P) = \frac{\mathsf{Ow}_i(N, v, P)}{v(N)} \quad \text{if } (N, v, P) \in \mathcal{MU}^+,$$

and $\rho_i^{\mathsf{Ow}}(N, v_0, P) = \frac{1}{m}\frac{1}{p_k}$, where $i \in P_k \in P$, $m = |M|$, and $p_k = |P_k|$.

Definition 3.2.2. The *Banzhaf-Owen share function*, ρ^{BO}, is the share function on $\mathcal{MU}$ defined for every $(N, v, P) \in \mathcal{MU}$ and $i \in N$ by

$$\rho_i^{\mathsf{BO}}(N, v, P) = \frac{\mathsf{BO}_i(N, v, P)}{\sum_{j \in N} \mathsf{BO}_j(N, v, P)} \quad \text{if } (N, v, P) \in \mathcal{MU}^+,$$

and $\rho_i^{\mathsf{BO}}(N, v_0, P) = \frac{1}{m}\frac{1}{p_k}$, where $i \in P_k \in P$, $m = |M|$, and $p_k = |P_k|$.

Definition 3.2.3. The *Symmetric coalitional Banzhaf share function*, ρ^{SCB}, is the share function on $\mathcal{MU}$ defined for every $(N, v, P) \in \mathcal{MU}$ and $i \in N$ by

$$\rho_i^{\mathsf{SCB}}(N, v, P) = \frac{\mathsf{SCB}_i(N, v, P)}{\sum_{j \in N} \mathsf{SCB}_j(N, v, P)} \quad \text{if } (N, v, P) \in \mathcal{MU}^+,$$

and $\rho_i^{\mathsf{SCB}}(N, v_0, P) = \frac{1}{m}\frac{1}{p_k}$, where $i \in P_k \in P$, $m = |M|$, and $p_k = |P_k|$.

However, as argued in van der Laan & van den Brink (2002) some of these definitions do not yield share functions on $\mathcal{MU}$ that satisfy the above described multiplication property. Hence, in what follows the class of share functions on $\mathcal{MU}$ that generalizes ρ^{μ} and satisfies the multiplication property will be introduced. For doing so, some concepts need to be formally introduced. First, following van der Laan & van den Brink (2002) two "internal" games among the players of each union will be defined. Let $(N, v, P) \in \mathcal{MU}$ and $i \in P_k \in P$, we define the so-called *Shapley internal-game* (P_k, v^{P_k}) and *Banzhaf internal-game* $(P_k, \overline{v}^{P_k})$ for every $S \subseteq P_k$ as follows,

$$\begin{aligned} v^{P_k}(S) &= \sum_{R \subseteq M \setminus k} \omega_{\mathsf{Sh}}^{m,r}\left[v(P_R \cup S) - v(P_R)\right], \text{ and} \\ \overline{v}^{P_k}(S) &= \sum_{R \subseteq M \setminus k} \omega_{\mathsf{Ba}}^{m,r}\left[v(P_R \cup S) - v(P_R)\right]. \end{aligned} \tag{3.14}$$

Recall that the weights ω_{Sh} and ω_{Ba} are the ones associated with μ^{Sh} and μ^{Ba} respectively and defined in Eq. (3.7) and (3.8). The Shapley internal-game is introduced back in Owen (1977) and studied together with the Banzhaf internal-game in van der Laan & van den Brink (2002) and van den Brink & van der Laan (2005). Next, the two internal games defined above are generalized and a family of internal games is considered.

Definition 3.2.4. Given a game with a priori unions $(N, v, P) \in \mathcal{MU}$ and a positive, additive, and anonymous function $\mu : \mathcal{M} \to \mathbb{R}$, we define for each $P_k \in P$ the *μ-internal game* $(P_k, v_{\mu}^{P_k})$, for every $S \subseteq P_k$, as follows

$$v_{\mu}^{P_k}(S) = \sum_{R \subseteq M \setminus k} \omega_{\mu}^{m,r}\left[v(P_R \cup S) - v(P_R)\right].$$

In some sense, the μ-internal game describes the possibilities of every coalition $S \subseteq P_k$ if it defects from P_k, assuming that the sharing is done according to μ. Note that the μ-internal game generalizes the Shapley and Banzhaf internal-games when μ^{Sh} and μ^{Ba} are considered, i.e., $(P_k, v_{\mu^{\mathsf{Sh}}}^{P_k}) = (P_k, v^{P_k})$ and

$(P_k, v^{P_k}_{\mu^{\mathsf{Ba}}}) = (P_k, \overline{v}^{P_k})$. Observe also that for every positive, additive, and anonymous $\mu : \mathcal{M} \to \mathbb{R}$ the μ-internal game described above is monotone whenever $(N, v) \in \mathcal{M}$.

Example 3.2.5. Consider the following monotone game with a priori unions: $(N, v, P) \in \mathcal{M}$, where $N = \{1, 2, 3, 4, 5\}$, $v = u_{\{1,2,4\}} + u_{\{3,5\}} + u_N$, where u_S is the unanimity game with carrier S, and $P = \{\{1, 2, 3\}, \{4, 5\}\}$. Then, the μ^{Sh}-internal game played among the members of coalition $P_1 = \{1, 2, 3\}$, that is, the Shapley internal-game of coalition P_1, is denoted by $(P_1, v^{P_1}_{\mu^{\mathsf{Sh}}})$, or simply by (P_1, v^{P_1}), and is defined as follows:

$$\begin{aligned}
&v^{P_1}(\{1\}) = v^{P_1}(\{2\}) = 0, \qquad v^{P_1}(\{3\}) = 1/2,\\
&v^{P_1}(\{1,2\}) = v^{P_1}(\{1,3\}) = v^{P_1}(\{2,3\}) = 1/2, \text{ and}\\
&v^{P_1}(\{1,2,3\}) = 3/2.
\end{aligned}$$

Note that when there are only two unions, i.e., $m = |P| = 2$, the Shapley and Banzhaf weights coincide, i.e., for every $r \in \{1, 2\}$, $\omega^{2,r}_{\mathsf{Sh}} = \omega^{2,r}_{\mathsf{Ba}} = 1/2$. Hence, in this example for every $k \in \{1, 2\}$, $(P_k, v^{P_k}) = (P_k, \overline{v}^{P_k})$. $\Diamond$

Next, a family of share functions on $\mathcal{MU}$ is defined. This family generalizes the family of share functions on $\mathcal{MU}$ defined in Theorem 3.1 of van den Brink & van der Laan (2005). It is a wider class since the use of two different share functions is allowed. First, a μ^1-share function is used to share among the unions and second, a μ^2-share function is used to share within each union.

Definition 3.2.6. Let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous. Then, the $\{\mu^1, \mu^2\}$-*share function on* $\mathcal{MU}$, ρ^{μ^1,μ^2}, is defined for every $(N, v, P) \in \mathcal{MU}$ and $i \in P_k \in P$, by

$$\rho^{\mu^1,\mu^2}_i(N, v, P) = \rho^{\mu^1}_k(M, v^P)\rho^{\mu^2}_i(P_k, v^{P_k}_{\mu^1}).$$

It can be easily checked that the above definition yields a share function on $\mathcal{MU}$, taking into account that ρ^{μ^1} and ρ^{μ^2} are share functions. Note that as mentioned above the class of share functions defined in Theorem 3.1 in van den Brink & van der Laan (2005) is a particular instance of the family of share functions on $\mathcal{MU}$ defined here. In fact, it is obtained taking $\mu^1 = \mu^2$. In Proposition 3.2.7 below some of the properties satisfied by the $\{\mu^1, \mu^2\}$-share functions are listed.

Proposition 3.2.7. *Let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous and $(N, v, P) \in \mathcal{MU}$. Then, ρ^{μ^1,μ^2} satisfies the following properties:*

1. $\sum_{i \in P_k} \rho_i^{\mu^1,\mu^2}(N, v, P) = \rho_k^{\mu^1}(M, v^P)$.

2. $\rho^{\mu^1,\mu^2}(N, v, P^N) = \rho^{\mu^2}(N, v)$.

3. $\rho^{\mu^1,\mu^2}(N, v, P^n) = \rho^{\mu^1}(N, v)$.

Proof. Property 1. is a direct consequence of ρ^{μ} being a share function. Property 3. is also straightforward since for $P = P^n$, $M = N$ and $v^P = v$. For Property 2., note that $(N, v^N_{\mu^1}) = (N, \omega^{0,1}_{\mu^1} v)$ and by definition of ρ^{μ}, $\rho^{\mu^2}(N, \omega^{0,1}_{\mu^1} v) = \rho^{\mu^2}(N, v)$. □

The Properties 2. and 3. show the fact that ρ^{μ^1,μ^2} generalizes the class of μ-share functions defined in Proposition 3.1.3. In the next proposition we show that the Owen and Symmetric coalitional Banzhaf share functions (Definitions 3.2.1 and 3.2.3) lie within the family of $\{\mu^1, \mu^2\}$-share functions for particular choices of μ^1 and μ^2.

Proposition 3.2.8. *The Owen and Symmetric coalitional Banzhaf share functions are particular instances of $\{\mu^1, \mu^2\}$-share functions. Indeed,*

$$\rho^{\mathsf{Ow}} = \rho^{\mu^{\mathsf{Sh}},\mu^{\mathsf{Sh}}}, \qquad \rho^{\mathsf{SCB}} = \rho^{\mu^{\mathsf{Ba}},\mu^{\mathsf{Sh}}}.$$

Proof. Let $(N, v, P) \in \mathcal{MU}$. If (N, v) is the null game, then, by Definitions 3.2.6, 3.2.1, and 3.2.3 and Proposition 3.1.3, for every $i \in P_k \in P$,

$$\rho_i^{\mu^{\mathsf{Sh}},\mu^{\mathsf{Sh}}}(N, v, P) = \rho_i^{\mathsf{Ow}}(N, v, P) = \frac{1}{m}\frac{1}{p_k} \qquad \text{and}$$
$$\rho_i^{\mu^{\mathsf{Ba}},\mu^{\mathsf{Sh}}}(N, v, P) = \rho_i^{\mathsf{SCB}}(N, v, P) = \frac{1}{m}\frac{1}{p_k}.$$

Let now $(N, v, P) \in \mathcal{MU}^+$ and $i \in P_k \in P$. If k is a null player in (M, v^P) by the monotonicity of (N, v) and Eq. (3.14), (P_k, v^{P_k}) and $(P_k, \overline{v}^{P_k})$ are both null games. Hence, on the one hand,

$$\rho_i^{\mu^{\mathsf{Sh}},\mu^{\mathsf{Sh}}}(N, v, P) = \rho_k^{\mu^{\mathsf{Sh}}}(M, v^P)\frac{1}{p_k} = 0 \qquad \text{and}$$
$$\rho_i^{\mu^{\mathsf{Ba}},\mu^{\mathsf{Sh}}}(N, v, P) = \rho_k^{\mu^{\mathsf{Ba}}}(M, v^P)\frac{1}{p_k} = 0.$$

On the other hand, since Sh and Ba satisfy $\mathcal{G}$:NPP, $\mathsf{Sh}_k(M, v^P) = \mathsf{Ba}_k(M, v^P) = 0$. Finally, it is easy to check that since Ow and SCB satisfy $\mathcal{GU}$:QGP and (N, v) is monotone, $\mathsf{Ow}_i(N,v,P)=0$ and $\mathsf{SCB}_i(N,v,P)=0$ and hence,

$$\rho_i^{\mathsf{Ow}}(N,v,P)=0 \qquad \text{and} \qquad \rho_i^{\mathsf{SCB}}(N,v,P)=0.$$

Finally, let $(N,v,P)\in\mathcal{MU}^+$ and $i\in P_k\in P$ where k is not a null player in (M,v^P). Then, by Eq. (3.14), (P_k,v^{P_k}) and $(P_k,\overline{v}^{P_k})$ are not null games, in particular, $v^{P_k}(P_k)>0$ and $\overline{v}^{P_k}(P_k)>0$. Then, by Definitions 3.2.6 and 3.2.4,

$$\begin{aligned}\rho_i^{\mu^{\mathsf{Sh}},\mu^{\mathsf{Sh}}}(N,v,P) &= \rho_k^{\mu^{\mathsf{Sh}}}(M,v^P)\rho_i^{\mu^{\mathsf{Sh}}}(P_k,v^{P_k}_{\mu^1}) = \rho_k^{\mathsf{Sh}}(M,v^P)\rho_i^{\mathsf{Sh}}(P_k,v^{P_k})\\ &= \frac{\mathsf{Sh}_k(M,v^P)}{v^P(M)}\frac{\mathsf{Sh}_i(P_k,v^{P_k})}{v^{P_k}(P_k)} = \frac{\mathsf{Sh}_k(M,v^P)}{v(N)}\frac{\mathsf{Ow}_i(N,v,P)}{\mathsf{Sh}_k(M,v^P)} = \rho_i^{\mathsf{Ow}}(N,v,P),\end{aligned}$$

where the last two equalities hold by the definitions of ρ^{Sh}, Sh, and (P_k,v^{P_k}) (Definitions 1.1.16 and 1.1.3 and Eq. (3.14), respectively).

Similarly, by Definitions 3.2.6 and 3.2.4 and Proposition 3.1.3,

$$\begin{aligned}\rho_i^{\mu^{\mathsf{Ba}},\mu^{\mathsf{Sh}}}(N,v,P) &= \rho_k^{\mu^{\mathsf{Ba}}}(M,v^P)\rho_i^{\mu^{\mathsf{Sh}}}(P_k,v^{P_k}_{\mu^{\mathsf{Ba}}}) = \frac{\mathsf{Ba}_k(M,v^P)}{\mu^{\mathsf{Ba}}(M,v^P)}\frac{\mathsf{Sh}_i(P_k,\overline{v}^{P_k})}{\overline{v}^{P_k}(P_k)}\\ &= \frac{\sum_{R\subseteq M\setminus k}\omega_{\mathsf{Ba}}^{m,r}\left[v^P(R\cup k)-v^P(R)\right]}{\mu^{\mathsf{Ba}}(M,v^P)}\frac{\sum_{T\subseteq P_k\setminus i}\omega_{\mathsf{Sh}}^{p_k,t}\left[\overline{v}^{P_k}(T\cup i)-\overline{v}^{P_k}(T)\right]}{\sum_{R\subseteq M\setminus k}\omega_{\mathsf{Ba}}^{m,r}\left[v(P_R\cup P_k)-v(P_R)\right]}\\ &= \frac{\sum_{T\subseteq P_k\setminus i}\omega_{\mathsf{Sh}}^{p_k,t}\left[\overline{v}^{P_k}(T\cup i)-\overline{v}^{P_k}(T)\right]}{\mu^{\mathsf{Ba}}(M,v^P)} = \frac{\mathsf{SCB}_i(N,v,P)}{\sum_{l\in M}\mathsf{Ba}_l(M,v^P)}\end{aligned}$$

From Theorem 2.1.7 we know that SCB satisfies the quotient game property ($\mathcal{GU}$:QGP). Hence,

$$\rho_i^{\mathsf{SCB}}(N,v,P)=\frac{\mathsf{SCB}_i(N,v,P)}{\sum_{l\in M}\sum_{i\in P_l}\mathsf{SCB}_i(N,v,P)}=\frac{\mathsf{SCB}_i(N,v,P)}{\sum_{l\in M}\mathsf{Ba}_l(M,v^P)}.$$

□

However, as we show in the remark below, the Banzhaf-Owen share function does not arise as a product of the Banzhaf share of the union in the external game and the Banzhaf share of the player in the Banzhaf internal-game.

Remark 3.2.9. The Banzhaf-Owen share function is not the $\{\mu^{\mathsf{Ba}},\mu^{\mathsf{Ba}}\}$-share function on $\mathcal{MU}$, i.e.,

$$\rho^{\mathsf{BO}}\neq\rho^{\mu^{\mathsf{Ba}},\mu^{\mathsf{Ba}}}.$$

Let $N=\{1,2,3,4\}$, $P=\{\{1,2,3\},\{4\}\}$, and $v=u_N$. Then, using the definitions we have that

$$\rho^{\mathsf{BO}}(N,v,P) = \left(\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{4}{7}\right) \neq \left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2}\right) = \rho^{\mu^{\mathsf{Ba}},\mu^{\mathsf{Ba}}}(N,v,P). \qquad \triangleleft$$

Next, we present the first characterization of the family of $\{\mu^1,\mu^2\}$-share functions by means of the aforementioned multiplication property. Note that, since the multiplication uses the μ-internal game $(P_k, v_\mu^{P_k})$, we will need to define the property for a given $\mu : \mathcal{M} \to \mathbb{R}$.

$\mathcal{MU}$:μ-MUL Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{MU}$, ρ, satisfies the *μ-multiplication property* if for every $(N,v,P) \in \mathcal{MU}$ and $i \in P_k \in P$,

$$\rho_i(N,v,P) = \rho_k(M, v^P, P^m)\rho_i(P_k, v_\mu^{P_k}, P^{P_k}).$$

Next, we define the null player property for share functions on $\mathcal{MU}$.

$\mathcal{MU}$:NPP A share function on $\mathcal{MU}$, ρ, satisfies the *null player property* if for every $(N,v,P) \in \mathcal{MU}^+$ and every null player $i \in N$ in (N,v),

$$\rho_i(N,v,P) = 0.$$

Note that, in line with $\mathcal{M}$:NPP, the null player property only requires that every null player in a monotone non-null game with a priori unions earns a zero payoff.

The next property is a slight modification of the individual symmetry property of van den Brink & van der Laan (2005). It states that two symmetric players in the original game get the same payoff in two situations, either, if they are in the same coalition or if they both form singleton coalitions.

$\mathcal{MU}$:SYM A share function on $\mathcal{MU}$, ρ, satisfies *symmetry* if for every $(N,v,P) \in \mathcal{MU}$ and every pair of symmetric players $i,j \in N$ in $(N,v) \in \mathcal{M}$ such that $i,j \in P_k \in P$ or $\{i\},\{j\} \in P$,

$$\rho_i(N,v,P) = \rho_j(N,v,P).$$

The last two properties extend the μ-additivity property ($\mathcal{M}$:μ-ADD) of share functions on $\mathcal{M}$ to the framework of monotone games with a priori unions.

$\mathcal{MU}$:μ-ADD(P^N) Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{MU}$, ρ, satisfies *μ-additivity for P^N* if for every pair of monotone games $(N, v), (N, w) \in \mathcal{M}$,

$$\mu(N, v + w)\rho(N, v + w, P^N) = \mu(N, v)\rho(N, v, P^N) + \mu(N, w)\rho(N, w, P^N).$$

$\mathcal{MU}$:μ-ADD(P^n) Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{MU}$, ρ, satisfies *μ-additivity for P^n* if for every pair of monotone games $(N, v), (N, w) \in \mathcal{M}$,

$$\mu(N, v + w)\rho(N, v + w, P^n) = \mu(N, v)\rho(N, v, P^n) + \mu(N, w)\rho(N, w, P^n).$$

The next result states that the class of $\{\mu^1, \mu^2\}$-share functions is characterized by the five properties above.

Theorem 3.2.10. *Let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous. Then, ρ^{μ^1,μ^2} is the unique share function on $\mathcal{MU}$ satisfying $\mathcal{MU}$:μ^1-MUL, $\mathcal{MU}$:NPP, $\mathcal{MU}$:SYM, $\mathcal{MU}$:μ^2-ADD(P^N), and $\mathcal{MU}$:μ^1-ADD(P^n).*

Proof.

(1) *Existence.* From the definition of ρ^{μ^1,μ^2} and Properties 2. and 3. of Proposition 3.2.7 we have that ρ^{μ^1,μ^2} satisfies $\mathcal{MU}$:μ^1-MUL.

Let $(N, v, P) \in \mathcal{MU}^+$ and $i \in P_k \in P$ a null player in (N, v). If k is a null player in (M, v^P), since $(M, v^P) \in \mathcal{M}^+$ and ρ^{μ^1} satisfies $\mathcal{M}$:NPP,

$$\rho_k^{\mu^1}(M, v^P) = 0.$$

Next, suppose that k is not a null player in (M, v^P), i.e., there is $R \subseteq M \setminus k$ such that $v(P_R \cup P_k) - v(P_R) > 0$. Then, $(P_k, v_{\mu^1}^{P_k})$ is not the null game because $v_{\mu^1}^{P_k}(P_k) \geq \omega_{\mu^1}^{m,r}\,[v(P_R \cup P_k) - v(P_R)] > 0$. On the other hand it is easy to check that i is a null player in $(P_k, v_{\mu^1}^{P_k})$ because it is so in (N, v). Since ρ^{μ^2} satisfies $\mathcal{M}$:NPP,

$$\rho_i^{\mu^2}(P_k, v_{\mu^1}^{P_k}) = 0.$$

That is, in any case $\rho_i^{\mu^1,\mu^2}(N, v, P) = 0$.

Next, let $i, j \in N$ be symmetric players in $(N, v) \in \mathcal{M}^+$. If $i, j \in P_k \in P$, then i and j are symmetric players in $(P_k, v_{\mu^1}^{P_k})$. Since ρ^{μ^2} satisfies $\mathcal{M}$:SYM,

$$\rho_i^{\mu^1,\mu^2}(N, v, P) = \rho_k^{\mu^1}(M, v^P)\rho_i^{\mu^2}(P_k, v_{\mu^1}^{P_k}) = \rho_k^{\mu^1}(M, v^P)\rho_j^{\mu^2}(P_k, v_{\mu^1}^{P_k}) = \rho_j^{\mu^1,\mu^2}(N, v, P).$$

If $\{i\}, \{j\} \in P$, there are $k, l \in M$ such that $P_k = \{i\}$ and $P_l = \{j\}$. Then, k and l

are symmetric players in (M, v^P). Since ρ^{μ^1} satisfies $\mathcal{M}$:SYM,

$$\rho_i^{\mu^1,\mu^2}(N,v,P) = \rho_k^{\mu^1}(M,v^P)\rho_i^{\mu^2}(\{i\},v_{\mu^1}^{P_k}) = \rho_k^{\mu^1}(M,v^P)$$
$$= \rho_l^{\mu^1}(M,v^P) = \rho_l^{\mu^1}(M,v^P)\rho_j^{\mu^2}(\{j\},v_{\mu^1}^{P_l}) = \rho_j^{\mu^1,\mu^2}(N,v,P).$$

Finally, Property 2. of Proposition 3.2.7 and the fact that ρ^{μ^2} satisfies $\mathcal{M}$:μ^2-ADD directly imply that ρ^{μ^1,μ^2} satisfies $\mathcal{MU}$:μ^2-ADD(P^N). And the same reasoning using Property 1. instead of Property 2. of Proposition 3.2.7 is valid and shows that ρ^{μ^1,μ^2} satisfies $\mathcal{MU}$:μ^1-ADD(P^n).

(2) *Uniqueness.* Suppose that ρ is a share function on $\mathcal{MU}$ satisfying the five properties. Then, $\mathcal{MU}$:SYM and the fact that ρ is a share function on $\mathcal{MU}$ imply that for every $i \in N$, $\rho_i(N, v_0, P^N) = \rho_i(N, v_0, P^n) = 1/n$. Next, for every $\emptyset \neq T \subseteq N$, let $(N, w_T) \in \mathcal{M}$ be given by $w_T = c_T u_T$, where $c_T > 0$ and (N, u_T) is the unanimity game of coalition T. For $i \in N \setminus T$, $\mathcal{MU}$:NPP implies that $\rho_i(N, w_T, P^N) = \rho_i(N, w_T, P^n) = 0$. For $i \in T$, $\mathcal{MU}$:SYM and the fact that ρ is a share function on $\mathcal{MU}$ imply that $\rho_i(N, w_T, P^N) = \rho_i(N, w_T, P^n) = 1/|T|$. Hence, $\rho(N, w_T, P^N)$ and $\rho(N, w_T, P^N)$ are uniquely determined.

For $(N, v) \in \mathcal{M}$, recall from the preliminaries (page 2) that $v + v^- = v^+$, with both v^- and v^+ being non negative linear combinations of unanimity games. Then, using $\mathcal{MU}$:μ^2-ADD(P^N) and $\mathcal{MU}$:μ^1-ADD(P^n) we know that $\rho(N, v^-, P^N)$, $\rho(N, v^+, P^N)$, $\rho(N, v^-, P^n)$, and $\rho(N, v^+, P^n)$ are uniquely determined. Finally, for an arbitrary $(N, v, P) \in \mathcal{MU}$, the uniqueness of $\rho(N, v, P)$ directly follows from $\mathcal{MU}$:μ^1-MUL. □

To end with, we propose a second characterization of ρ^{μ^1,μ^2} based on a property which is stronger than the two additivity properties used in the characterization above. Next, let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous, we extend the previous properties of μ^2-additivity for P^N ($\mathcal{MU}$:μ-ADD(P^N)) and μ^1-additivity for P^n ($\mathcal{MU}$:μ-ADD(P^n)) to the new (μ^1, μ^2)-additivity which is stated for an arbitrary $P \in P(N)$.

$\mathcal{MU}$:(μ^1, μ^2)-ADD Let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{MU}$, ρ, satisfies (μ^1, μ^2)-*additivity* if for every $(N, v, P), (N, w, P) \in \mathcal{MU}$, $z = v+w$, and every $i \in P_k \in P$,

$$\mu^1(M, z^P)\mu^2(P_k, z_{\mu^1}^{P_k})\rho_i(N, z, P)$$
$$= \left[\mu^2(P_k, v_{\mu^1}^{P_k})\rho_i(P_k, v_{\mu^1}^{P_k}, P^{P_k}) + \mu^2(P_k, w_{\mu^1}^{P_k})\rho_i(P_k, w_{\mu^1}^{P_k}, P^{P_k})\right]$$
$$\times \left[\mu^1(M, v^P)\rho_k(M, v^P, P^m) + \mu^1(M, w^P)\rho_k(M, w^P, P^m)\right].$$

We obtain the next characterization replacing $\mathcal{MU}$:μ^1-MUL, $\mathcal{MU}$:μ^2-ADD(P^N), and $\mathcal{MU}$:μ^1-ADD(P^n) by $\mathcal{MU}$:(μ^1, μ^2)-ADD.

Corollary 3.2.11. *Let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous. Then ρ^{μ^1,μ^2} is the unique share function on $\mathcal{MU}$ satisfying $\mathcal{MU}$:NPP, $\mathcal{MU}$:SYM, and $\mathcal{MU}$:(μ^1, μ^2)-ADD.*

Proof. (1) *Existence.* By Definition 3.2.6 and the $\mathcal{M}$:μ-ADD property of ρ^μ it is straightforward to check that ρ^{μ^1,μ^2} satisfies $\mathcal{MU}$:(μ^1, μ^2)-ADD. Finally, ρ^{μ^1,μ^2} satisfies the remaining properties by Theorem 3.2.10.

(2) *Uniqueness.* It is a consequence of the uniqueness in Theorem 3.2.10. Note that $\mathcal{MU}$:(μ^1, μ^2)-ADD includes $\mathcal{MU}$:μ^2-ADD(P^N), $\mathcal{MU}$:μ^1-ADD(P^n), and $\mathcal{MU}$:μ^1-MUL. Indeed, if we take $P = P^n$, $\mathcal{MU}$:(μ^1, μ^2)-ADD becomes $\mathcal{MU}$:μ^1-ADD(P^n) and if we take $P = P^N$, $\mathcal{MU}$:(μ^1, μ^2)-ADD becomes $\mathcal{MU}$:μ^2-ADD(P^N). Finally, if we take $w = v^0$, $\mathcal{MU}$:(μ^1, μ^2)-ADD becomes $\mathcal{MU}$:μ^1-MUL. □

To conclude, we check that the properties used in the characterizations are independent.

Remark 3.2.12. The properties considered in Theorem 3.2.10 are independent as the following examples show. Let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous.

(i) Let ρ^1 be the share function on $\mathcal{MU}$ defined for every $(N, v, P) \in \mathcal{MU}$ as follows:

- If $P = P^n$, $\rho^1(N, v, P) = \rho^{\mu^1}(N, v)$,
- Otherwise, $\rho^1(N, v, P) = \rho^{\mu^2}(N, v)$.

Then, ρ^1 satisfies $\mathcal{MU}$:NPP, $\mathcal{MU}$:SYM, $\mathcal{MU}$:μ^2-ADD(P^N), and $\mathcal{MU}$:μ^1-ADD(P^n), but not $\mathcal{MU}$:μ^1-MUL.

(ii) The share function on $\mathcal{MU}$, ρ^2, defined for every $(N, v, P) \in \mathcal{MU}$ and $i \in N$ with $i \in P_k$ and $k \in M$, by

$$\rho_i^2(N, v, P) = \frac{1}{m \cdot p_k},$$

satisfies $\mathcal{MU}$:μ^1-MUL, $\mathcal{MU}$:SYM, $\mathcal{MU}$:μ^2-ADD(P^N), and $\mathcal{MU}$:μ^1-ADD(P^n), but not $\mathcal{MU}$:NPP.

(iii) Let a and b be two fixed and different players. If $N = \{a, b\}$, for every $(N, v) \in \mathcal{M}$ let $\lambda_1, \lambda_2 > 0$ be such that $\mu^2(N, v) = \lambda_1 v(a) + \lambda_1 v(b) + \lambda_2 v(N)$. Define the share function on $\mathcal{MU}$, ρ^3, for every $(N, v, P) \in \mathcal{MU}$ as follows:

- If $N = \{a, b\}$, $P = P^N$, and $(N, v) \in \mathcal{M}^+$

$$\begin{cases} \rho_a^3(N,v,P) & = \frac{\lambda_1 v(a)+\lambda_2(v(N)-v(b))}{\mu^2(N,v)} \\ \rho_b^3(N,v,P) & = \frac{(\lambda_1+\lambda_2)v(b)}{\mu^2(N,v)} \end{cases}$$

- If $N \neq \{a, b\}$ and $P = P^N$, $\rho^3(N, v, P) = \rho^{\mu^2}(N, v)$.
- If $P = P^n$, $\rho^3(N, v, P) = \rho^{\mu^1}(N, v)$.
- Otherwise, for every $i \in N$ with $i \in P_k$ and $k \in M$,

$$\rho_i^3(N,v,P) = \rho_k^3(M, v^P, P^m)\rho_i^3(P_k, v_{\mu^1}^{P_k}, P^{P_k}).$$

Then, ρ^3 satisfies $\mathcal{MU}$:μ^1-MUL, $\mathcal{MU}$:NPP, $\mathcal{MU}$:μ^2-ADD(P^N), and $\mathcal{MU}$:μ^1-ADD(P^n), but not $\mathcal{MU}$:SYM.

(iv) The share function on $\mathcal{MU}$, ρ^4, defined for every $(N, v, P) \in \mathcal{MU}$ by

$$\rho^4(N,v,P) = \rho^{\mu^1,\mu^1}(N,v,P),$$

satisfies $\mathcal{MU}$:μ^1-MUL, $\mathcal{MU}$:NPP, $\mathcal{MU}$:SYM, and $\mathcal{MU}$:μ^1-ADD(P^n), but not $\mathcal{MU}$:μ^2-ADD(P^N).

(v) Let ρ^5 be a share function on $\mathcal{MU}$ defined for every $(N, v, P) \in \mathcal{MU}$ as follows:

- If $P = P^n$ or $P = P^N$, $\rho^5(N, v, P) = \rho^{\mu^2}(N, v)$,
- Otherwise, for every $i \in N$ with $i \in P_k$ and $k \in M$,

$$\rho_i^5(N,v,P) = \rho_k^5(M, v^P, P^m)\rho_i^5(P_k, v_{\mu^1}^{P_k}, P^{P_k}).$$

Then, ρ^5 satisfies $\mathcal{MU}$:μ^1-MUL, $\mathcal{MU}$:NPP, $\mathcal{MU}$:SYM, and $\mathcal{MU}$:μ^2-ADD (P^N), but not $\mathcal{MU}$:μ^1-ADD(P^n). $\triangleleft$

Remark 3.2.13. The properties considered in Corollary 3.2.11 are independent as the following examples show. Let $\mu^1, \mu^2 : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous.

(i) The share function on $\mathcal{MU}$, ρ^1, defined above satisfies $\mathcal{MU}$:NPP and $\mathcal{MU}$:SYM, but not $\mathcal{MU}$:(μ^1, μ^2)-ADD.

(ii) The share function on $\mathcal{MU}$, ρ^2, defined above satisfies $\mathcal{MU}$:SYM and $\mathcal{MU}$:(μ^1, μ^2)-ADD, but not $\mathcal{MU}$:NPP.

(iii) The share function on $\mathcal{MU}$, ρ^3, defined above satisfies $\mathcal{MU}$:NPP and $\mathcal{MU}$:(μ^1, μ^2)-ADD, but not $\mathcal{MU}$:SYM. $\triangleleft$

3.3 Share functions on $\mathcal{ML}$

In the present section we introduce a family of share functions for monotone games with levels structures of cooperation. The different share functions depend only on the choice of a positive, additive, and anonymous real valued function $\mu : \mathcal{M} \to \mathbb{R}$. Therefore, we generalize the family of share functions on $\mathcal{M}$, ρ^{μ}, (Section 3.1.2) and the family of share functions on $\mathcal{MU}$, $\rho^{\mu,\mu}$, (Section 3.2) to monotone games with an arbitrary number of levels structures of cooperation. Let $\mathcal{ML}$ be the set monotone games with leveles structure of cooperation, i.e., $\mathcal{ML} = \{(N, v, \underline{B}) \in \mathcal{GL} : (N, v) \in \mathcal{M}\}$.

A *share function on* $\mathcal{ML}$ is a map, ρ, that assigns a share vector $\rho(N, v, \underline{B})$ to every monotone game with levels structure of cooperation $(N, v, \underline{B}) \in \mathcal{ML}$. We generalize the results of van den Brink & van der Laan (2005) to monotone games with an arbitrary number of levels structure of cooperation. That is, if in Section 3.2 we take $\mu^1 = \mu^2$, all the results of Section 3.2 will be generalized to monotone games with an arbitrary number of levels structure of cooperation. To do so, for each $i \in N$ and $r \in \{0, 1, \dots, k\}$, let $N_r^i(N, \underline{B}) = \{U : U \in B_r, U \subseteq U_{r+1}^i\}$ be the set of all unions of the r^{th} level of cooperation that form the union U_{r+1}^i of the $(r+1)^{\text{th}}$ level of cooperation. Recall that, for each $i \in N$ and $r \in \{0, \dots, k+1\}$, $U_r^i \in B_r$ are such that $\{i\} = U_0^i \subseteq U_1^i \subseteq \cdots \subseteq U_k^i \subseteq U_{k+1}^i = N$. In order to ease the notation, we write N_r^i or $N_r^i(\underline{B})$ when no confusion may arise.

Example 3.3.1. Recall the levels structure of cooperation of Example 2.2.1, i.e., $N = \{1, 2, 3, 4, 5, 6\}$ and $\underline{B} = \{B_0, B_1, B_2, B_3\}$ given by

$$
\begin{aligned}
B_3 &= \{\{1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6\}\},\\
B_2 &= \{\{1, \quad 2, \quad 3\}, \{4, \quad 5, \quad 6\}\},\\
B_1 &= \{\{1, \quad 2\}, \{3\}, \{4\}, \{5, \quad 6\}\}, \text{ and}\\
B_0 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.
\end{aligned}
$$

Consider player $i = 1 \in N$, then,

$$\overbrace{\{1\}}^{U_0^i} \subseteq \overbrace{\{1,2\}}^{U_1^i} \subseteq \overbrace{\{1,2,3\}}^{U_2^i} \subseteq \overbrace{N}^{U_3^i}, \text{ and}$$

$$N_0^i = \{\{1\}, \{2\}\}, \quad N_1^i = \{\{1, 2\}, \{3\}\}, \text{ and} \quad N_2^i = \{\{1, 2, 3\}, \{4, 5, 6\}\}. \qquad \Diamond$$

Following the philosophy behind the definition of the internal and external (or quotient) games for games with coalition structure, given a game with k levels structure of cooperation we define $k + 1$ different games for each player.

Definition 3.3.2. Let μ be positive, additive, and anonymous. For every $(N, v, \underline{B}) \in \mathcal{ML}$, $i \in N$, and $r \in \{0, 1, \ldots, k\}$, the r^{th} *level game of* i *with respect to* μ, $(N_r^i(N, \underline{B}), v_\mu^{i,r}) \in \mathcal{M}$, is given for every $T \subseteq N_r^i$, by

$$v_\mu^{i,r}(T) = \sum_{\substack{S_k \subseteq N_k^i \\ U_k^i \notin S_k}} \cdots \sum_{\substack{S_{r+1} \subseteq N_{r+1}^i \\ U_{r+1}^i \notin S_{r+1}}} \omega_\mu^{n_k^i, s_k} \cdots \omega_\mu^{n_{r+1}^i, s_{r+1}} \left[v(S_{k,r+1} \cup T) - v(S_{k,r+1}) \right], \tag{3.15}$$

where $S_{k,r} = S_k \cup S_{k-1} \cup \cdots \cup S_r$ and lowercase letters denote cardinalities, i.e., $n_r^i = |N_r^i|$ and $s_r = |S_r|$ for every $r \in \{0, \ldots k\}$.

Note that in the definition above we abuse notation and simply write $v(S_{k,r+1})$ to denote $v(\{i \in U : U \in S_l$ for some $l \in \{r+1, \ldots, k\}\})$. This kind of notational abuse is done in many places throughout this section. Further, we assume that the empty set always belongs to the summation in Eq. (3.15). Therefore the k^{th} level game of any player with respect to any μ coincides with the k^{th} union level game (Definition 2.2.2 on page 29) or the quotient game (Definition 2.1.1 on page 23) when there is only one level, i.e., we have $v_\mu^{i,k}(T) = v^k(T)$ for every $T \subseteq N_k^i = \{U \in B_k : U \subseteq N\} = B_k$. Moreover, it is easy to check that, when $k = 1$, that is, when $\underline{B} = \{B_0, B_1, B_2\} = \{\{\{i\} : i \in N\}, B_1, \{N\}\}$ with $B_1 = P \in P(N)$, for every $i \in P_k \in P$, $(N_0^i, v_\mu^{i,0}) = (P_k, v_\mu^{P_k})$ and $(N_1^i, v_\mu^{i,1}) = (M, v^P)$. It is as well straightforward to check that for every $i \in N$ and $r \in \{0, \ldots, k\}$, $(N_r^i, v^{i,r}) \in \mathcal{M}$ whenever $(N, v) \in \mathcal{M}$. The r^{th} level game of i with respect to μ, describes the possibilities of each coalition of unions of N_r^i if it defects form U_{r+1}^i and forms its own union in the $(r+1)^{\text{th}}$ level of cooperation, considering that the sharing is carried out according to μ. Observe that given a positive, additive, and anonymous μ and $(N, v, \underline{B}) \in \mathcal{ML}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$, for every $i, j \in U \in B_r$ and $l \geq r$, $(N_l^i(N, \underline{B}), v_\mu^{i,l}) = (N_l^j(N, \underline{B}), v_\mu^{j,l})$. Hence, we are actually defining as many internal games as unions in levels $1, \ldots, k+1$.

Next, in line with Theorem 3.1 in van den Brink & van der Laan (2005) (which corresponds to Definition 3.2.6 of the present document restricted to the case $\mu^1 = \mu^2$), a class of share functions on $\mathcal{ML}$ is defined based on positive, additive, and anonymous μ functions. Each member of this class of share functions on $\mathcal{ML}$ is denoted by $\underline{\rho}^\mu$ and is defined as a product of the shares in each of the internal games.

Definition 3.3.3. Let $\mu : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous. Then, the *μ-share function on $\mathcal{ML}$, $\underline{\rho}^\mu$*, is defined for every $(N, v, \underline{B}) \in \mathcal{ML}$ and $i \in N$, by

$$\underline{\rho}_i^\mu(N, v, \underline{B}) = \prod_{r=0}^{k} \rho^\mu_{U_r^i}(N_r^i, v_\mu^{i,r}), \tag{3.16}$$

where ρ^μ is as defined in Proposition 3.1.3.

It can be easily checked that the above definition indeed yields a share function on $\mathcal{ML}$. It is a consequence of ρ^μ being a share function on $\mathcal{M}$ and the aforementioned fact that for each $r \in \{1, \ldots, k\}$ and $U \in B_r$, $(N_r^i, v_\mu^{i,r})$ is the same game for every $i \in U$. In Proposition 3.3.4 below it is shown that μ-share functions on $\mathcal{ML}$ satisfy some other properties, especially that each μ-share function on $\mathcal{ML}$ coincides, when the levels structure of cooperation is trivial, with the μ-share function on $\mathcal{M}$ defined in Proposition 3.1.3 for games without levels structure of cooperation.

Proposition 3.3.4. *Let $\mu : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous. Then*

(i) For every $(N, v) \in \mathcal{M}$,

$$\underline{\rho}^\mu(N, v, \underline{B_0}) = \rho^\mu(N, v).$$

(ii) For every $(N, v, \underline{B}) \in \mathcal{ML}^+$ and every null player $i \in N$ in (N, v),

$$\underline{\rho}_i^\mu(N, v, \underline{B}) = 0.$$

(iii) For every $(N, v, \underline{B}) \in \mathcal{ML}$ and every pair $i, j \in N$ of symmetric players in (N, v) that are in the same union of every level, i.e., given that $\underline{B} = \{B_0, \ldots, B_{k+1}\}$, $i, j \in U \in B_1$,

$$\underline{\rho}_i^\mu(N, v, \underline{B}) = \underline{\rho}_j^\mu(N, v, \underline{B}).$$

Proof. Let $\mu : \mathcal{M} \to \mathbb{R}$ be positive, additive, and anonymous.

To prove *(i)*, let $(N, v) \in \mathcal{M}$ and $(N, \underline{B})$ be the trivial levels structure of cooperation, i.e., $\underline{B} = \underline{B_0} = \{B_0, B_1\}$. Then, for every $i \in N$, $N_0^i = N$ and for every positive, additive, and anonymous $\mu : \mathcal{M} \to \mathbb{R}$, $v_\mu^{i,0} = kv$ for some $k \in \mathbb{R}_+$. Hence, by Definition 3.3.3,

$$\underline{\rho}_i^\mu(N, v, \underline{B_0}) = \rho^\mu_{U_0^i}(N_0^i, v_\mu^{i,0}) = \rho_i^\mu(N, v),$$

where the last equality holds by Proposition 3.1.3.

To prove *(ii)*, let $(N, v, \underline{B}) \in \mathcal{ML}^+$ and $i \in N$ a null player in (N, v). First of all, and by Definition 3.3.2, $(N, v) \in \mathcal{M}^+$ implies that $(N^i_k, v^{i,k}_\mu) \in \mathcal{M}^+$. We will show that for every $r \in \{1, \dots, k\}$, if U^i_r is not a null player in $(N^i_r, v^{i,k}_\mu)$, then $(N^i_{r-1}, v^{i,r-1}_\mu) \in \mathcal{M}^+$. We first show it for $r = k$ because it is the easiest case. Let $U^i_k \in N^i_k$ be a non null player in $(N^i_k, v^{i,k}_\mu)$, hence, there is $S^*_k \subseteq N^i_k \setminus U^i_k$ such that

$$v^{i,k}_\mu(S^*_k \cup U^i_k) - v^{i,k}_\mu(S^*_k) = v(S^*_k \cup U^i_k) - v(S^*_k) > 0. \tag{3.17}$$

Next, we show that $(N^i_{k-1}, v^{i,k-1}_\mu)$ is not a null game. Note that by the monotonicity of (N, v),

$$v^{i,k-1}_\mu(N^i_{k-1}) = \sum_{\substack{S_k \subseteq N^i_k \\ U^i_k \notin S_k}} \omega^{n^i_k, s_k}_\mu \left[v(S_k \cup U^i_k) - v(S_k)\right] \geq 0. \tag{3.18}$$

Observe that in the summation of Eq. (3.18) we also take S^*_k and, hence, by Eq. (3.17), $v^{i,k-1}_\mu(N^i_{k-1}) > 0$.

Next, let U^i_r be a non null player in $(N^i_r, v^{i,r}_\mu)$, i.e., there is $S^*_r \subseteq N^i_r \setminus U^i_r$ such that

$$\begin{aligned} & v^{i,r}_\mu(S^*_r \cup U^i_r) - v^{i,r}_\mu(S^*_r) \\ & = \sum_{\substack{S_k \subseteq N^i_k \\ U^i_k \notin S_k}} \cdots \sum_{\substack{S_{r+1} \subseteq N^i_{r+1} \\ U^i_{r+1} \notin S_{r+1}}} \omega^{n^i_k, s_k}_\mu \cdots \omega^{n^i_{r+1}, s_{r+1}}_\mu \left[v(S_{k,r+1} \cup S^*_r \cup U^i_r) - v(S_{k,r+1} \cup S^*_r)\right] > 0, \end{aligned} \tag{3.19}$$

we show that $(N^i_{r-1}, v^{i,r-1}_\mu)$ is not a null game. Again, by the monotonicity of (N, v),

$$v^{i,r-1}_\mu(N^i_{r-1}) = \sum_{\substack{S_k \subseteq N^i_k \\ U^i_k \notin S_k}} \cdots \sum_{\substack{S_r \subseteq N^i_r \\ U^i_r \notin S_r}} \omega^{n^i_k, s_k}_\mu \cdots \omega^{n^i_r, s_r}_\mu \left[v(S_{k,r} \cup U^i_r) - v(S_{k,r})\right] \geq 0. \tag{3.20}$$

Finally, observe that in the summation of Eq. (3.20) we also take S^*_r and, hence, by Eq. (3.19), $v^{i,r-1}_\mu(N^i_{r-1}) > 0$.

Hence, going over again, we have seen first that $(N^i_k, v^{i,k}_\mu)$ is not a null game, and second, that for every $r \in \{1, \dots, k\}$ if U^i_r is not a null player in $(N^i_r, v^{i,k}_\mu)$, then $(N^i_{r-1}, v^{i,r-1}_\mu) \in \mathcal{M}^+$. In other words, two cases may arise,

- either there is $r \in \{1, \dots, k\}$ such that U^i_r is a null player in a non null game $(N^i_r, v^{i,r}_\mu)$, which by $\mathcal{M}$:NPP of ρ^μ means that $\rho^\mu_{U^i_r}(N^i_r, v^{i,r}_\mu) = 0$,
- or, for every $r \in \{1, \dots, k\}$, $(N^i_{r-1}, v^{i,r-1}_\mu)$ is not a null game. In particular

$(N_0^i, v_\mu^{i,0}) \in \mathcal{M}^+$. Finally, since i is a null player in (N, v) it is also a null player in $(N_0^i, v_\mu^{i,0})$, which means that $\rho^\mu_{U_0^i}(N_0^i, v_\mu^{i,0}) = 0$.

Therefore, in any case $\underline{\rho_i^\mu}(N, v, \underline{B}) = 0$.

To show *(iii)*, let $(N, v, \underline{B}) \in \mathcal{ML}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $i, j \in N$ a pair of symmetric players in (N, v) such that $i, j \in U \in B_1$. From the latter condition it follows that for every $r \in \{1, \ldots, k\}$, $U_r^i = U_r^j$ and $(N_r^i, v_\mu^{i,r}) = (N_r^j, v_\mu^{j,r})$. Then, by Definition 3.3.3

$$\underline{\rho_i^\mu}(N, v, \underline{B}) - \underline{\rho_j^\mu}(N, v, \underline{B}) = \left[\rho_i^\mu(N_0^i, v_\mu^{i,0}) - \rho_j^\mu(N_0^j, v_\mu^{j,0})\right] \prod_{r=1}^{k} \rho^\mu_{U_r^i}(N_r^i, v_\mu^{i,r}). \tag{3.21}$$

Finally, note that $(N_0^i, v_\mu^{i,0}) = (N_0^j, v_\mu^{j,0})$ and since i, j are symmetric players in (N, v), they are also symmetric players in $(N_0^i, v_\mu^{i,0})$. By $\mathcal{M}$:SYM of ρ^μ,

$$\rho_i^\mu(N_0^i, v_\mu^{i,0}) - \rho_j^\mu(N_0^j, v_\mu^{j,0}) = 0$$

which together with Eq. (3.21) means that $\underline{\rho_i^\mu}(N, v, \underline{B}) - \underline{\rho_j^\mu}(N, v, \underline{B}) = 0$ and concludes the proof. □

Property *(i)* shows that the μ-share function on $\mathcal{ML}$, $\underline{\rho^\mu}$, extends the μ-share function on $\mathcal{M}$, ρ^μ. Properties *(ii)* and *(iii)* are standard and apply to any game with levels structure of cooperation. Although only weaker versions of these two properties will be needed in the forthcoming characterization results, it is important to point out that these stronger versions are also satisfied.

In the last part of this section we prove that the share function on $\mathcal{ML}$ defined from the Shapley levels value, Sh^{L}, belongs to the family of μ-share functions on $\mathcal{ML}$ introduced in Definition 3.3.2. In order to do so, we give an alternative expression of the Shapley levels value which will be very useful henceforth. We will use two of the properties satisfied by the Shapley levels value (see Theorem 2.2.9) which are recalled next.

$\mathcal{GL}$:LGP A value on $\mathcal{GL}$, f, satisfies the *level game property* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $U \in B_r$ for some $r \in \{1, \ldots, k\}$,

$$\sum_{i \in U} \mathsf{f}_i(N, v, \underline{B}) = \mathsf{f}_U(B_r, v^r, \underline{B_r}).$$

$\mathcal{GL}$:LBC A value on $\mathcal{GL}$, f, satisfies *level balanced contributions* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $i, j \in U \in B_1$,

$$\mathsf{f}_i(N, v, \underline{B}) - \mathsf{f}_i(N, v, \underline{B^{-j}}) = \mathsf{f}_j(N, v, \underline{B}) - \mathsf{f}_j(N, v, \underline{B^{-i}}).$$

Lemma 3.3.5. *Let* $(N, v, \underline{B}) \in \mathcal{GL}$ *be a game with levels structure of cooperation. Then, for each* $i \in N$,

$$\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B}) = \sum_{S_k \subseteq N^i_k \setminus U^i_k} \cdots \sum_{S_0 \subseteq N^i_0 \setminus U^i_0} \omega_{\mathsf{Sh}}^{n^i_k, s_k} \cdots \omega_{\mathsf{Sh}}^{n^i_0, s_0} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right]. \tag{3.22}$$

Proof. We prove it by induction on the number k of levels of $(N, \underline{B})$. The case $k = 1$ is a consequence of Sh^{L} being a generalization of Ow. Hence, suppose that the Shapley levels value $\mathsf{Sh}^{\mathsf{L}}(N, v, \underline{B})$ is obtained from Eq. (3.22) for every $(N', v', \underline{B'}) \in \mathcal{GL}$ such that $(N, \underline{B'})$ has at most $k - 1$ levels and let $(N, v, \underline{B}) \in \mathcal{GL}$ be such that $(N, \underline{B})$ has k levels, i.e., $\underline{B} = \{B_0, \ldots, B_{k+1}\}$. Let $i \in N$, we prove that $\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B})$ is obtained from Eq. (3.22) by a second induction on $|U^i_1|$ (recall that $U^i_1 \in B_1$ is such that $i \in U^i_1$). If $u = 1$, i.e. $U^i_1 = \{i\}$, since Sh^{L} satisfies $\mathcal{GL}$:LGP, we have

$$\begin{aligned}
\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B}) &= \sum_{i \in U^i_1} \mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B}) = \mathsf{Sh}^{\mathsf{L}}_{U^i_1}(B_1, v^1, \underline{B_1}) \\
&= \sum_{\substack{S_k \subseteq N^i_k(\underline{B_1}) \\ U^i_k \notin S_k}} \cdots \sum_{\substack{S_1 \subseteq N^i_1(\underline{B_1}) \\ U^i_1 \notin S_1}} \omega_{\mathsf{Sh}}^{n^i_k(\underline{B_1}), s_k} \cdots \omega_{\mathsf{Sh}}^{n^i_1(\underline{B_1}), s_1} \left[v^1(S_{k,1} \cup U^i_1) - v^1(S_{k,1})\right] \\
&= \sum_{\substack{S_k \subseteq N^i_k \\ U^i_k \notin S_k}} \cdots \sum_{\substack{S_0 \subseteq N^i_0 \\ U^i_0 \notin S_0}} \omega_{\mathsf{Sh}}^{n^i_k, s_k} \cdots \omega_{\mathsf{Sh}}^{n^i_0, s_0} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right],
\end{aligned}$$

where the third equality holds by the first induction hypothesis since $(B_1, \underline{B_1})$ is a levels structure of cooperation with $k - 1$ levels and the fourth equality holds since $N^i_0(N, \underline{B}) \setminus U^i_0 = \emptyset$ and for every $r \in \{1, \ldots, k\}$, $N^i_r(B_1, \underline{B_1}) = N^i_r(N, \underline{B})$.

Now assume that $\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B})$ is obtained from Eq. (3.22) for every $(N', v', \underline{B'})$ such that $(N, \underline{B'})$ has at most $k - 1$ levels and every $i \in N$ such that $|U^i_1| < u$. Next, let $(N, v, \underline{B}) \in \mathcal{GL}$ be such that $(N, \underline{B})$ has k levels, i.e., $\underline{B} = \{B_0, \ldots, B_{k+1}\}$, and $i \in N$ be such that $|U^i_1| = u$. Since Sh^{L} satisfies $\mathcal{GL}$:LBC, for every $j \in U^i_1$,

$$\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B}) - \mathsf{Sh}^{\mathsf{L}}_j(N, v, \underline{B}) = \mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B^{-j}}) - \mathsf{Sh}^{\mathsf{L}}_j(N, v, \underline{B^{-i}}), \tag{3.23}$$

adding up Eq. (3.23) for all $j \in U_1^i$,

$$|U_1^i|\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B}) - \sum_{j \in U_1^i} \mathsf{Sh}_j^{\mathsf{L}}(N,v,\underline{B})$$
$$= \sum_{j \in U_1^i} \left[\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B^{-j}}) - \mathsf{Sh}_j^{\mathsf{L}}(N,v,\underline{B^{-i}})\right] = \sum_{j \in U_1^i \setminus i} \left[\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B^{-j}}) - \mathsf{Sh}_j^{\mathsf{L}}(N,v,\underline{B^{-i}})\right]. \tag{3.24}$$

Now, since Sh^{L} satisfies $\mathcal{GL}$:LGP we can rewrite Eq. (3.24) as

$$n_0^i \mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B}) - \mathsf{Sh}_{U_1^i}^{\mathsf{L}}(B_1,v^1,\underline{B_1}) = \sum_{j \in U_1^i \setminus i} \left[\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B^{-j}}) - \mathsf{Sh}_j^{\mathsf{L}}(N,v,\underline{B^{-i}})\right]. \tag{3.25}$$

Observe that according to the double induction hypothesis, for every $j \in U_1^i \setminus i$, $\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B^{-j}})$, $\mathsf{Sh}_j^{\mathsf{L}}(N,v,\underline{B^{-i}})$, and $\mathsf{Sh}_{U_1^i}^{\mathsf{L}}(B_1,v^1,\underline{B_1})$ can be obtained from Eq. (3.22). In particular, for every $j \in U_1^i \setminus i$,

$$\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B^{-j}})$$
$$= \sum_{\substack{S_k \subseteq N_k^i(\underline{B^{-j}}) \\ U_k^i \notin S_k}} \cdots \sum_{\substack{S_0 \subseteq N_0^i(\underline{B^{-j}}) \\ U_0^i \notin S_0}} \omega_{\mathsf{Sh}}^{n_k^i(\underline{B^{-j}}),s_k} \cdots \omega_{\mathsf{Sh}}^{n_0^i(\underline{B^{-j}}),s_0} \left[v(S_{k,0} \cup i) - (v(S_{k,0})\right]$$
$$= \sum_{\substack{S_k \subseteq N_k^i(\underline{B^{-j}}) \\ \{j\}, U_k^i \notin S_k}} \cdots \sum_{\substack{S_0 \subseteq N_0^i(\underline{B^{-j}}) \\ U_0^i \notin S_0}} \Bigg\{ \omega_{\mathsf{Sh}}^{n_k^i(\underline{B^{-j}}),s_k} \cdots \omega_{\mathsf{Sh}}^{n_0^i(\underline{B^{-j}}),s_0} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right]$$
$$+ \frac{s_k+1}{n_k^i(\underline{B^{-j}}) - s_k - 1} \cdot \omega_{\mathsf{Sh}}^{n_k^i(\underline{B^{-j}}),s_k} \cdots \omega_{\mathsf{Sh}}^{n_0^i(\underline{B^{-j}}),s_0} \left[v(S_{k,0} \cup j \cup i) - v(S_{k,0} \cup j)\right] \Bigg\},$$

where the first equality holds by the induction hypothesis and the second equality is obtained by distinguishing the cases $U_k^i \in S_k$ and $U_k^i \notin S_k$. Notice that for every $r \in \{1,\ldots,k-1\}$, $N_r^i(\underline{B^{-j}}) = N_r^i \setminus U_r^i \cup (U_r^i \setminus j)$, $N_k^i(\underline{B^{-j}}) = N_k^i \setminus U_k^i \cup (U_k^i \setminus j) \cup j$, and $N_0^i(\underline{B^{-j}}) = N_0^i \setminus j$. Hence,

$$\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B^{-j}})$$
$$= \sum_{\substack{S_k \subseteq N_k^i \\ U_k^i \notin S_k}} \cdots \sum_{\substack{S_0 \subseteq N_0^i \\ \{i\},\{j\} \notin S_0}} \Bigg\{ \frac{n_k^i - s_k}{n_k^i + 1} \frac{n_0^i}{n_0^i - s_0 - 1} \omega_{\mathsf{Sh}}^{n_k^i,s_k} \cdots \omega_{\mathsf{Sh}}^{n_0^i,s_0} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right]$$
$$+ \frac{s_k+1}{n_k^i+1} \frac{n_0^i}{n_0^i - s_0 - 1} \omega_{\mathsf{Sh}}^{n_k^i,s_k} \cdots \omega_{\mathsf{Sh}}^{n_0^i,s_0} \left[v(S_{k,0} \cup j \cup i) - v(S_{k,0} \cup j)\right] \Bigg\}.$$

Using twice the above expression, exchanging i and j, we obtain,

$$\begin{aligned}
&\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B^{-j}}) - \mathsf{Sh}^{\mathsf{L}}_j(N, v, \underline{B^{-i}}) \\
&= \sum_{\substack{S_k \subseteq N^i_k \\ U^i_k \notin S_k}} \cdots \sum_{\substack{S_0 \subseteq N^i_0 \\ \{i\},\{j\} \notin S_0}} \frac{n^i_0}{n^i_0 - s_0 - 1} \omega^{n^i_k, s_k}_{\mathsf{Sh}} \cdots \omega^{n^i_0, s_0}_{\mathsf{Sh}} \left[v(S_{k,0} \cup i) - v(S_{k,0} \cup j) \right].
\end{aligned}$$

Then, by the above equation and using the induction hypothesis on $\mathsf{Sh}^{\mathsf{L}}_{U^i_1}(B_1, v^1, \underline{B_1})$, Eq. (3.25) can be rewritten as

$$\begin{aligned}
\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B}) = \frac{1}{n^i_0} \sum_{\substack{S_k \subseteq N^i_k \\ U^i_k \notin S_k}} \cdots \sum_{\substack{S_1 \subseteq N^i_1 \\ U^i_1 \notin S_1}} \omega^{n^i_k, s_k}_{\mathsf{Sh}} \cdots \omega^{n^i_1, s_1}_{\mathsf{Sh}} \Bigg\{ &\left[v(S_{k,1} \cup U^i_1) - v(S_{k,1}) \right] \\
&+ \sum_{j \in U^i_1 \setminus i} \sum_{\substack{S_0 \subseteq N^i_0 \\ \{i\},\{j\} \notin S_0}} \frac{n^i_0}{n^i_0 - s_0 - 1} \omega^{n^i_0, s_0}_{\mathsf{Sh}} \left[v(S_{k,0} \cup i) - v(S_{k,0} \cup j) \right] \Bigg\}
\end{aligned}$$

Thus, it suffices to prove that, given $S_k \subseteq N^i_k \setminus U^i_k, \ldots, S_1 \subseteq N^i_1 \setminus U^i_1$,

$$\begin{aligned}
\frac{1}{n^i_0} \left[v(S_{k,1} \cup U^i_1) - v(S_{k,1}) \right] + \sum_{j \in U^i_1 \setminus i} \sum_{\substack{S_0 \subseteq N^i_0 \\ \{i\},\{j\} \notin S_0}} \frac{s_0!(n^i_0 - s_0 - 2)!}{n^i_0!} \left[v(S_{k,0} \cup i) - v(S_{k,0} \cup j) \right] \\
= \sum_{\substack{S_0 \subseteq N^i_0 \\ \{i\} \notin S_0}} \frac{s_0!(n^i_0 - s_0 - 1)!}{n^i_0!} \left[v(S_{k,0} \cup i) - v(S_{k,0}) \right].
\end{aligned}$$

We prove the above claim by rearranging the addend in the left-hand side of the expression. Note that

$$\begin{aligned}
&\sum_{j \in U^i_1 \setminus i} \sum_{\substack{S_0 \subseteq N^i_0 \\ \{i\},\{j\} \notin S_0}} \frac{s_0!(n^i_0 - s_0 - 2)!}{n^i_0!} \left[v(S_{k,0} \cup i) - v(S_{k,0} \cup j) \right] \\
&\qquad = \sum_{\substack{S_0 \subseteq N^i_0 \setminus \{i\} \\ S_0 \neq N^i_0 \setminus \{i\}}} (n^i_0 - s_0 - 1) \frac{s_0!(n^i_0 - s_0 - 2)!}{n^i_0!} v(S_{k,0} \cup i) \\
&\qquad\qquad - \sum_{\substack{S_0 \subseteq N^i_0 \setminus \{i\} \\ S_0 \neq \emptyset}} s_0 \frac{(s_0 - 1)!(n^i_0 - s_0 - 1)!}{n^i_0!} v(S_{k,0}),
\end{aligned}$$

where the equality holds by observing that, given $S_0 \subsetneq N^i_0 \setminus i$, the number of different players $j \in N_0^i \setminus i$ such that $j \notin S_0$ is $n_0^i - s_0 - 1$, whereas, given $\emptyset \neq S_0 \subseteq N_0^i \setminus i$, the number of different players $j \in N_0^i \setminus i$ such that $j \in S_0$ is s_0.

Finally,

$$\frac{1}{n_0^i}\left[v(S_{k,1} \cup U_1^i) - v(S_{k,1}) + v(S_{k,1} \cup i) - v(S_{k,1} \cup U_1^i \setminus i)\right]$$

$$+ \sum_{\substack{S_0 \subseteq N_0^i \setminus \{i\} \\ S_0 \neq \emptyset, N_0^i \setminus \{i\}}} \frac{s_0!(n_0^i - s_0 - 1)!}{n_0^i!} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right]$$

$$= \sum_{S_0 \subseteq N_0^i \setminus U_0^i} \frac{s_0!(n_0^i - s_0 - 1)!}{n_0^i!} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right],$$

which concludes the proof. □

Next, we introduce the share function on $\mathcal{ML}$ associated to the Shapley levels value. Note that since Sh^{L} is efficient, we only need to divide the value by $v(N)$.

Definition 3.3.6. The *Shapley levels share function*, $\rho^{\mathsf{Sh}^{\mathsf{L}}}$, is the share function on $\mathcal{ML}$ defined for every $(N, v, \underline{B}) \in \mathcal{ML}$ and $i \in N$ by

$$\rho_i^{\mathsf{Sh}^{\mathsf{L}}}(N, v, \underline{B}) = \frac{\mathsf{Sh}_i^{\mathsf{L}}(N, v, \underline{B})}{v(N)} \qquad \text{if } (N, v) \in \mathcal{M}^+,$$

and $\rho_i^{\mathsf{Sh}^{\mathsf{L}}}(N, v_0, \underline{B}) = \prod_{r=0}^{k} \frac{1}{|N_r^i|}$.

Next we prove that the Shapley levels share function, $\rho^{\mathsf{Sh}^{\mathsf{L}}}$, lies within the class of μ-share functions on $\mathcal{ML}$ (Definition 3.3.3), i.e., that there is a positive, additive, and anonymous real-valued function, μ, such that $\underline{\rho^{\mathsf{Sh}^{\mathsf{L}}}} = \underline{\rho^{\mu}}$. Furthermore, $\mu = \mu^{\mathsf{Sh}}$.

Proposition 3.3.7. *Let $(N, v, \underline{B}) \in \mathcal{ML}$ be a monotone game with levels structure of cooperation. Then,*

$$\rho^{\mathsf{Sh}^{\mathsf{L}}}(N, v, \underline{B}) = \underline{\rho^{\mu^{\mathsf{Sh}}}}(N, v, \underline{B}). \tag{3.26}$$

Proof. Let $(N, v, \underline{B}) \in \mathcal{ML}$, $i \in N$, and $\mu = \mu^{\mathsf{Sh}}$. First of all, note that if (N, v) is a null game the result is straightforward. Then, in the sequel we assume that $(N, v) \in \mathcal{M}^+$. If i is a null player in (N, v), by $\mathcal{M}$:NPP $\rho_i^{\mathsf{Sh}^{\mathsf{L}}}(N, v, \underline{B}) = 0$ and by (ii) of Proposition 3.3.4 $\underline{\rho_i^{\mu^{\mathsf{Sh}}}}(N, v, \underline{B}) = 0$. Thus, we can also assume that i is not a null player.

From the proof of (ii) in Proposition 3.3.4, we know that for every $r \in \{1, \ldots, k\}$ if U_r^i is not a null player in $(N_r^i, v_\mu^{i,r})$ then $(N_{r-1}^i, v_\mu^{i,r-1})$ is not a null game. Since

$(N_k^i, v_\mu^{i,k}) = (B_k, v^k)$, we know that it is not a null game. Hence, if for some $r \in \{1,\dots,k\}$, U_r^i is a null player it is so in a non null game, $(N_r^i, v_\mu^{i,r})$. Suppose that for some $r \in \{1,\dots,k\}$, U_r^i is a null player in $(N_r^i, v_\mu^{i,r}) \in \mathcal{M}^+$. Then, on the one hand by $\mathcal{M}$:NPP we have that $\rho^\mu_{U_r^i}(N_r^i, v_\mu^{r,i}) = 0$ and, hence, $\rho^\mu_{\underline{i}}(N, v, \underline{B}) = 0$ too. On the other hand, for every $S_r \subseteq N_r^i \setminus U_r^i$,

$$v_\mu^{i,r}(S_r \cup U_r^i) = v_\mu^{i,r}(S_r), \tag{3.27}$$

it can be checked that in such case U_r^i is a null player also in (B_r, v^r), in particular it is a dummy player. Then, since Sh^L satisfies $\mathcal{GL}$:DPP*,

$$\mathsf{Sh}^\mathsf{L}_{U_r^i}(B_r, v^r, \underline{B_r}) = v^r(U_r^i) = 0, \tag{3.28}$$

where the second equality holds by Eq. (3.27). Finally, taking into account that $(N, v) \in \mathcal{M}^+$, for every $j \in N$, $\mathsf{Sh}^\mathsf{L}_j(N, v, \underline{B}) \geq 0$. Which by Eq. (3.28) and the fact that Sh^L satisfies $\mathcal{GL}$:LGP implies that $\mathsf{Sh}^\mathsf{L}_i(N, v, \underline{B}) = 0$ and, hence, $\rho_i^{\mathsf{Sh}^\mathsf{L}}(N, v, \underline{B}) = 0$ and the result is proved. So, we can assume that for every $r \in \{0,\dots,k\}$, $(N_r^i, v_\mu^{r,i}) \in \mathcal{M}^+$.

Using Definition 3.3.3 and Proposition 3.1.3, it is enough to check that,

$$\rho_i^{\mathsf{Sh}^\mathsf{L}}(N, v, \underline{B}) = \prod_{r=0}^{k} \frac{\mathsf{Sh}_{U_r^i}(N_r^i, v_\mu^{i,r})}{v_\mu^{i,r}(N_r^i)}. \tag{3.29}$$

On the one hand, note that by Definitions 1.1.3 and 3.3.2, for each $r \in \{0,\dots,k-1\}$,

$$\mathsf{Sh}_{U_{r+1}^i}(N_{r+1}^i, v_\mu^{i,r+1}) = v_\mu^{i,r}(N_r^i). \tag{3.30}$$

On the other hand, using Lemma 3.3.5, we have

$$\begin{aligned}
\mathsf{Sh}^\mathsf{L}_i(N, v, \underline{B}) &= \sum_{S_k \subseteq N_k^i \setminus U_k^i} \cdots \sum_{S_0 \subseteq N_0^i \setminus U_0^i} \omega_{\mathsf{Sh}}^{n_k^i, s_k} \cdots \omega_{\mathsf{Sh}}^{n_0^i, s_0} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right] \\
&= \sum_{\substack{S_0 \subseteq N_0^i \\ U_0^i \notin S_0}} \omega_{\mathsf{Sh}}^{n_0^i, s_0} \left\{ \sum_{\substack{S_k \subseteq N_k^i \\ U_k^i \notin S_k}} \cdots \sum_{\substack{S_1 \subseteq N_1^i \\ U_1^i \notin S_1}} \omega_{\mathsf{Sh}}^{n_k^i, s_k} \cdots \omega_{\mathsf{Sh}}^{n_1^i, s_1} \left[v(S_{k,0} \cup i) - v(S_{k,0})\right] \right\} = \mathsf{Sh}_i(N_0^i, v_\mu^{i,0}) \\
&= \mathsf{Sh}_{U_0^i}(N_0^i, v_\mu^{i,0})
\end{aligned} \tag{3.31}$$

where the last equality holds by Definition 3.2.4.

Finally, by Eq. (3.30) (for the third equality), Eq. (3.31) and the fact that $v_\mu^{i,k}(N_k^i) = v(N)$ (for the fifth equality), and the definition of ρ^{Sh} (for the last equality),

$$\rho_i^{\mathsf{Sh}^{\mathsf{L}}}(N,v,\underline{B}) = \frac{\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B})}{v(N)} = \frac{\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B})}{v(N)} \prod_{r=0}^{k-1} \frac{v_\mu^{i,r}(N_r^i)}{v_\mu^{i,r}(N_r^i)}$$
$$= \frac{\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B})}{v(N)} \prod_{r=0}^{k-1} \frac{\mathsf{Sh}_{U_{r+1}^i}(N_{r+1}^i, v_\mu^{i,r+1})}{v_\mu^{i,r}(N_r^i)} = \frac{\mathsf{Sh}_i^{\mathsf{L}}(N,v,\underline{B})}{v(N)} \prod_{r=1}^{k-1} \frac{\mathsf{Sh}_{U_r^i}(N_r^i, v_\mu^{i,r})}{v_\mu^{i,r}(N_r^i)} \frac{\mathsf{Sh}_{U_k^i}(N_k^i, v_\mu^{i,k})}{v_\mu^{i,0}(N_0^i)}$$
$$= \frac{\mathsf{Sh}_{U_0^i}(N_0^i, v_\mu^{i,0})}{v_\mu^{i,0}(N_0^i)} \prod_{r=1}^{k-1} \frac{\mathsf{Sh}_{U_r^i}(N_r^i, v_\mu^{i,r})}{v_\mu^{i,r}(N_r^i)} \frac{\mathsf{Sh}_{U_k^i}(N_k^i, v_\mu^{i,k})}{v_\mu^{i,k}(N_k^i)} = \prod_{r=0}^{k} \rho_{U_r^i}^{\mathsf{Sh}}(N_r^i, v_\mu^{i,r}).$$

□

In the remaining part of this chapter we propose sets of properties that characterize the class of μ-share functions on $\mathcal{ML}$. Two kind of properties are considered. The first ones apply only to games with the trivial levels structure of cooperation. The second type of properties involve games with arbitrary levels structures of cooperation.

In the first place, let us consider some properties of the first type.

$\mathcal{ML}$:NPP^0 A share function on $\mathcal{ML}$, ρ, satisfies the *null player property for the trivial levels structures of cooperation* if, for every $(N,v) \in \mathcal{M}^+$ and every null player $i \in N$ in (N,v),

$$\rho_i(N,v,\underline{B_0}) = 0.$$

$\mathcal{ML}$:SYM^0 A share function on $\mathcal{ML}$, ρ, satisfies *symmetry for the trivial levels structures of cooperation* if, for every $(N,v) \in \mathcal{M}$ and every pair $i,j \in N$ of symmetric players in (N,v),

$$\rho_i(N,v,\underline{B_0}) = \rho_j(N,v,\underline{B_0}).$$

$\mathcal{ML}$:μ-ADD^0 Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{ML}$, ρ, satisfies *μ-additivity for the trivial levels structures of cooperation* if for every pair of games $(N,v),(N,w) \in \mathcal{M}$,

$$\mu(N,v+w)\rho(N,v+w,\underline{B_0}) = \mu(N,v)\rho(N,v,\underline{B_0}) + \mu(N,w)\rho(N,w,\underline{B_0}).$$

The properties above are formulated only for the trivial levels structure of cooperation, which means that the possible restrictions to the cooperation are not taken into account. Hence, they are reformulations of $\mathcal{M}$:NPP, $\mathcal{M}$:SYM, and $\mathcal{M}$:μ-ADD properties defined in Section 3.1.2 for share functions on $\mathcal{ML}$.

In the second place, let us consider some properties of a second type.

$\mathcal{ML}$:μ-MUL Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{ML}$, ρ, satisfies the *μ-multiplication property* if for every $(N, v, \underline{B}) \in \mathcal{ML}$ and $i \in N$,

$$\rho_i(N, v, \underline{B}) = \prod_{r=0}^{k} \rho_{U_r^i}(N_r^i, v_\mu^{i,r}, \underline{B_0}).$$

$\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$ Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{ML}$, ρ, satisfies *μ-additivity for arbitrary levels structure of cooperation* if for every pair of monotone games $(N, v), (N, w) \in \mathcal{M}$, every levels structure of cooperation $(N, \underline{B}) \in \mathcal{L}(N)$, and every player $i \in N$,

$$\rho_i(N, z, \underline{B}) \prod_{r=0}^{k} \mu(N_r^i, z_\mu^{i,r}) = \prod_{r=0}^{k} \left[\mu(N_r^i, v_\mu^{i,r})\rho_{U_r^i}(N_r^i, v_\mu^{i,r}, \underline{B_0}) + \mu(N_r^i, w_\mu^{i,r})\rho_{U_r^i}(N_r^i, w_\mu^{i,r}, \underline{B_0})\right],$$

where $z = v + w$.

$\mathcal{ML}$:CON A share function on $\mathcal{ML}$, ρ, satisfies *consistency* if for every $(N, v, \underline{B}) \in \mathcal{ML}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $U \in B_r$ for some $r \in \{1, \cdots, k\}$,

$$\sum_{i \in U} \rho_i(N, v, \underline{B}) = \rho_U(B_r, v^r, \underline{B_r}).$$

$\mathcal{ML}$:μ-DIF Let $\mu : \mathcal{M} \to \mathbb{R}$. A share function on $\mathcal{ML}$, ρ, satisfies *μ-difference* if for every $(N, v, \underline{B}) \in \mathcal{ML}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and every union of the first level $U \in B_1$ there is a unique scalar $K^\mu_{U,(B_1,v^1,\underline{B_1})} \in \mathbb{R}_+$ such that for every $i, j \in U$,

$$\rho_i(N, v, \underline{B}) - \rho_j(N, v, \underline{B}) = K^\mu_{U,(B_1,v^1,\underline{B_1})} \left[\rho_i(U, v_\mu^{i,0}, \underline{B_0}) - \rho_j(U, v_\mu^{j,0}, \underline{B_0})\right].$$

The $\mathcal{ML}$:μ-MUL property is a generalization of the $\mathcal{MU}$:μ-MUL property (see Section 3.2) in line with the way in which the class of μ-share function on $\mathcal{ML}$ has been constructed. It states that the share of a player in a monotone game with levels structure of cooperation is the product of the shares of the unions she belongs to in the "internal" games played among the unions with the trivial levels structure of cooperation. The $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$ property generalizes the $\mathcal{ML}$:μ-ADD^0 property defined above and the $\mathcal{MU}$:(μ^1, μ^2)-ADD property when $\mu^1 = \mu^2$. It relates the share of a player in a sum game with levels structure of cooperation with shares of players in games with the trivial levels structure of cooperation obtained from the original ones. The $\mathcal{ML}$:CON property states that the joint share of the players that make up a union of a given level equals the share of that union in the corresponding union level game. The property is a reformulation of the $\mathcal{GL}$:LGP considered in Section 2.2.1 in terms of share functions on $\mathcal{ML}$ instead of values on $\mathcal{GL}$. Finally, the $\mathcal{ML}$:μ-DIF property describes the difference between the shares of two players that lie in the same union at every level. In fact, it states that this difference only depends on the particular union they belong to in the first level, U, the first level union game played among the unions of the first level, and the difference among the shares of the players in the internal game played among the members of U.

Next, the first characterization result is stated and proved.

Theorem 3.3.8. *Let $\mu : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous. Then, $\underline{\rho^\mu}$ is the unique share function on $\mathcal{ML}$ satisfying $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:μ-ADD^0, and $\mathcal{ML}$:μ-MUL.*

Proof. Let $\mu : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous.

(1) *Existence.* From Proposition 3.3.4 we know that $\underline{\rho^\mu}$ satisfies $\mathcal{ML}$:NPP^0 and $\mathcal{ML}$:SYM^0. We also have that $\underline{\rho^\mu}$ generalizes ρ^μ (part (iii) of Proposition 3.3.4). Since ρ^μ satisfies $\mathcal{M}$:μ-ADD, $\underline{\rho^\mu}$ satisfies $\mathcal{ML}$:μ-ADD^0. Finally, by Definition 3.3.3 and the fact that $\underline{\rho^\mu}$ generalizes ρ^μ, $\underline{\rho^\mu}$ satisfies $\mathcal{ML}$:μ-MUL.

(2) *Uniqueness.* Let ρ be a share function on $\mathcal{ML}$ satisfying the properties above. Note that by Proposition 3.1.3, ρ is unique for games with the trivial levels structure of cooperation by $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, and $\mathcal{ML}$:μ-ADD^0. Finally, the $\mathcal{ML}$:μ-MUL property relates the share of any player in any game with levels structure of cooperation to shares of players in games with the trivial levels structure of cooperation. Hence, by $\mathcal{ML}$:μ-MUL and the uniqueness for games with the trivial levels structure cooperation, ρ is unique. $\square$

Theorem 3.3.8 upgrades Theorem 3.2.10 (when $\mu^1 = \mu^2$) in the sense that, on the one hand, it extends the characterization result of the class of share functions from one-level structures to an arbitrary number of levels structures of cooperation, and, on the other hand, it replaces for both characterizations three of the properties by weaker versions.

Corollary 3.3.9. *Let $\mu : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous. Then, $\underline{\rho^\mu}$ is the unique share function on $\mathcal{ML}$ satisfying $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, and $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$.*

Proof.

(1) *Existence.* Using Proposition 3.3.4 it only remains to prove $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$. However, note that $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$ directly follows using $\mathcal{ML}$:μ-MUL and $\mathcal{ML}$:NPP^0.

(2) *Uniqueness.* Note that $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$ property includes $\mathcal{ML}$:μ-ADD^0, by taking $\underline{B} = \underline{B_0}$, and $\mathcal{ML}$:μ-MUL, by taking $w = v_0$. Hence, the uniqueness follows from the uniqueness result of Theorem 3.3.8 above. □

Corollary 3.3.9 upgrades Theorem 3.2.11 in the sense that, on the one hand, it extends the characterization result of the class of share functions from one-level structures to an arbitrary number of levels structures of cooperation, and, on the other hand, it replaces two of the axioms by weaker versions. Moreover, if we face the result above with Theorem 5.3 in van den Brink & van der Laan (2005) we realize that the consistency property can be dropped without changing the result.

Lastly, another characterization of $\underline{\rho^\mu}$ is presented using a different set of properties.

Theorem 3.3.10. *Let $\mu : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous. Then, $\underline{\rho^\mu}$ is the unique share function on $\mathcal{ML}$ satisfying $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:μ-ADD^0, $\mathcal{ML}$:CON, and $\mathcal{ML}$:μ-DIF.*

Proof. Let $\mu : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous.

(1) *Existence.* By Proposition 3.3.4, it only remains to prove that $\underline{\rho^\mu}$ satisfies $\mathcal{ML}$:CON and $\mathcal{ML}$:μ-DIF.

Let $(N, v, \underline{B}) \in \mathcal{ML}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $U \in B_r$ for some $r \in \{1, \ldots, k\}$. Note that for every $i, j \in U$ and $l \in \{r, \ldots, k\}$, $U_l^i = U_l^j$ and $(N_l^i, v_\mu^{i,l}) = (N_l^j, v_\mu^{j,l})$. Then, by Definition 3.3.3,

$$\begin{aligned}\sum_{i\in U} \underline{\rho_i^\mu}(N, v, \underline{B}) &= \sum_{i\in U} \prod_{l=0}^{k} \rho^\mu_{U_r^i}(N_l^i, v_\mu^{i,l}) = \prod_{l=r}^{k} \rho^\mu_{U_r^i}(N_l^i, v_\mu^{i,l}) \sum_{i\in U} \prod_{l=0}^{r-1} \rho^\mu_{U_r^i}(N_l^i, v_\mu^{i,l}) \\ &= \underline{\rho_U^\mu}(B_r, v^r, \underline{B_r}) \sum_{i\in U} \underline{\rho_i^\mu}(U, v_{|U}, \{B_{0|U}, \ldots, B_{r-1|U}\}) = \underline{\rho_U^\mu}(B_r, v^r, \underline{B_r}),\end{aligned}$$

where the last equality holds because $\underline{\rho^\mu}$ is a share function on $\mathcal{ML}$.

Let $(N, v, \underline{B}) \in \mathcal{ML}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $i, j \in U \in B_1$. Note that for every $i, j \in U$ and $l \in \{1, \ldots, k\}$, $U_l^i = U_l^j$ and $(N_l^i, v_\mu^{i,l}) = (N_l^j, v_\mu^{j,l})$. Then, by Definition 3.3.3,

$$\underline{\rho_i^\mu}(N, v, \underline{B}) - \underline{\rho_j^\mu}(N, v, \underline{B}) = \prod_{r=0}^{k} \left[\rho^\mu_{U_r^i}(N_r^i, v_\mu^{i,r}) - \rho^\mu_{U_r^j}(N_r^j, v_\mu^{j,r})\right]$$

$$= \prod_{r=1}^{k} \rho^\mu_{U_r^i}(N_l^i, v_\mu^{i,l}) \begin{bmatrix} \rho_i^\mu(U, v_\mu^{i,0}) \\ -\rho_j^\mu(U, v_\mu^{j,0}) \end{bmatrix} = \underline{\rho_U^\mu}(B_1, v^1, \underline{B_1}) \left[\underline{\rho_i^\mu}(U, v_\mu^{i,0}, \underline{B_0}) - \underline{\rho_j^\mu}(U, v_\mu^{j,0}, \underline{B_0})\right].$$

Thus, $\underline{\rho^\mu}$ satisfies $\mathcal{ML}$:μ-DIF.

(2) *Uniqueness.* Let $\mu : \mathcal{M} \to \mathbb{R}$ be an additive, positive, and anonymous mapping. Suppose that ρ^1 and ρ^2 are two share functions on $\mathcal{ML}$ satisfying the properties. We show that for every $(N, v, \underline{B}) \in \mathcal{ML}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$, $\rho^1(N, v, \underline{B}) = \rho^2(N, v, \underline{B})$ by induction on the number of levels k.

Let $k = 0$, i.e., $\underline{B} = \underline{B_0}$. In this case from Proposition 3.1.3 and using that ρ^1 and ρ^2 satisfy $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, and $\mathcal{ML}$:μ-ADD^0 we have that

$$\rho^1(N, v, \underline{B_0}) = \rho^2(N, v, \underline{B_0}) = \rho^\mu(N, v).$$

Next, assume that for every $(N', v', \underline{B'}) \in \mathcal{ML}$ such that $(N, \underline{B'})$ has at most l levels, $\rho^1(N', v', \underline{B'}) = \rho^2(N', v', \underline{B'})$. Let $(N, v, \underline{B}) \in \mathcal{ML}$ such that $(N, \underline{B})$ has $l + 1$ levels, i.e., $\underline{B} = \{B_0, \ldots, B_{l+1}, B_{l+2}\}$. Let $i \in N$ and $j \in U_1^i \setminus i$, then by $\mathcal{ML}$:μ-DIF there is a unique scalar $K^\mu_{U,(B_1,v^1,\underline{B_1})} \in \mathbb{R}_+$ such that,

$$\begin{aligned}\rho_i^1(N, v, \underline{B}) - \rho_j^1(N, v, \underline{B}) &= K^\mu_{U,(B_1,v^1,\underline{B_1})} \left[\rho_i^1(U, v_\mu^{i,0}, \underline{B_0}) - \rho_j^1(U, v_\mu^{j,0}, \underline{B_0})\right] \\ &= K^\mu_{U,(B_1,v^1,\underline{B_1})} \left[\rho_i^2(U, v_\mu^{i,0}, \underline{B_0}) - \rho_j^2(U, v_\mu^{j,0}, \underline{B_0})\right] = \rho_i^2(N, v, \underline{B}) - \rho_j^2(N, v, \underline{B}), \end{aligned} \tag{3.32}$$

where the second equality is a consequence of the uniqueness for monotone games with trivial levels structure of cooperation and the induction hypothesis applied to the scalar $K^\mu_{U,(B_1,v^1,\underline{B_1})}$ since $(B_1, \underline{B_1})$ is a levels structure of cooperation with l levels. Next, adding up Eq. (3.32) for every $j \in U_1^i \setminus i$,

$$|U_1^i|\rho_i^1(N, v, \underline{B}) - \sum_{j \in U_1^i} \rho_j^1(N, v, \underline{B}) = |U_1^i|\rho_i^2(N, v, \underline{B}) - \sum_{j \in U_1^i} \rho_j^2(N, v, \underline{B}). \tag{3.33}$$

Finally, by $\mathcal{ML}$:CON

$$\sum_{j \in U_1^i} \rho^1(N, v, \underline{B}) = \rho^1(B_1, v^1, \underline{B_1}) = \rho^2(B_1, v^1, \underline{B_1}) = \sum_{j \in U_1^i} \rho^2(N, v, \underline{B}), \tag{3.34}$$

where the second equality is due to the induction hypothesis, since $(B_1, \underline{B_1})$ is a levels structure of cooperation with l levels. This last equation together with Eq. (3.33) concludes the proof. □

Remark 3.3.11. The properties considered in Theorem 3.3.8 are independent as the following examples show. Let $\mu : \mathcal{M} \to \mathbb{R}$ be additive, positive, and anonymous.

(i) The share function on $\mathcal{ML}$, ρ^1, defined for every $(N, v, \underline{B}) \in \mathcal{ML}$ by

$$\rho^1(N, v, \underline{B}) = \rho^\mu(N, v),$$

satisfies $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, and $\mathcal{ML}$:μ-ADD^0, but not $\mathcal{ML}$:μ-MUL.

(ii) The share function on $\mathcal{ML}$, ρ^2, defined for every $(N, v, \underline{B}) \in \mathcal{ML}$ and $i \in N$ by

$$\rho_i^1(N, v, \underline{B}) = \prod_{r=0}^{k} \frac{1}{|N_r^i|},$$

satisfies $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:μ-ADD^0, and $\mathcal{ML}$:μ-MUL, but not $\mathcal{ML}$:NPP^0.

(iii) Let a and b be two distinct, fixed, and indivisible players. Recall that in this context, by indivisible we mean that there are no players $i_1, \ldots, i_l$ such that $a = \{i_1, \ldots, i_l\}$ or $b = \{i_1, \ldots, i_l\}$. If $N = \{a, b\}$, for every $(N, v) \in \mathcal{M}$ let $\lambda_1, \lambda_2 > 0$ be such that $\mu(N, v) = \lambda_1 v(a) + \lambda_1 v(b) + \lambda_2 v(N)$. Define the share function on $\mathcal{ML}$, ρ^3, for every $(N, v, \underline{B}) \in \mathcal{ML}$ as follows:

- If $N = \{a, b\}$, $\underline{B} = \underline{B_0}$, and $(N, v) \in \mathcal{M}^+$,

$$\begin{cases} \rho_a^3(N, v, \underline{B}) & = \frac{\lambda_1 v(a) + \lambda_2 (v(N) - v(b))}{\mu(N, v)} \\ \rho_b^3(N, v, \underline{B}) & = \frac{(\lambda_1 + \lambda_2) v(b)}{\mu(N, v)} \end{cases}$$

- If $N \neq \{a, b\}$ and $\underline{B} = \underline{B_0}$, $\rho^3(N, v, \underline{B}) = \rho^\mu(N, v)$.

- If $\underline{B} \neq \underline{B_0}$, for every $i \in N$, $\rho_i^3(N, v, \underline{B}) = \prod\limits_{r=0}^{k} \rho^3_{U_r^i}(N_r^i, v_\mu^{i,r}, \underline{B_0})$.

Then, ρ^3 satisfies $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:μ-ADD^0, and $\mathcal{ML}$:μ-MUL, but not $\mathcal{ML}$:SYM^0.

(iv) Let $\mu^0 : \mathcal{M} \to \mathbb{R}$ be additive, positive, anonymous, and different from μ. The share function on $\mathcal{ML}$, ρ^4, defined for every $(N, v, \underline{B}) \in \mathcal{ML}$ and $i \in N$ by

$$\rho_i^4(N, v, \underline{B}) = \prod_{r=0}^{k} \rho^{\mu^0}_{U_r^i}(N_r^i, v_\mu^{i,r}),$$

satisfies $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:NPP^0, and $\mathcal{ML}$:μ-MUL, but not $\mathcal{ML}$:μ-ADD^0. $\triangleleft$

Remark 3.3.12. The properties considered in Corolary 3.3.9 are independent as the following examples show:

(i) The share function on $\mathcal{ML}$, ρ^1, defined above satisfies $\mathcal{ML}$:NPP^0 and $\mathcal{ML}$:SYM^0, but not $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$.

(ii) The share function on $\mathcal{ML}$, ρ^2, defined above satisfies $\mathcal{ML}$:SYM^0 and $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$, but not $\mathcal{ML}$:NPP^0.

(iii) The share function on $\mathcal{ML}$, ρ^3, defined above satisfies $\mathcal{ML}$:NPP^0 and $\mathcal{ML}$:μ-ADD$^{\mathsf{L}}$, but not $\mathcal{ML}$:SYM^0. $\triangleleft$

Remark 3.3.13. The properties considered in Theorem 3.3.10 are independent as the following examples show:

(i) The share function on $\mathcal{ML}$, ρ^2, defined as above satisfies $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:μ-ADD^0, $\mathcal{ML}$:CON, and $\mathcal{ML}$:μ-DIFF, but not $\mathcal{ML}$:NPP^0.

(ii) The share function on $\mathcal{ML}$, ρ^3, defined as above satisfies $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:μ-ADD^0, $\mathcal{ML}$:CON, and $\mathcal{ML}$:μ-DIFF, but not $\mathcal{ML}$:SYM^0.

(iii) The share function on $\mathcal{ML}$, ρ^4, defined as above satisfies $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:CON, and $\mathcal{ML}$:μ-DIFF, but not $\mathcal{ML}$:μ-ADD^0.

(iv) Let ρ^5 be the share function on $\mathcal{ML}$ defined for every $(N, v, \underline{B}) \in \mathcal{ML}$ as follows:

- If $\underline{B} = \underline{B_0}$, $\rho^5(N, v, \underline{B}) = \rho^\mu(N, v)$.
- Otherwise, for every $i \in N$ $\rho^5_i(N, v, \underline{B}) = \underline{\rho^\mu_{U^i_1}}(B_1, v^1, \underline{B_1})\frac{1}{|U^i_1|}$.

Then ρ^5 satisfies $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:μ-ADD^0, and $\mathcal{ML}$:μ-DIFF, but not $\mathcal{ML}$:CON.

(v) Let ρ^6 be the share function on $\mathcal{ML}$ defined for every $(N, v, \underline{B}) \in \mathcal{ML}$ as follows:

- If $\underline{B} = \underline{B_0}$, $\rho^6(N, v, \underline{B}) = \rho^\mu(N, v)$.
- If N is a set of indivisible players. For every $U \in B_1$ let $i_U \in U$ be a randomly selected particular agent, then

$$\begin{cases} \rho^6_{i_U}(N, v, \underline{B}) = \underline{\rho^\mu_U}(B_1, v^1, \underline{B_1}) & \\ \rho^6_i(N, v, \underline{B}) = 0 & \text{for every } i \in U \setminus i_U \end{cases}$$

- Otherwise, $\rho^6_i(N, v, \underline{B}) = \underline{\rho^\mu_i}(N, v, \underline{B})$.

Then ρ^6 satisfies $\mathcal{ML}$:NPP^0, $\mathcal{ML}$:SYM^0, $\mathcal{ML}$:μ-ADD^0, and $\mathcal{ML}$:CON, but not $\mathcal{ML}$:μ-DIFF. $\triangleleft$

4

Games with a priori unions and graph restricted communication

Myerson (1977) depicted partial cooperation through a graph. Each link of the graph indicates that direct communication, and hence cooperation, is possible between agents located at each end. Also, communication occurs between agents joined via a path. In this framework, the game and the graph define a new game, called the communication game. In this paper, an allocation of the total gains is proposed and characterized, the so-called Myerson value. This value coincides with the Shapley value of the communication game. Owen (1986) proposed a new value for the family of games with graph restricted communication. In this case, this value coincides with the Banzhaf value of the corresponding communication game and it is called the Banzhaf graph value. The Banzhaf graph value is characterized in Alonso-Meijide & Fiestras-Janeiro (2006) where a comparison between the properties satisfied by the Myerson value and the Banzhaf graph value is provided.

The model of games with a priori unions introduced in Section 2.1 is compatible with the model of games with graph restricted communication. In fact, Vázquez-Brage et al. (1996) study both generalizations jointly, that is, they consider games with a priori unions and graph restricted communication. For this family of games, they propose and characterize a generalization of the Shapley value which extends both the Owen and Myerson values. We refer to it as the Owen graph value. The properties used in the characterizations are modifications of the properties considered in previous characterizations of the Owen and Myerson values.

In this chapter, the model with the two restrictions to the cooperation mentioned above is considered. Two new values for this family of games are defined and characterized. The first proposal is an extension of the Banzhaf graph value and the Banzhaf-Owen value. The second one is an extension of the Banzhaf graph value and the Symmetric coalitional Banzhaf value. Besides, we define a new game, the communication quotient game. This game is built following the ideas behind the quotient game and the communication game. Finally, a new characterization of the Owen graph value is provided together with a comparison of the properties satisfied by the three values considered for the class of games with graph restricted communication and a priori unions.

The results contained in this chapter are a joint work with my supervisors José M. Alonso-Meijide and M. Gloria Fiestras-Janeiro and have been published in Mathematical Social Sciences (Alonso-Meijide et al. 2009a). The chapter is organized as follows. In Section 4.1, the model of games with graph restricted communication is introduced. Extensions of the Shapley and Banzhaf values in this context are presented together with the main characterization results. The main results are contained in Section 4.2. First, the model of games with a priori unions and graph restricted communication is presented and the existing literature on such models is recalled. Next, two new values for this family of games are proposed and parallel characterizations of them provided. Finally, in Section 4.2.1 a real example coming from the political field is used to illustrate the differences among the values.

4.1 Games with graph restricted communication

An *undirected graph without loops* on N is a set C of unordered pairs of distinct elements of N. Each pair $(i:j) \in C$ with $i \neq j$ is a *link*. Given $i, j \in S \subseteq N$, we say that i and j are *connected in S by C* if there is a path in S connecting them, i.e., there is some $k \geq 1$ and a subset $\{i_0, i_1, \ldots, i_k\} \subseteq S$ such that $i_0 = i$, $i_k = j$ and $(i_{h-1} : i_h) \in C$, for every $h = 1, \ldots, k$. Denote by S/C the set of connected components of S determined by C, i.e., the set of maximal subsets of elements connected in S by C. Observe that S/C is a partition of S. We denote by $\mathcal{C}^N$ the set of all undirected graphs without loops on N. Given $C \in \mathcal{C}^N$, C^* denotes the *dual graph of C*, i.e., $C^* = \{(i:j) : i, j \in N, i \neq j, (i:j) \notin C\}$. Then, $\emptyset^*$ denotes the *complete graph* on N, i.e., $\emptyset^* = \{(i:j) : i, j \in N, i \neq j\}$.

Given $C \in \mathcal{C}^N$ we say that agent $i \in N$ is an isolated agent with respect to the graph C if there is no $j \in N$ such that $(i:j) \in C$, that is, if $\{i\} \in N/C$. Given a link $(i:j) \in C$, the graph $C^{-ij} \in \mathcal{C}^N$ is defined as the resulting graph after the elimination of the link $(i:j)$, that is $C^{-ij} = C \setminus (i:j)$. For every $i \in N$, we denote by C^{-i} the element of $\mathcal{C}^N$ obtained from C by breaking the links where agent i is involved, i.e., $C^{-i} = \{(j : h) \in C : j \neq i \text{ and } h \neq i\}$. A *game with graph restricted communication* is a triple (N, v, C) where $(N, v) \in \mathcal{G}$ and $C \in \mathcal{C}^N$. We denote by $\mathcal{GC}$ the set of all such games. Next, a game which will play a crucial role in the sequel is defined.

Definition 4.1.1. Given $(N, v, C) \in \mathcal{GC}$, the *communication game* $(N, v^C) \in \mathcal{G}$ is defined for every $S \subseteq N$ by

$$v^C(S) = \sum_{T \in S/C} v(T).$$

Notice that when $C = \emptyset^*$, we have $v^C = v$ and when $C = \emptyset$, $(N, v^C) \in \mathcal{G}$ is an additive game.

The definition of the communication game can be understood as follows. Consider a coalition of players $S \subseteq N$. If coalition S is internally connected, i.e., if all players in S can communicate with one another (directly or indirectly) without the help of players in $N \setminus S$, then they can fully coordinate their actions and obtain the worth $v(S)$. Nevertheless, if coalition S is not internally connected, then not all players in S can communicate with each other without the help of outsiders. Coalition S will then be split into communication components according to the partition S/C. The best that players in S can accomplish under these conditions is to coordinate their actions within each of these components. Players in different components cannot coordinate their actions and hence, the components will operate independently.

A *value on* $\mathcal{GC}$ is a map f that assigns a vector $\mathsf{f}(N, v, C) \in \mathbb{R}^N$ to every game with graph restricted communication $(N, v, C) \in \mathcal{GC}$. In this context there are two well known values on $\mathcal{GC}$ based on the Shapley and Banzhaf values which are presented in the next definitions.

Definition 4.1.2. (Myerson 1977). The *Myerson value*, Sh^C, is the value on $\mathcal{GC}$ defined for every $(N, v, C) \in \mathcal{GC}$ by

$$\mathsf{Sh}^\mathsf{C}(N, v, C) = \mathsf{Sh}(N, v^C).$$

Definition 4.1.3. (Owen 1986). The *Banzhaf graph value*, Ba^C, is the value on $\mathcal{GC}$ defined for every $(N, v, C) \in \mathcal{GC}$ by

$$\mathsf{Ba}^\mathsf{C}(N, v, C) = \mathsf{Ba}(N, v^C).$$

If two players are in different communication components of a graph restricted game $(N, v, C) \in \mathcal{GC}$, then they do not interact with each other at all.

Consequently, it seems reasonable to expect that the values on $\mathcal{GC}$ of coalitions that include players that are not connected to player $i \in N$ as well as links involving such players do not influence the payoff of player i. This requirement is satisfied by both the Myerson and the Banzhaf graph values, which are component decomposable as it is shown in van den Nouweland (1993) and Alonso-Meijide & Fiestras-Janeiro (2006).

$\mathcal{GC}$:CDE A value on $\mathcal{GC}$, f, satisfies *component decomposability* if for every $(N, v, C) \in \mathcal{GC}$ and every player $i \in N$,

$$\mathsf{f}_i(N, v, C) = \mathsf{f}_i(S, v_{|S}, C_{|S}),$$

where $S \in N/C$ such that $i \in S$, and $(S, v_{|S}, C_{|S}) \in \mathcal{GC}$ is the graph restricted game obtained from $(N, v, C) \in \mathcal{GC}$ when the player set is restricted to S.

Myerson (1977) defines the Myerson value axiomatically, that is, he obtains the Myerson value looking for an allocation rule that satisfies component efficiency and fairness. These two properties are formally introduced below.

$\mathcal{GC}$:CEF A value on $\mathcal{GC}$, f, satisfies *component efficiency* if for every $(N, v, C) \in \mathcal{GC}$ and every $S \in N/C$,

$$\sum_{i \in S} \mathsf{f}_i(N, v, C) = v(S).$$

$\mathcal{GC}$:FAI A value on $\mathcal{GC}$, f, satisfies *fairness* if for every $(N, v, C) \in \mathcal{GC}$ and every $i, j \in N$ such that $(i : j) \in C$,

$$\mathsf{f}_i(N, v, C) - \mathsf{f}_i(N, v, C^{-ij}) = \mathsf{f}_j(N, v, C) - \mathsf{f}_j(N, v, C^{-ij}).$$

A value on $\mathcal{GC}$ satisfies $\mathcal{GC}$:CEF if the payoffs of the players in a maximal connected component add up to the worth of that component. Using a component efficient value on $\mathcal{GC}$, the players distribute the worth of this component among themselves. $\mathcal{GC}$:FAI is a reciprocity property. It reflects the equal gains equity principle that two players should gain or loose equally from their bilateral agreement. Moreover, it is a reciprocity property that resembles the balanced contributions properties studied in Chapter 2 ($\mathcal{GU}$:BCO and $\mathcal{GL}$:LBC).

In this setting, other properties that a value on $\mathcal{GC}$ might satisfy have been proposed in the literature. Next, we define four more properties which will be used to characterize the Myerson and Banzhaf graph values.

$\mathcal{GC}$:BCG A value on $\mathcal{GC}$, f, satisfies *balanced contributions in the graph* if for every $(N, v, C) \in \mathcal{GC}$ and every $i, j \in N$,

$$\mathsf{f}_i(N, v, C) - \mathsf{f}_i(N, v, C^{-j}) = \mathsf{f}_j(N, v, C) - \mathsf{f}_j(N, v, C^{-i}).$$

$\mathcal{GC}$:GIS A value on $\mathcal{GC}$, f, satisfies *graph isolation* if for every $(N, v, C) \in \mathcal{GC}$ and any $i \in N$ isolated agent,

$$\mathsf{f}_i(N, v, C) = v(i).$$

$\mathcal{GC}$:CTP A value on $\mathcal{GC}$, f, satisfies *component total power* if for every $(N, v, C) \in \mathcal{GC}$ and every $S \in N/C$,

$$\sum_{i \in S} \mathsf{f}_i(N, v, C) = \frac{1}{2^{s-1}} \sum_{i \in S} \sum_{T \subseteq S \setminus i} \left[v^C(T \cup i) - v^C(T) \right].$$

$\mathcal{GC}$:2-EFF A value on $\mathcal{GC}$, f, satisfies *2-efficiency* if for every $(N, v, C) \in \mathcal{GC}$ and every pair of players $i, j \in N$ such that $(i : j) \in C$,

$$\mathsf{f}_i(N, v, C) + \mathsf{f}_j(N, v, C) = \mathsf{f}_p(N^{ij}, v^{ij}, C^{ij}),$$

where (N^{ij}, v^{ij}, C^{ij}) is the game obtained from (N, v, C) when players i and j merge in a new player p, i.e., (N^{ij}, v^{ij}) is as defined in Section 1.1.2 and given $l, k \in N^{ij}$,

$$(l : k) \in C^{ij} \text{ if and only if } \begin{cases} (l : k) \in C \text{ with } l, k \in N \setminus \{i, j\} \\ (l : i) \in C \text{ or } (l : j) \in C \text{ and } k = p \\ (i : k) \in C \text{ or } (j : k) \in C \text{ and } l = p. \end{cases}$$

Note that $\mathcal{GC}$:BCG generalizes $\mathcal{GC}$:FAI. The balanced contributions in the graph is a reciprocity property that states that a player's isolation from the graph affects another player in the same amount as if it happens the other way around. $\mathcal{GC}$:GIS is very similar to the dummy player property, in fact it is just a weaker version of the dummy player property applied to the communication game (N, v^C) since an isolated agent in the graph becomes a dummy in the communication game. $\mathcal{GC}$:GIS states that a player who cannot communicate to any other player should be given what she can obtain on her own. The $\mathcal{GC}$:CTP property indicates the amount that a connected component will receive. Finally, $\mathcal{GC}$:2-EFF is just the generalization of the $\mathcal{G}$:2-EFF property defined in Section 1.1.2 for a pair of agents that are directly connected by the graph. It states that a value on $\mathcal{GC}$ satisfying it is immune against artificial merging or splitting of directly communicated players.

To conclude the section the main characterizations of the Myerson and Banzhaf graph values are presented below. The two characterizations by Myerson have been widely used in the literature.

Theorem 4.1.4. (Myerson 1977). *The Myerson value,* Sh^{C}*, is the unique value on* $\mathcal{GC}$ *satisfying* $\mathcal{GC}$:CEF *and* $\mathcal{GC}$:FAI.

Theorem 4.1.5. (Myerson 1980). *The Myerson value,* Sh^{C}*, is the unique value on* $\mathcal{GC}$ *satisfying* $\mathcal{GC}$:CEF *and* $\mathcal{GC}$:BCG.

More recently, the Banzhaf graph value has been characterized using a similar set of properties. The difference between the Shapley and the Banzhaf value, described in Section 1.1.2, is transferred to this context. The characterizations of the Banzhaf graph value replace $\mathcal{GC}$:CEF of the Myerson value either by $\mathcal{GC}$:CTP or by $\mathcal{GC}$:GIS and $\mathcal{GC}$:2-EFF.

Theorem 4.1.6. (Alonso-Meijide & Fiestras-Janeiro 2006).

- *The Banzhaf graph value,* Ba^{C}*, is the unique value on* $\mathcal{GC}$ *satisfying* $\mathcal{GC}$:CTP *and* $\mathcal{GC}$:FAI.
- *The Banzhaf graph value,* Ba^{C}*, is the unique value on* $\mathcal{GC}$ *satisfying* $\mathcal{GC}$:CTP *and* $\mathcal{GC}$:BCG.

Theorem 4.1.7. (Alonso-Meijide & Fiestras-Janeiro 2006).

- *The Banzhaf graph value,* Ba^{C}*, is the unique value on* $\mathcal{GC}$ *satisfying* $\mathcal{GC}$:GIS*,* $\mathcal{GC}$:2-EFF*, and* $\mathcal{GC}$:FAI.
- *The Banzhaf graph value,* Ba^{C}*, is the unique value on* $\mathcal{GC}$ *satisfying* $\mathcal{GC}$:GIS*,* $\mathcal{GC}$:2-EFF*, and* $\mathcal{GC}$:BCG.

Finally, we depict in Table 4.1 the characterizations of Sh^{C} and Ba^{C} in short.

	Sh^{C}	Ba^{C}		
Myerson (1977)	$\mathcal{GC}$:CEF $\mathcal{GC}$:FAI	$\mathcal{GC}$:CTP $\mathcal{GC}$:FAI	$\mathcal{GC}$:2-EFF $\mathcal{GC}$:GIS $\mathcal{GC}$:FAI	Alonso-Meijide and Fiestras-Janeiro (2006)
Myerson (1980)	$\mathcal{GC}$:CEF $\mathcal{GC}$:BCG	$\mathcal{GC}$:CTP $\mathcal{GC}$:BCG	$\mathcal{GC}$:2-EFF $\mathcal{GC}$:GIS $\mathcal{GC}$:BCG	Alonso-Meijide and Fiestras-Janeiro (2006)

Table 4.1: Parallel characterizations of Sh^{C} and Ba^{C}

4.2 Games with a priori unions and graph restricted communication

In this section games with a priori unions and graph restricted communication are studied. First of all, the results in Vázquez-Brage et al. (1996) are recalled. They propose and characterize the Owen graph value, which generalizes the Owen and Myerson values to this framework. Next, two new values are introduced, namely the Banzhaf Owen graph value and the Symmetric coalitional Banzhaf graph value. Finally, parallel characterizations of these three values are proposed.

A *game with a priori unions and graph restricted communication* is a quadruple (N, v, C, P), where $(N, v) \in \mathcal{G}$, $C \in \mathcal{C}^N$, and $P \in P(N)$. We denote by $\mathcal{GUC}$ the set of all such games. Let $(N, v, C, P) \in \mathcal{GUC}$ and $P_k, P_s \in P$, then $C^{-(P_k,P_s)} \in \mathcal{C}^N$ is the graph obtained from C when all links between members of P_k and P_s are deleted, i.e., $C^{-(P_k,P_s)} = C \setminus \{(i : j) \in C : i \in P_k \text{ and } j \in P_s\}$. A *value on* $\mathcal{GUC}$ is a map f that assigns a vector $\mathsf{f}(N, v, C, P) \in \mathbb{R}^N$ to every game with a priori unions and graph restricted communication $(N, v, C, P) \in \mathcal{GUC}$.

Definition 4.2.1. (Vázquez-Brage et al. 1996). The *Owen graph value*, Ow^{C}, is the value on $\mathcal{GUC}$ defined for every $(N, v, C, P) \in \mathcal{GUC}$ by

$$\mathsf{Ow}^{\mathsf{C}}(N, v, C, P) = \mathsf{Ow}(N, v^C, P).$$

Next we recall a characterization of this value based on the following properties.

$\mathcal{GUC}$:CEF A value on $\mathcal{GUC}$, f, satisfies *component efficiency* if for every $(N, v, C, P) \in \mathcal{GUC}$ and every $T \in N/C$,

$$\sum_{i \in T} \mathsf{f}_i(N, v, C, P) = v(T).$$

$\mathcal{GUC}$:FAQ A value on $\mathcal{GUC}$, f, satisfies *fairness in the quotient* if for every $(N, v, C, P) \in \mathcal{GUC}$ and every $P_k, P_s \in P$,

$$\sum_{i \in P_k} \mathsf{f}_i(N, v, C, P) - \sum_{i \in P_k} \mathsf{f}_i(N, v, C^{-(P_k,P_s)}, P) = \sum_{i \in P_s} \mathsf{f}_i(N, v, C, P) - \sum_{i \in P_s} \mathsf{f}_i(N, v, C^{-(P_k,P_s)}, P).$$

$\mathcal{GUC}$:BCU A value on $\mathcal{GUC}$, f, satisfies *balanced contributions within the unions* if for every $(N, v, C, P) \in \mathcal{GUC}$, $P_k \in P$, and $i, j \in P_k$,

$$\mathsf{f}_i(N, v, C, P) - \mathsf{f}_i(N, v, C, P_{-j}) = \mathsf{f}_j(N, v, C, P) - \mathsf{f}_j(N, v, C, P_{-i}).$$

$\mathcal{GUC}$:CEF states that a value on $\mathcal{GUC}$ should be efficient inside each connected coalition. It is a generalization of $\mathcal{GC}$:CEF to the framework of games with a priori unions and graph restricted communication. The most demanding property may be $\mathcal{GUC}$:FAQ. It states that the joint profit or loss of the agents in a union when this union is disconnected from another union is the same for each of the unions involved. It studies the implications of the way in which unions are connected among them, hence it takes into account both the communication graph and the structure of a priori unions. Finally, $\mathcal{GUC}$:BCU is a generalization of the $\mathcal{GU}$:BCU property studied in Section 2.1. It stipulates that the gain (or loss) inflicted to a player by another player's withdrawal from the union they both lie in is the same as if it happens the other way around.

Theorem 4.2.2. (Vázquez-Brage et al. 1996). *The Owen graph value,* Ow^{C}*, is the unique value on* $\mathcal{GUC}$ *satisfying* $\mathcal{GUC}$:CEF*,* $\mathcal{GUC}$:FAQ*, and* $\mathcal{GUC}$:BCU*.*

Note that as one might expect, this value generalizes the Shapley, Owen, and Myerson values. Table 4.2 depicts these generalizations for particular instances of systems of a priori unions and communication graphs.

graph \ unions	$P = P^n$	$P = P^N$	$P \in P(N)$
$C = \emptyset^*$	Sh	Sh	Ow
$C \in \mathcal{C}^N$	Sh^{C}	Sh^{C}	Ow^{C}

Table 4.2: The Owen graph value

Following the way in which the Owen graph value is defined, two new values on $\mathcal{GUC}$ that generalize the Banzhaf value are introduced.

Definition 4.2.3. (Alonso-Meijide et al. 2009a) The *Banzhaf-Owen graph value*, BO^{C}, is the value on $\mathcal{GUC}$ defined for every $(N, v, C, P) \in \mathcal{GUC}$ by

$$\mathsf{BO}^{\mathsf{C}}(N, v, C, P) = \mathsf{BO}(N, v^C, P).$$

Definition 4.2.4. (Alonso-Meijide et al. 2009a) The *Symmetric coalitional Banzhaf graph value*, $\mathsf{SCB}^{\mathsf{C}}$, is the value on $\mathcal{GUC}$ defined for every $(N, v, C, P) \in \mathcal{GUC}$ by

$$\mathsf{SCB}^{\mathsf{C}}(N, v, C, P) = \mathsf{SCB}(N, v^C, P).$$

Again, note that these two values on $\mathcal{GUC}$ generalize Sh, Ba, $\mathsf{Sh^C}$, $\mathsf{Ba^C}$, BO, and SCB in the way shown in Table 4.3 and Table 4.4. Observe in Table 4.4 that $\mathsf{SCB^C}$ generalizes the Symmetric coalitional Banzhaf value, the Myerson and Banzhaf graph values, and the Banzhaf and Shapley values.

graph \ unions	$P = P^n$	$P = P^N$	$P \in P(N)$
$C = \emptyset^*$	Ba	Ba	BO
$C \in \mathcal{C}^N$	$\mathsf{Ba^C}$	$\mathsf{Ba^C}$	$\mathsf{BO^C}$

Table 4.3: The Banzhaf-Owen graph value

graph \ unions	$P = P^n$	$P = P^N$	$P \in P(N)$
$C = \emptyset^*$	Ba	Sh	SCB
$C \in \mathcal{C}^N$	$\mathsf{Ba^C}$	$\mathsf{Sh^C}$	$\mathsf{SCB^C}$

Table 4.4: The Symmetric coalitional Banzhaf graph value

Once the new values on $\mathcal{GUC}$ are introduced we now switch the attention to their characterizations. Let us consider the following properties for a value on $\mathcal{GUC}$. Some of the properties only apply to games with the trivial singleton system of a priori unions P^n.

$\mathcal{GUC}$:GIS A value on $\mathcal{GUC}$, f, satisfies *graph isolation* if for every $(N, v, C) \in \mathcal{GC}$ and every $i \in N$ such that i is an isolated agent, i.e, $\{i\} \in N/C$,

$$\mathsf{f}_i(N, v, C, P^n) = v(i).$$

The idea behind this property is that an isolated agent with respect to the communication situation given the trivial singleton coalition structure will only receive the utility she can obtain on her own, because she will not be able to communicate with any other agent. $\mathcal{GUC}$:GIS is based on $\mathcal{GC}$:GIS, which is presented in Section 4.1.

$\mathcal{GUC}$:2-EFF A value on $\mathcal{GUC}$, f, satisfies *2-efficiency* if for every $(N, v, C) \in \mathcal{GC}$ and every $i, j \in N$ such that $(i : j) \in C$,

$$\mathsf{f}_i(N, v, C, P^n) + \mathsf{f}_j(N, v, C, P^n) = \mathsf{f}_p(N^{ij}, v^{ij}, C^{ij}, P^{n-1}).$$

Property $\mathcal{GUC}$:2-EFF states that a value satisfying it is immune against artificial merging or splitting of two directly connected players in $C \in \mathcal{C}^N$ under the trivial singleton coalition structure. It generalizes $\mathcal{GC}$:2-EFF as well as $\mathcal{G}$:2-EFF.

$\mathcal{GUC}$:FAG A value on $\mathcal{GUC}$, f, satisfies *fairness in the graph* if for every $(N, v, C) \in \mathcal{GC}$ and every $i, j \in N$ such that $(i : j) \in C$,

$$\mathsf{f}_i(N, v, C, P^n) - \mathsf{f}_i(N, v, C^{-ij}, P^n) = \mathsf{f}_j(N, v, C, P^n) - \mathsf{f}_j(N, v, C^{-ij}, P^n).$$

The fairness in the graph property establishes that, given the trivial singleton coalition structure, if a player's payoff increases or decreases when breaking the link with another player, this other player should gain or lose the same amount. $\mathcal{GUC}$:FAG is based on $\mathcal{GC}$:FAI property introduced in Myerson (1977).

The following properties are stated for games with arbitrary structures of a priori unions. All of them are based on properties introduced in Section 2.1.

$\mathcal{GUC}$:NID A value on $\mathcal{GUC}$, f, satisfies *neutrality under individual desertion* if for every $(N, v, C, P) \in \mathcal{GUC}$ and $i, j \in N$ such that $i, j \in P_k \in P$,

$$\mathsf{f}_i(N, v, C, P) = \mathsf{f}_i(N, v, C, P_{-j}).$$

The neutrality under individual desertion property states that the desertion of an agent from an a priori union does not affect the payoff of the remaining members of the union. Property $\mathcal{GUC}$:NID is just a stronger version of $\mathcal{GUC}$:BCU. It is based on $\mathcal{GU}$:NID.

In order to present the last two properties we need a game which combines the ideas behind both the quotient game, (N, v^P), and the communication game, (N, v^C). It is formally introduced next.

Definition 4.2.5. Given $(N, v, C, P) \in \mathcal{GUC}$, the *communication quotient game* $(M, v^{CP}) \in \mathcal{G}^M$ is defined for every $R \subseteq M$, by

$$v^{CP}(R) = \sum_{L \in P_R/C} v(L),$$

where, recall that for every $R \subseteq M$, $P_R = \cup_{r \in R} P_r$.

$\mathcal{GUC}$:1-QGP A value on $\mathcal{GUC}$, f, satisfies *1-quotient game property* if for every $(N, v, C, P) \in \mathcal{GUC}$ and $i \in N$ such that $\{i\} = P_k \in P$,

$$\mathsf{f}_i(N, v, C, P) = \mathsf{f}_k(M, v^{CP}, \emptyset^*, P^m).$$

Property $\mathcal{GUC}$:1-QGP states that, using the value on $\mathcal{GUC}$ in the original game, any isolated agent with respect to the system of a priori unions gets the same payoff as the union she forms in the communication quotient game with the trivial singleton coalition structure and the complete graph.

$\mathcal{GUC}$:QGP A value on $\mathcal{GUC}$, f, satisfies the *quotient game property* if for every $(N, v, C, P) \in \mathcal{GUC}$ and every $P_k \in P$,

$$\sum_{i \in P_k} \mathsf{f}_i(N, v, C, P) = \mathsf{f}_k(M, v^{CP}, \emptyset^*, P^m).$$

The $\mathcal{GUC}$:QGP property states that the total payoff obtained by the members of a union in the original game, is the amount obtained by the union itself in the communication quotient game with the trivial singleton system of a priori unions and the complete graph. In the case where $P_k = \{i\}$ properties $\mathcal{GUC}$:1-QGP and $\mathcal{GUC}$:QGP are equivalent. Hence, $\mathcal{GUC}$:QGP implies $\mathcal{GUC}$:1-QGP.

At this point the properties needed to characterize the values on $\mathcal{GUC}$ proposed in Definitions 4.2.3 and 4.2.4 have been introduced.

Theorem 4.2.6. (Alonso-Meijide et al. 2009a). *The Banzhaf-Owen graph value, BO^{C}, is the unique value on $\mathcal{GUC}$ satisfying $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:NID, and $\mathcal{GUC}$:1-QGP.*

Proof. As depicted in Table 4.3, $\mathsf{BO}^{\mathsf{C}}(N, v, C, P^n) = \mathsf{Ba}^{\mathsf{C}}(N, v, C)$. Then, using the characterization of Ba^{C} depicted in Theorem 4.1.7 it follows that BO^{C} is characterized by $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, and $\mathcal{GUC}$:FAG when the system of a priori unions is P^n. Then, we just need to prove that BO^{C} satisfies $\mathcal{GUC}$:NID and $\mathcal{GUC}$:1-QGP and the uniqueness for every $(N, v, C, P) \in \mathcal{GUC}$ with $P \neq P^n$.

(1) *Existence.* First we show that BO^{C} satisfies $\mathcal{GUC}$:1-QGP. Take $(N, v, C, P) \in \mathcal{GUC}$, $i \in N$ and $k \in M$ such that $P_k = \{i\}$ and consider the communication quotient game $(M, v^{CP}, \emptyset^*, P^m) \in \mathcal{GUC}$. Since $(v^{CP})^{\emptyset^*} = v^{CP}$,

$$\begin{aligned}\mathsf{BO}^{\mathsf{C}}_k(M, v^{CP}, \emptyset^*, P^m) &= \sum_{R \subseteq M \setminus k} \frac{1}{2^{m-1}} \Big[(v^{CP})^{\emptyset^*}(R \cup k) - (v^{CP})^{\emptyset^*}(R)\Big] \\ &= \sum_{R \subseteq M \setminus k} \frac{1}{2^{m-1}} \big[v^{CP}(R \cup k) - v^{CP}(R)\big] = \mathsf{BO}^{\mathsf{C}}_i(N, v, C, P).\end{aligned}$$

Next, we show that BO^{C} satisfies $\mathcal{GUC}$:NID. Let $i, j \in P_k$, then we can write

$$\mathsf{BO}^{\mathsf{C}}_i(N, v, C, P) = 2^{2-m-p_k} \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} \left[\begin{array}{c} v^C(P_R \cup T \cup i \cup j) - v^C(P_R \cup T) \\ +v^C(P_R \cup T \cup i) - v^C(P_R \cup T \cup j) \end{array}\right].$$

Take $P_{-j} = \{P'_1, \ldots, P'_{m+1}\}$ where $P'_k = P_k \setminus j$, $P'_{m+1} = \{j\}$, and for every $l \in \{1, \ldots, k-1, k+1, \ldots, m\}$, $P'_l = P_l$. Let $M' = \{1, \ldots, m+1\}$. Then,

$$\begin{aligned}
\mathsf{BO}_i^{\mathsf{C}}(N,v,C,P_{-j}) &= \sum_{R\subseteq M'\setminus k} \frac{1}{2^m} \sum_{T\subseteq P'_k\setminus i} \frac{1}{2^{p_k-2}} \left[v^C(P_R \cup T \cup i) - v^C(P_R \cup T)\right] \\
&= \sum_{R\subseteq M\setminus k} \frac{1}{2^m} \sum_{T\subseteq P_k\setminus\{i,j\}} \frac{1}{2^{p_k-2}} \left[v^C(P_R \cup T \cup i) - v^C(P_R \cup T)\right] \\
&\quad + \sum_{R\subseteq M\setminus k} \frac{1}{2^m} \sum_{T\subseteq P_k\setminus\{i,j\}} \frac{1}{2^{p_k-2}} \left[v^C(P_R \cup T \cup j \cup i) - v^C(P_R \cup T \cup j)\right] \\
&= 2^{2-m-p_k} \sum_{R\subseteq M\setminus k} \sum_{T\subseteq P_k\setminus\{i,j\}} \left[\begin{array}{c} v^C(P_R \cup T \cup i \cup j) - v^C(P_R \cup T) \\ +v^C(P_R \cup T \cup i) - v^C(P_R \cup T \cup j) \end{array}\right].
\end{aligned}$$

(2) *Uniqueness.* Suppose that there are two different values on $\mathcal{GUC}$, f^1 and f^2, that satisfy the properties. Then, there exists $(N,v,C,P) \in \mathcal{GUC}$ such that $\mathsf{f}^1(N,v,C,P) \neq \mathsf{f}^2(N,v,C,P)$. Which means that $P \neq P^n$. We may suppose that for the triple (N,v,C), P is a system of a priori unions with the maximum number of unions for which $\mathsf{f}^1(N,v,C,P) \neq \mathsf{f}^2(N,v,C,P)$ holds. Take $i \in N$ such that $\mathsf{f}_i^1(N,v,C,P) \neq \mathsf{f}_i^2(N,v,C,P)$. Two cases may arise.

- $\{i\} = P_k \in P$. By $\mathcal{GUC}$:1-QGP and the uniqueness for the trivial coalition structure,

$$\mathsf{f}_i^1(N,v,C,P) = \mathsf{f}_k^1(M, v^{CP}, \emptyset^*, P^m) = \mathsf{f}_k^2(M, v^{CP}, \emptyset^*, P^m) = \mathsf{f}_i^2(N,v,C,P).$$

- There is $j \neq i$ such that $i,j \in P_k$. Then, by $\mathcal{GUC}$:NID and the maximality of P,

$$\mathsf{f}_i^1(N,v,C,P) = \mathsf{f}_i^1(N,v,C,P_{-j}) = \mathsf{f}_i^2(N,v,C,P_{-j}) = \mathsf{f}_i^2(N,v,C,P).$$

Which contradicts the inequality in the beginning, and hence, the result is proved. □

With a similar scheme, a characterization of the Symmetric coalitional Banzhaf graph value is obtained, we just need to replace properties $\mathcal{GUC}$:NID and $\mathcal{GUC}$:1-QGP by $\mathcal{GUC}$:BCU and $\mathcal{GUC}$:QGP.

Theorem 4.2.7. (Alonso-Meijide et al. 2009a) *The Symmetric coalitional Banzhaf graph value,* $\mathsf{SCB}^{\mathsf{C}}$, *is the unique value on* $\mathcal{GUC}$ *satisfying* $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:BCU, *and* $\mathcal{GUC}$:QGP.

Proof. As depicted in Table 4.4, $\mathsf{SCB}^{\mathsf{C}}(N,v,C,P^n) = \mathsf{Ba}^{\mathsf{C}}(N,v,C)$. Then, using the characterization of Ba^{C} depicted in Theorem 4.1.7 it follows that $\mathsf{SCB}^{\mathsf{C}}$ is characterized by $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, and $\mathcal{GUC}$:FAG when the system of a priori unions is P^n. Then, we just need to prove that SCB^C satisfies $\mathcal{GUC}$:BCU and $\mathcal{GUC}$:QGP and the uniqueness for every $(N, v, C, P) \in \mathcal{GUC}$ with $P \neq P^n$.

(1) *Existence.* By Definition 4.2.4 we know that SCB^C satisfies $\mathcal{GUC}$:BCU and $\mathcal{GUC}$:QGP because SCB satisfies $\mathcal{GU}$:BCU and $\mathcal{GU}$:QGP (see Theorem 2.1.6).

(2) *Uniqueness.* Suppose that there are two different values on $\mathcal{GUC}$, f^1 and f^2, that satisfy the properties. Then, there is $(N, v, C, P) \in \mathcal{GUC}$ such that $\mathsf{f}^1(N, v, C, P) \neq \mathsf{f}^2(N, v, C, P)$. Which means that $P \neq P^n$. We may suppose that for the triple (N, v, C), P is a system of a priori unions with the maximum number of unions for which $\mathsf{f}^1(N, v, C, P) \neq \mathsf{f}^2(N, v, C, P)$ holds. Then, there is $i \in N$ such that $\mathsf{f}^1_i(N, v, C, P) \neq \mathsf{f}^2_i(N, v, C, P)$. Two cases may arise.

- $\{i\} = P_k \in P$. By $\mathcal{GUC}$:QGP and the uniqueness for the trivial singleton coalition structure,

  $$\mathsf{f}^1_i(N, v, C, P) = \mathsf{f}^1_k(M, v^{CP}, \emptyset^*, P^m) = \mathsf{f}^2_k(M, v^{CP}, \emptyset^*, P^m) = \mathsf{f}^2_i(N, v, C, P).$$

- There is $j \in P_k \setminus i$. Then, by $\mathcal{GUC}$:BCU and the maximality of P,

  $$\mathsf{f}^1_i(N, v, C, P) - \mathsf{f}^1_j(N, v, C, P) = \mathsf{f}^2_i(N, v, C, P) - \mathsf{f}^2_j(N, v, C, P). \tag{4.1}$$

  Using $\mathcal{GUC}$:QGP and Eq. (4.1),

  $$p_k \cdot \mathsf{f}^1_i(N, v, C, P) = p_k \cdot \mathsf{f}^2_i(N, v, C, P).$$

Hence, we obtain a contradiction in both cases and the proof concludes. □

Lastly, we give a characterization of Ow^C which will be useful to discuss the differences between the values on $\mathcal{GUC}$ considered in this section.

Theorem 4.2.8. (Alonso-Meijide et al. 2009a). *The Owen graph value,* Ow^C*, is the unique value on $\mathcal{GUC}$ satisfying $\mathcal{GUC}$:CEF(for P^n), $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:BCU, and $\mathcal{GUC}$:QGP.*

Proof.

(1) *Existence.* Using that Ow^C generalizes Sh^C, that is, using that for every $(N, v, C, P^n) \in \mathcal{GUC}$, $\mathsf{Ow}^\mathsf{C}(N, v, C, P^n) = \mathsf{Sh}^\mathsf{C}(N, v, C)$ and Theorem 4.1.4, Ow^C satisfies $\mathcal{GUC}$:CEF (for P^n) and $\mathcal{GUC}$:FAG. From Vázquez-Brage et al. (1997) we know that Ow^C satisfies both $\mathcal{GUC}$:BCU and $\mathcal{GUC}$:QGP.

(2) *Uniqueness.* We know that the Myerson value is characterized by $\mathcal{GC}$:CEF and $\mathcal{GC}$:FAI (see Theorem 4.1.5). Hence, $\mathcal{GUC}$:CEF(for P^n) and $\mathcal{GUC}$:FAG uniquely determine Ow^C for every $(N, v, C, P^n) \in \mathcal{GUC}$. It remains to prove the uniqueness

for every $(N, v, C, P) \in \mathcal{GUC}$ with $P \neq P^n$. The reasoning used in the proof of the uniqueness of Theorem 4.2.7 can be repeated here and the result is proved. □

In Table 4.5 the characterization results of Theorems 4.2.6, 4.2.7, and 4.2.8 are summarized.

Ow^{C}	$\mathsf{SCB}^{\mathsf{C}}$	BO^{C}
$\mathcal{GUC}$:CEF	$\mathcal{GUC}$:2-EFF $\mathcal{GUC}$:GIS	$\mathcal{GUC}$:2-EFF $\mathcal{GU}$:GIS
$\mathcal{GUC}$:FAG	$\mathcal{GUC}$:FAG	$\mathcal{GUC}$:FAG
$\mathcal{GUC}$:QGP $\mathcal{GUC}$:BCU	$\mathcal{GUC}$:QGP $\mathcal{GUC}$:BCU	$\mathcal{GUC}$:1-QGP $\mathcal{GUC}$:NID

Table 4.5: Parallel characterizations of Ow^{C}, $\mathsf{SCB}^{\mathsf{C}}$, and BO^{C}

Note that, the characterizations are based on two sets of properties. The first set of properties applies only to games with a priori unions and graph restricted communication where the system of a priori unions is the trivial singleton coalition structure. This first set of properties determines which value on $\mathcal{GC}$, either the Myerson value or the Banzhaf graph value, does the value on $\mathcal{GUC}$ generalize. Moreover, from the proofs above it follows that this first set of properties can be replaced by any other set of properties that characterizes the corresponding value on $\mathcal{GC}$. The second set of properties applies to arbitrary games with a priori unions and graph restricted communication. This set of properties reveals the way in which the corresponding value on $\mathcal{GUC}$ deals with the system of a priori unions. Besides, the properties of this second set are logically related which eases the comparation among the different values on $\mathcal{GUC}$. Finally, we check that the properties used in the characterizations are indeed independent.

Remark 4.2.9. The properties considered in Theorem 4.2.6 are independent as the following examples show:

(i) The value on $\mathcal{GUC}$, g^1, defined for every $(N, v, C, P) \in \mathcal{GUC}$, by

$$\mathrm{g}^1(N, v, C, P) = 0,$$

satisfies $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:NID, and $\mathcal{GUC}$:1-QGP, but not $\mathcal{GUC}$:GIS.

(ii) The value on $\mathcal{GUC}$, g^2, defined for every $(N, v, C, P) \in \mathcal{GUC}$ and $i \in N$, by

$$\mathrm{g}^2_i(N, v, C, P) = v(i),$$

satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:NID, and $\mathcal{GUC}$:1-QGP, but not $\mathcal{GUC}$:2-EFF.

(iii) Let a, b be two distinct, fixed, and indivisible players. In this context by indivisible we mean first, that a and b are not obtained after a merging of two players and second, that neither a nor b are representatives of a union of a coalition structure. Define the value on $\mathcal{GUC}$, g^3, for every $(N, v, C, P) \in \mathcal{GUC}$ as follows:

- If $(N, v, C, P) = (\{a, b\}, \emptyset^*, P^n)$,

$$\mathrm{g}^3_a(N, v, C, P) = \frac{3}{4}\left[v(N) - v(b)\right] + \frac{1}{4}v(a) \quad \text{and}$$
$$\mathrm{g}^3_b(N, v, C, P) = \frac{1}{4}\left[v(N) - v(a)\right] + \frac{3}{4}v(b).$$

- Otherwise, $\mathrm{g}^3(N, v, C, P) = \mathsf{BO}^{\mathsf{C}}(N, v, C, P)$.

Then, g^3 satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:NID, and $\mathcal{GUC}$:1-QGP, but not $\mathcal{GUC}$:FAG.

(iv) The $\mathsf{SCB}^{\mathsf{C}}$ value satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, and $\mathcal{GUC}$:1-QGP, but not $\mathcal{GUC}$:NID.

(v) The value on $\mathcal{GUC}$, g^4, defined for every $(N, v, C, P) \in \mathcal{GUC}$, by

$$\mathrm{g}^4(N, v, C, P) = \mathsf{Ba}^{\mathsf{C}}(N, v, C),$$

satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, and $\mathcal{GUC}$:NID, but not $\mathcal{GUC}$:1-QGP. $\triangleleft$

Remark 4.2.10. The properties considered in Theorem 4.2.7 are independent as the following examples show:

(i) The value on $\mathcal{GUC}$, g^1, defined above satisfies $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:BCU, and $\mathcal{GUC}$:QGP, but not $\mathcal{GUC}$:GIS.

(ii) The Ow^{C} value satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:BCU, and $\mathcal{GUC}$:QGP, but not $\mathcal{GUC}$:2-EFF.

(iii) Again, let a, b be two distinct, fixed, and indivisible players. Define the value on $\mathcal{GUC}$, g^5, for every $(N, v, C, P) \in \mathcal{GUC}$ as follows:

- If $(N, v, C, P) = (\{a, b\}, \emptyset^*, P^n)$,

$$\mathrm{g}^5_a(N, v, C, P) = \frac{3}{4}\left[v(N) - v(b)\right] + \frac{1}{4}v(a) \quad \text{and}$$
$$\mathrm{g}^5_b(N, v, C, P) = \frac{1}{4}\left[v(N) - v(a)\right] + \frac{3}{4}v(b).$$

- Otherwise, $\mathrm{g}^5(N, v, C, P) = \mathsf{SCB}^{\mathsf{C}}(N, v, C, P)$.

Then, g^5 satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:BCU, and $\mathcal{GUC}$:QGP, but not $\mathcal{GUC}$:FAG.

(iv) The value on $\mathcal{GUC}$, g^6, defined for every $(N,v,C,P) \in \mathcal{GUC}$ and every $i \in N$, by

$$\mathrm{g}_i^6(N,v,C,P) = \mathsf{Ba}_k(M,v^{CP})/|P_k|,$$

where $i \in P_k$ and $k \in M$, satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, and $\mathcal{GUC}$:QGP, but not $\mathcal{GUC}$:BCU.

(v) The BO^{C} value, satisfies $\mathcal{GUC}$:GIS, $\mathcal{GUC}$:2-EFF, $\mathcal{GUC}$:FAG, and $\mathcal{GUC}$:BCU, but not $\mathcal{GUC}$:QGP. $\triangleleft$

Remark 4.2.11. The properties considered in Theorem 4.2.8 are independent as the following examples show:

(i) The value on $\mathcal{GUC}$, g^1, defined above satisfies $\mathcal{GUC}$:FAG, $\mathcal{GUC}$:BCU, and $\mathcal{GUC}$:QGP, but not $\mathcal{GUC}$:CEF (for P^n).

(ii) Again, let a, b be two distinct, fixed, and indivisible players. Define the value on $\mathcal{GUC}$, g^7, for every $(N,v,C,P) \in \mathcal{GUC}$ as follows:

- If $(N,v,C,P) = (\{a,b\}, v, \emptyset^*, P^n)$,

$$\mathrm{g}_a^7(N,v,C,P) = \frac{3}{4}\left[v(N) - v(b)\right] + \frac{1}{4}v(a) \quad \text{and}$$
$$\mathrm{g}_b^7(N,v,C,P) = \frac{1}{4}\left[v(N) - v(a)\right] + \frac{3}{4}v(b).$$

- Otherwise, $\mathrm{g}^7(N,v,C,P) = \mathsf{Ow}^{\mathsf{C}}(N,v,C,P)$.

Then, g^7 satisfies $\mathcal{GUC}$:CEF (for P^n), $\mathcal{GUC}$:BCU, and $\mathcal{GUC}$:QGP, but not $\mathcal{GUC}$:FAG.

(iii) The value on $\mathcal{GUC}$, g^8, defined for every $(N,v,C,P) \in \mathcal{GUC}$ and $i \in N$, by

$$\mathrm{g}_i^8(N,v,C,P) = \mathsf{Sh}_k(M,v^{CP})/|P_k|,$$

where $i \in P_k$ and $k \in M$, satisfies $\mathcal{GUC}$:CEF (for P^n), $\mathcal{GUC}$:FAG, and $\mathcal{GUC}$:QGP, but not $\mathcal{GUC}$:BCU.

(iv) The value on $\mathcal{GUC}$, g^9, defined for every $(N,v,C,P) \in \mathcal{GUC}$, by

$$\mathrm{g}^9(N,v,C,P) = \mathsf{Sh}^{\mathsf{C}}(N,v,C),$$

satisfies $\mathcal{GUC}$:CEF (for P^n), $\mathcal{GUC}$:BCU, and $\mathcal{GUC}$:FAG, but not $\mathcal{GUC}$:QGP. $\triangleleft$

To conclude this section the studied values on $\mathcal{GUC}$ are illustrated by means of an example coming from the political field. Indeed, the values on $\mathcal{GUC}$ considered are used as power indices to measure the decisiveness of the political parties with parliamentary representation.

4.2.1 A political example

The Parliament of the Basque Country, one of Spain's seventeen regions, is constituted by 75 members. Since most decisions are taken by majority, the characteristic function of the game played by the parties with parliamentary representation is as follows, unity for any coalition adding up to 38 or more members, and zero for the rest. Since elections in 2005, the Parliament was composed by 22 members of the Basque nationalist conservative party EAJ/PNV, "A", 18 members of the Spanish socialist party PSE-EE/PSOE, "B", 15 members of the Spanish conservative party PP, "C", 9 members of the Basque nationalist left-wing party EHAK/PCTV, "D", 7 members of the Basque nationalist social democrat party EA, "E", 3 members of the Spanish left-wing party EB/IU, "F", and 1 member of the Basque nationalist moderated left-wing party Aralar, "J".

In order to build a model that takes into account the ideology of the political parties involved we propose a communication graph defined in Figure 4.1. The graph is based on the relations between the parties in such a way that we put a link between two agents whenever these parties had reached agreements in the past.

Finally, we propose a cooperation structure in terms of a system of a priori unions. Since the government was formed by A, E, and F before the elections, we consider the system of a priori unions $P = \{P_1, P_2, P_3, P_4, P_5\}$, where $P_1 = \{A, E, F\}$, $P_2 = \{B\}$, $P_3 = \{C\}$, $P_4 = \{D\}$, and $P_5 = \{G\}$. The proposed coalition and communication structures are jointly depicted in Figure 4.1.

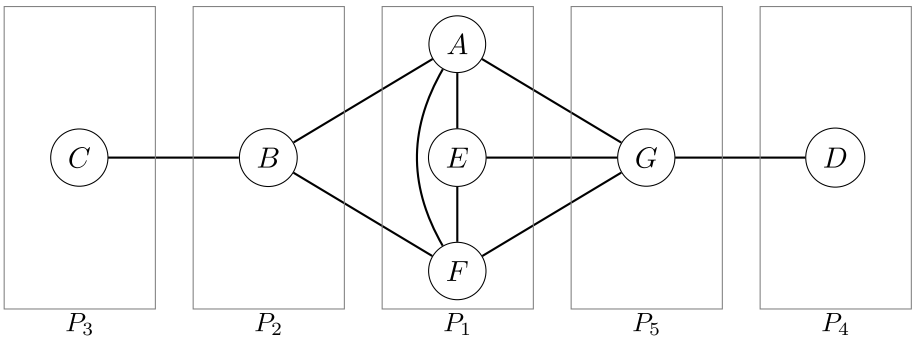

Figure 4.1: The communication graph and the system of a priori of unions

First, in Table 4.6 we compute the power of each party with no restriction to the cooperation and the power of each party taking into account the graph restricted communication. Concerning the results presented in Table 4.6, the most remarkable effect of the communication graph restriction is that the most powerful player switches from A to B. In general those parties which have more links to other parties increase their payoffs at the expense of the parties located at the extremes of the graph.

Party	Seats	Sh	Ba	$\overline{\text{Ba}}$	Sh^C	Ba^C	$\overline{\text{Ba}^C}$
A	22	.352	.594	.345	.302	.484	.248
B	18	.252	.406	.236	.369	.531	.272
C	15	.186	.344	.200	.035	.125	.064
D	9	.086	.156	.091	.069	.187	.096
E	7	.086	.156	.091	.086	.250	.128
F	3	.019	.031	.018	.069	.187	.096
G	1	.019	.031	.018	.069	.187	.096

Table 4.6: The Shapley, Banzhaf, Myerson, and Banzhaf graph values

Next, we compute in Tables 4.7 and 4.8 the different measures of power of the parties taking into account only the restrictions given by the coalition structure, on Table 4.7, and the restrictions given by both the coalition structure and the communication graph, on Table 4.8.

Party	Seats	Ow	BO	$\overline{\text{BO}}$	SCB	$\overline{\text{SCB}}$
A	22	.383	.594	.388	.583	.389
B	18	.167	.250	.163	.250	.167
C	15	.167	.250	.163	.250	.167
D	9	.167	.250	.163	.250	.167
E	7	.100	.156	.102	.146	.097
F	3	.017	.031	.020	.021	.014
G	1	0	0	0	0	0

Table 4.7: The Owen, Banzhaf-Owen, and Symmetric coalitional Banzhaf values

Table 4.7 shows the distribution of power based on the different values on $\mathcal{GU}$ studied in Section 2.1 given the system of a priori unions P. This approach gives more power to A, which is the biggest party in the union P_1. On the other hand the next three parties, B, C, and D are each allotted with the same power. Finally, the smallest party becomes a null player.

When we consider the communication graph together with the system of a priori unions (Table 4.8) the distribution of power changes significantly. The most remarkable change is that player C, third most voted one, becomes irrele-

Party	Seats	Ow^{C}	BO^{C}	$\overline{\mathsf{BO}^{\mathsf{C}}}$	$\mathsf{SCB}^{\mathsf{C}}$	$\overline{\mathsf{SCB}^{\mathsf{C}}}$
A	22	.381	.469	.357	.448	.352
B	18	.250	.375	.286	.375	.295
C	15	0	0	0	0	0
D	9	.083	.125	.095	.125	.098
E	7	.072	.125	.095	.104	.082
F	3	.131	.094	.071	.094	.074
G	1	.083	.125	.095	.125	.098

Table 4.8: The Owen, Banzhaf-Owen, and Symmetric coalitional Banzhaf graph values

vant and that G, which has only one representative, is not null anymore. This shows that the studied model is different from the models presented in Section 2.1 and Section 4.1, and that considering the two restrictions together enriches the model. Last, but not least, if we focus on the values on $\mathcal{GUC}$, the difference between Ow^{C} and the other two lies in the power of parties E and F. The Owen graph value gives more weight to coalitions formed by many players (also by few players), while the other two (mostly BO^{C}) are not so sensible to the sizes of the coalitions where there are swings. This is the reason why parties E and F switch their order.

5

Games with incompatible players

In this chapter we study games in which there are incompatible players. So, we consider another class of games with restricted cooperation. The incompatibilities among the players are described by means of an undirected graph without loops. A link between two agents means that the agents are incompatible, and hence they can not productively cooperate. The first work concerning cooperative games where there are players who cannot be together in a coalition may be Kilgour (1974). However, the approach of this work is rather cumbersome. The first work in which the incompatibilities among the players are described by an undirected graph is Carreras (1991). In this paper a joint model which takes into account the affinities, described by a communication graph, and incompatibilities among the players is proposed for simple games. In Carreras & Owen (1996) a political application is provided taking into account the existence of incompatible players.

In Bergantiños (1993) and Bergantiños et al. (1993) the model of games with incompatibilities is extended to the class of TU games. The existence of incompatible players is much more restrictive than the restrictions to the cooperation arriving from the affinities among the players (communication graph), since players which are not connected by the affinities graph can still cooperate if there is a path connecting them while if two players are incompatible they could never be in the same coalition. In the aforesaid works a generalization of the Shapley value to this framework is proposed and characterized. In this setting we saw the lack of a generalization of the Banzhaf value, which we will introduce and characterize here.

The joint model of games with affinities and incompatibilities is considered in Amer & Carreras (1995a). In this work the authors define the cooperation index, which is a map $p : 2^N \to [0, 1]$ that describes quantitative restrictions to the cooperation. The cooperation index is capable of modeling situations in which the affinities among players have different intensities, the only requirement is that for every $i \in N$, $p(i) = 1$ (non schizophrenic players). If $p(S) = 0$, it means that there are incompatible players on S, while $p(S) > 0$ means that players in S can communicate, and hence, cooperate. In Amer & Carreras (1995a) the Shapley value is extended and characterized for games with cooperation indices. The model studied in Section 4.1 is included in this new model if we consider the cooperation index p_C, given by $p_C(S) = 1$ if $S \subseteq N$ is connected by C and $p_C(S) = 0$ otherwise. In a similar way, the model that will be described in this chapter is included in the approach of Amer & Carreras (1995a) as we will soon see.

As the previous chapter, the work contained in this chapter is a joint work with my supervisors, José M. Alonso-Meijide and M. Gloria Fiestras-Janeiro. The results contained in this chapter constitute my second article (Alonso-Meijide et al. 2009b) published in Homo Oeconomicus. The outline of the rest of the chapter is as follows. In Section 5.1, we present the games with incompatibilities and recall the main results in this framework. Then, in Section 5.2 a new value for this class of games is defined and two characterizations of it are proposed. Finally, Section 5.3 concludes illustrating the new value by means of an example coming from the political field.

5.1 The model of games with incompatibilities

A *game with incompatibilities* is a triple (N, v, I) where $(N, v) \in \mathcal{G}$ is a game and $I \in \mathcal{C}^N$ is the incompatibility graph, i.e., $i, j \in N$ are incompatible if $(i : j) \in I$. We denote by $\mathcal{GI}$ the set of all such games. Given $(N, v, I) \in \mathcal{GI}$ we say that a coalition $S \subseteq N$ is *I-admissible* if there are not incompatible players contained on it. By $P(S, I)$ we will denote the set of all partitions of S whose elements are I-admissible coalitions. Recall some notation defined in Section 4.1. Given $I \in \mathcal{C}^N$, I^* denotes the *dual graph of* I, i.e., $I^* = \{(i : j) : i, j \in N, i \neq j, (i : j) \notin I\}$. Then, $\emptyset^*$ denotes the *complete graph* on N, i.e., $\emptyset^* = \{(i : j) : i, j \in N, i \neq j\}$. Finally, let $I \in \mathcal{C}^N$ and $i \in N$, I^{+i} denotes the graph obtained from I when player i becomes incompatible with the rest of the players, i.e., $I^{+i} = I \cup \{(i : j) : j \in N \backslash i\}$.

Definition 5.1.1. Given $(N, v, I) \in \mathcal{GI}$, the *I-restricted game*, $(N, v^I) \in \mathcal{G}$, is defined for every $S \subseteq N$ by

$$v^I(S) = \max_{P \in P(S,I)} \sum_{T \in P} v(T).$$

The I-restricted game is in correspondence with the communication game defined in Section 4.1. The idea behind the I-restricted game is that players of a coalition S form I-admissible subcoalitions (which are the only feasible coalitions) and they choose them in such a way that the joint worth of the subcoalitions of S is maximized.

Bergantiños (1993) shows that the game with incompatibilities $(N, v, I) \in \mathcal{GI}$ is not in general equal to the game with graph restricted communication, $(N, v, I^*) \in \mathcal{GC}$, restricted by the dual graph. This fact will be shown in Example 5.1.2. In Bergantiños (1993) it is also shown that the I-restricted game is always superadditive. In fact, Example 5.1.2 illustrates the way in which the I-restricted game is built and the difference between the I-restricted game and the communication game associated with the dual graph.

Example 5.1.2. Let $(N, v, I) \in \mathcal{GI}$ be the game with incompatibilities where $N = \{1, 2, 3\}$, $I = \{(1 : 2)\}$, and the characteristic function is defined as follows:

$$\begin{aligned}
&v(i) = 0 \quad \text{for every } i \in N,\\
&v(\{1,2\}) = v(\{1,3\}) = 1, \quad v(\{2,3\}) = 2, \text{ and}\\
&v(N) = 10.
\end{aligned}$$

Let us compute the I-restricted game v^I following its definition.

$$\begin{aligned}
&v^I(i) = 0 \quad \text{for every } i \in N,\\
&v^I(\{1,2\}) = v(1) + v(2) = 0,\\
&v^I(\{1,3\}) = v(\{1,3\}) = 1,\\
&v^I(\{2,3\}) = v(\{2,3\}) = 2, \text{ and}\\
&v^I(N) = \max_{P \in P(N,I)} \sum_{T \in P} v(T) = v(\{2,3\}) + v(1) = 2.
\end{aligned}$$

Next, let us consider the dual graph of I given by $I^* = \{(1 : 3), (2 : 3)\}$. Then, the communication game $(N, v^{I^*}) \in \mathcal{G}$ is given by,

$$\begin{aligned}
&v^{I^*}(i) = 0 \quad \text{for every } i \in N,\\
&v^{I^*}(\{1,2\}) = v(1) + v(2) = 0,\\
&v^{I^*}(\{1,3\}) = v(\{1,3\}) = 1,\\
&v^{I^*}(\{2,3\}) = v(\{2,3\}) = 2, \text{ and}\\
&v^{I^*}(N) = \sum_{S \in N/I^*} v(S) = v(\{1,2,3\}) = 10.
\end{aligned}$$

Hence, in the I-restricted game the grand coalition is not feasible since it contains incompatible players. Nevertheless, the grand coalition N can cooperate jointly in the communication game (N, v^{I^*}) since players 1 and 2 can communicate through player 3. $\Diamond$

After having introduced the model of games with incompatibilities, we come now to the matter of how to allocate the benefits of the cooperation.

A *value on* $\mathcal{GI}$ is a map, f, that assigns a vector $\mathsf{f}(N, v, I) \in \mathbb{R}^N$ to every game with incompatibilities $(N, v, I) \in \mathcal{GI}$. In Bergantiños (1993) a generalization of the Shapley value for this kind of games is proposed and characterized. As in the case of the Myerson value, that is just the Shapley value of the communication game, the incompatibility Shapley value is the Shapley value of the I-restricted game. Its formal definition is given next.

Definition 5.1.3. (Bergantiños 1993). The *incompatibility Shapley value*, Sh^I, is the value on $\mathcal{GI}$ defined for every $(N, v, I) \in \mathcal{GI}$ by

$$\mathsf{Sh}^\mathsf{I}(N, v, I) = \mathsf{Sh}(N, v^I).$$

Example 5.1.4. Consider the game with incompatibilities presented in Example 5.1.2. We saw that the I-restricted game, (N, v^I), differs from the communication game of the dual graph, (N, v^{I^*}). Next, we show that the incompatibility Shapley value of $(N, v, I) \in \mathcal{GI}$ is also different from the Myerson value of $(N, v, I^*) \in \mathcal{GC}$.

$$\mathsf{Sh}^\mathsf{I}(N, v, I) = (1/6, 4/6, 7/6), \quad \mathsf{Sh}^\mathsf{C}(N, v, I^*) = (17/6, 20/6, 23/6). \qquad \Diamond$$

The incompatibility Shapley value is characterized in Bergantiños (1993) and more recently in Alonso-Meijide & Casas-Méndez (2007). In order to present such characterizations we first introduce the following properties.

$\mathcal{GI}$:CEF A value on $\mathcal{GI}$, f, satisfies *component efficiency* if for every $S \in N/I^*$,

$$\sum_{i \in S} \mathsf{f}_i(N, v, I) = \max_{P \in P(S,I)} \sum_{T \in P} v(T) = v^I(S).$$

$\mathcal{GI}$:FAI A value on $\mathcal{GI}$, f, satisfies *fairness* if for every pair of incompatible players $i, j \in N$ such that $(i : j) \notin I$,

$$\mathsf{f}_i(N, v, I) - \mathsf{f}_i(N, v, I \cup (i : j)) = \mathsf{f}_j(N, v, I) - \mathsf{f}_j(N, v, I \cup (i : j)).$$

$\mathcal{GI}$:BCG A value on $\mathcal{GI}$, f, satisfies *balanced contributions for the graph* if for every $i, j \in N$,

$$\mathsf{f}_i(N, v, I) - \mathsf{f}_i(N, v, I^{+j}) = \mathsf{f}_j(N, v, I) - \mathsf{f}_j(N, v, I^{+i}).$$

The $\mathcal{GI}$:CEF property is similar to the component efficiency proposed by Myerson (1977) and presented in Section 4.1 ($\mathcal{GC}$:CEF). It states that the joint payoff of the players of a connected component in the dual graph equals the worth of that component. The idea behind the $\mathcal{GI}$:FAI property keeps a close relation with $\mathcal{GC}$:FAI (Section 4.1). It is a reciprocity property that states that if two players are not incompatible anymore, both gain or loss the same amount. Finally, the $\mathcal{GI}$:BCO property is in line with the balanced contributions in the graph property defined in Section 4.1 ($\mathcal{GC}$:BCG). It is a strengthening of the $\mathcal{GI}$:FAI property and states that if a player i becomes incompatible with the rest of the players this benefits or damages another player j in the same amount as if it happens the other way around.

In the next theorem the first characterization of the incompatibility Shapley value is presented.

Theorem 5.1.5. (Bergantiños 1993). *The incompatibility Shapley value,* Sh^{I}*, is the unique value on* $\mathcal{GI}$ *satisfying* $\mathcal{GI}$:CEF *and* $\mathcal{GI}$:FAI.

Theorem 5.1.5 above reveals that, as in the case of the Myerson value Sh^{C}, the incompatibility Shapley value is characterized by means of only two properties. More recently, Alonso-Meijide & Casas-Méndez (2007) present an alternative characterization of Sh^{I}, using the $\mathcal{GI}$:BCO property. Hence, these two characterizations are the counterparts of Theorems 4.1.4 and 4.1.5 in the context of games with incompatibilities.

Theorem 5.1.6. (Alonso-Meijide & Casas-Méndez 2007). *The incompatibility Shapley value,* Sh^{I}*, is the unique value on* $\mathcal{GI}$ *satisfying* $\mathcal{GI}$:CEF *and* $\mathcal{GI}$:BCG.

5.2 The incompatibility Banzhaf value

In this section we introduce and characterize a new value on $\mathcal{GI}$ which generalizes the Banzhaf value. This is done using the I-restricted game.

Definition 5.2.1. The *incompatibility Banzhaf value*, Ba^{I}, is the value on $\mathcal{GI}$ defined for every $(N, v, I) \in \mathcal{GI}$ by

$$\mathsf{Ba}^{\mathsf{I}}(N, v, I) = \mathsf{Ba}(N, v^I).$$

First of all, we come to the discussion on the properties that this new value on $\mathcal{GI}$ satisfies. To do so, we first define another property, which is the natural modification of $\mathcal{G}$:TPP (see Section 1.1.2) for this scenario.

$\mathcal{GI}$:CTP A value on $\mathcal{GI}$, f, satisfies the *component total power property* if for every $S \in N/I^*$,

$$\sum_{i \in S} \mathsf{f}_i(N, v, I) = \frac{1}{2^{s-1}} \sum_{i \in S} \sum_{L \subseteq S \setminus i} \left[v^I(L \cup i) - v^I(L) \right].$$

The $\mathcal{GI}$:CTP determines the amount that the players in a connected component of the dual graph obtain jointly. Next, we check that the incompatibility Banzhaf value satisfies all the properties introduced in this chapter so far but $\mathcal{GI}$:CEF.

Lemma 5.2.2. *The incompatibility Banzhaf value satisfies $\mathcal{GI}$:FAI, $\mathcal{GI}$:BCG, and $\mathcal{GC}$:CTP.*

Proof. We check that Ba^{I} satisfies the three properties.

$\mathcal{GI}$:FAI. Let $(N, v, I) \in \mathcal{GI}$ and $i, j \in N$ such that $(i : j) \notin I$, then

$$\begin{aligned} 2^{n-1} &\left[\mathsf{Ba}^{\mathsf{I}}_i(N, v, I) - \mathsf{Ba}^{\mathsf{I}}_i(N, v, I \cup (i : j)) \right] \\ &= \sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^I(S \cup j) + v^I(S \cup i) - v^I(S) \right] \\ &- \sum_{S \subseteq N \setminus \{i,j\}} \left[v^{I \cup (i:j)}(S \cup i \cup j) - v^{I \cup (i:j)}(S \cup j) + v^{I \cup (i:j)}(S \cup i) - v^{I \cup (i:j)}(S) \right]. \end{aligned}$$

Since for every $S \subseteq N \setminus \{i, j\}$,

$$v^I(S) = v^{I \cup (i:j)}(S),\ v^I(S \cup i) = v^{I \cup (i:j)}(S \cup i), \text{ and } v^I(S \cup j) = v^{I \cup (i:j)}(S \cup j).$$

$$\begin{aligned} \mathsf{Ba}^{\mathsf{I}}_i(N, v, I) - \mathsf{Ba}^{\mathsf{I}}_i(N, v, I \cup (i : j)) &= \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^{I \cup (i:j)}(S \cup i \cup j) \right] \\ &= \mathsf{Ba}^{\mathsf{I}}{}_j(N, v, I) - \mathsf{Ba}^{\mathsf{I}}{}_j(N, v, I \cup (i : j)). \end{aligned}$$

$\mathcal{GI}$:BCG. Let $(N, v, I) \in \mathcal{GI}$ and $i, j \in N$, then

$$\begin{aligned} 2^{n-1} &\left[\mathsf{Ba}^{\mathsf{I}}_i(N, v, I) - \mathsf{Ba}^{\mathsf{I}}_i(N, v, I^{+j}) \right] \\ &= \sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^I(S \cup j) + v^I(S \cup i) - v^I(S) \right] \\ &- \sum_{S \subseteq N \setminus \{i,j\}} \left[v^{I^{+j}}(S \cup i \cup j) - v^{I^{+j}}(S \cup j) + v^{I^{+j}}(S \cup i) - v^{I^{+j}}(S) \right]. \end{aligned}$$

Since for all $S \subseteq N \setminus j$,

$$v^I(S) = v^{I^{+j}}(S) \text{ and } v^{I^{+j}}(S \cup j) = v^I(S) + v(j).$$

$$\begin{aligned}
&\mathsf{Ba}^{\mathsf{I}}_i(N, v, I) - \mathsf{Ba}^{\mathsf{I}}_i(N, v, I^{+j}) \\
&\qquad = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^I(S \cup i) - v^I(S \cup j) + v^I(S) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \mathsf{Ba}^{\mathsf{I}}_j(N, v, I) - \mathsf{Ba}^{\mathsf{I}}_j(N, v, I^{+i}).
\end{aligned}$$

$\mathcal{GI}$:CTP. Let $(N, v, I) \in \mathcal{GI}$, $S \in N/I^*$ and take $(N, v^{I,S}) \in \mathcal{G}$ defined for every $T \subseteq N$ as follows:

$$v^{I,S}(T) = \max_{P \in P(T \cap S, I)} \sum_{L \in P} v(L).$$

First of all we will see that $v^I = \sum_{S \in N/I^*} v^{I,S}$.

Let $P \in P(T, I)$ and $L \in P$. As L is I-admissible it follows that L is a connected component of I^* on N. Then, there exists $S' \in N/I^*$ such that $L \subseteq S'$. Hence, for every $S \in N/I^*$, each partition $P \in P(T, I)$ induces another partition $P \in P(T \cap S, I)$. Taking into account the definition of the I-restricted game we conclude that for every $T \subseteq N$,

$$v^I(T) \leq \sum_{S \in N/I^*} v^{I,S}(T).$$

On the other hand, let $T \subseteq N$. If $N/I^* = \{S_1, \ldots, S_m\}$, for every $j \in \{1, \ldots, m\}$ we may take $P_j \in P(T \cap S_j, I)$ and a partition P of T defined by those P_j's. As $P \in P(T, I)$, we conclude that for every $T \subseteq N$,

$$v^I(T) \geq \sum_{S \in N/I^*} v^{I,S}(T).$$

Then, the stated equality is proved. Given $S \in N/I^*$ and using the additivity of the Banzhaf value ($\mathcal{G}$:ADD) we have

$$\sum_{j \in S} \mathsf{Ba}_j(N, v^I) = \sum_{j \in S} \sum_{T \in N/I^*} \mathsf{Ba}_j(N, v^{I,T}) = \sum_{T \in N/I^*} \sum_{j \in S} \mathsf{Ba}_j(N, v^{I,T}).$$

For every $i \in N \setminus T$, i is a null player in $(N, v^{I,T}) \in \mathcal{G}$. Hence,

$$\sum_{T \in N/I^*} \sum_{j \in S} \mathsf{Ba}_j(N, v^{I,T}) = \sum_{j \in S} \mathsf{Ba}_j(N, v^{I,S}) = \sum_{j \in S} \mathsf{Ba}_j(S, v^{I,S}).$$

Lastly, using that the Banzhaf value satisfies the total power property ($\mathcal{G}$:TPP) we conclude that,

$$\sum_{j \in S} \mathsf{Ba}_j(S, v^{I,S}) = \frac{1}{2^{s-1}} \sum_{j \in S} \sum_{L \subseteq S \setminus j} \left[v^I(L \cup j) - v^I(L) \right].$$

$\square$

At this point we have defined the concepts and results needed to characterize the incompatibility Banzhaf value.

Theorem 5.2.3. *The incompatibility Banzhaf value,* Ba^{I}, *is the unique value on* $\mathcal{GI}$ *satisfying* $\mathcal{GI}$:BCG *and* $\mathcal{GI}$:CTP.

Proof.

(1) *Existence.* It is shown in Lemma 5.2.2.

(2) *Uniqueness.* Let f be a value on $\mathcal{GI}$ satisfying the properties. Let also $(N, v, \emptyset^*) \in \mathcal{GI}$, then, $N/I^* = \{\{1\}, \{2\}, \dots, \{n\}\}$. By $\mathcal{GI}$:CTP we have

$$\mathsf{f}(N, v, I) = (v(1), \dots, v(n)),$$

and, hence, f is unique. Suppose that there are two different values on $\mathcal{GI}$, namely f^1 and f^2, satisfying the properties. Then there exists $(N, v, I) \in \mathcal{GI}$ such that $\mathsf{f}^1(N, v, I) \neq \mathsf{f}^2(N, v, I)$ and $I \neq \emptyset^*$. Hence, we can take $I \in \mathcal{C}^N$ with the maximum number of links for which the inequality above holds. Thus, there is $i \in N$ such that $\mathsf{f}^1_i(N, v, I) \neq \mathsf{f}^2_i(N, v, I)$.

If for every $j \in N \setminus i$, $(i : j) \in I$, then $\{i\} \in N/I^*$. Applying $\mathcal{GI}$:CTP we come to a contradiction.

If there is $j \in N \setminus i$ such that $(i : j) \notin I$, then by $\mathcal{GI}$:BCG, the fact that $|I^{+i}| \geq |I|$, and the maximality of I,

$$\begin{aligned} \mathsf{f}^1_i(N, v, I) - \mathsf{f}^1_j(N, v, I) &= \mathsf{f}^1_i(N, v, I^{+j}) - \mathsf{f}^1_j(N, v, I^{+i}) = \\ &= \mathsf{f}^2_i(N, v, I^{+j}) - \mathsf{f}^2_j(N, v, I^{+i}) = \mathsf{f}^2_i(N, v, I) - \mathsf{f}^2_j(N, v, I). \end{aligned} \tag{5.1}$$

Moreover, let $S \in N/I^*$ be such that $i \in S$, then for each $j \in S \setminus i$ either $(i : j) \notin I$ or there are $\{i_1, i_2, \dots, i_k\} \subseteq S$ such that $i = i_1$, $j = i_k$, and $(i_l : i_{l+1}) \notin I$ for every $l = 1, \dots, k-1$. Hence by Eq. (5.1),

$$\begin{aligned} \mathsf{f}^1_{i_1}(N, v, I) - \mathsf{f}^1_{i_2}(N, v, I) &= \mathsf{f}^2_{i_1}(N, v, I) - \mathsf{f}^2_{i_2}(N, v, I) \\ &\vdots \\ \mathsf{f}^1_{i_{k-1}}(N, v, I) - \mathsf{f}^1_{i_k}(N, v, I) &= \mathsf{f}^2_{i_{k-1}}(N, v, I) - \mathsf{f}^2_{i_k}(N, v, I). \end{aligned}$$

Adding up both sides,

$$\mathsf{f}_i^1(N, v, I) - \mathsf{f}_j^1(N, v, I) = \mathsf{f}_i^2(N, v, I) - \mathsf{f}_j^2(N, v, I). \tag{5.2}$$

On the other hand, using $\mathcal{GI}$:CTP,

$$\sum_{i \in S} \mathsf{f}_i^1(N, v, I) = \frac{1}{2^{s-1}} \sum_{i \in S} \sum_{T \subseteq S \setminus i} \left[v^I(T \cup i) - v^I(T) \right] = \sum_{i \in S} \mathsf{f}_i^2(N, v, I). \tag{5.3}$$

By Eq. (5.2) and (5.3) it follows that

$$s \cdot \mathsf{f}_i^1(N, v, I) = s \cdot \mathsf{f}_i^2(N, v, I).$$

Hence, we come to contradiction and the result is proved. □

In a similar way, we can obtain a characterization of the incompatibility Banzhaf value by means of $\mathcal{GI}$:FAG and $\mathcal{GI}$:CTP.

Theorem 5.2.4. *The incompatibility Banzhaf value,* Ba^I*, is the unique value on* $\mathcal{GI}$ *satisfying* $\mathcal{GI}$:FAI *and* $\mathcal{GI}$:CTP.

Proof.

(1) *Existence.* It is shown in Lemma 5.2.2.

(2) *Uniqueness.* It can be easily proved following the lines of the proof of the uniqueness in Theorem 5.2.3. Note that we can use $\mathcal{GI}$:FAI in all the steps in which $\mathcal{GI}$:BCG is used. □

The parallel characterizations of the incompatibility Shapley and Banzhaf values are summarized in Table 5.1. The difference between Sh^I and Ba^I lies on the fact that the former satisfies $\mathcal{GI}$:CEF while the latter satisfies $\mathcal{GI}$:CTP. Hence, the differences between Sh and Ba are transferred to situations with incompatibilities. Finally, we show that the pairs of properties used in both characterizations are indeed independent.

	Sh^I	Ba^I	
Bergantiños (1993)	$\mathcal{GC}$:CEF $\mathcal{GC}$:FAI	$\mathcal{GC}$:CTP $\mathcal{GC}$:FAI	Alonso-Meijide et al. (2009b)
Alonso-Meijide et al. (2007)	$\mathcal{GC}$:CEF $\mathcal{G}$:BCG	$\mathcal{GC}$:CTP $\mathcal{G}$:BCG	Alonso-Meijide et al. (2009b)

Table 5.1: Parallel characterizations of Sh^I and Ba^I

Remark 5.2.5. The properties considered in Theorem 5.2.3 are independent as the following examples show:

(i) The incompatibility Shapley value satisfies $\mathcal{GI}$:BCG but not $\mathcal{GI}$:CTP.

(ii) Let a and b be two fixed and different players. Define the value on $\mathcal{GI}$, g, for every $(N, v, I) \in \mathcal{GI}$ as follows:

- If $N = \{a, b\}$ and $I = \emptyset$,

$$\mathrm{g}_a(N, v, I) = g_b(N, v, I) = \frac{v(N)}{2}.$$

- Otherwise, $\mathrm{g}(N, v, I) = \mathsf{Ba}^{\mathsf{I}}(N, v, I)$.

Then, g satisfies $\mathcal{GI}$:CTP but not $\mathcal{GI}$:BCG. ◁

Remark 5.2.6. The properties considered in Theorem 5.2.4 are independent as the following examples show:

(i) The incompatibility Shapley value satisfies $\mathcal{GI}$:FAI but not $\mathcal{GI}$:CTP.

(ii) The value on $\mathcal{GI}$, g, defined above satisfies $\mathcal{GI}$:CTP but not $\mathcal{GI}$:FAI. ◁

5.3 A political example

In this section we study the power distribution in the Parliament of the Basque Country, illustrating in this way the use of the different solution concepts studied in this chapter. However, this time we consider the situation in the Parliament after the elections in November 1986, because it has been studied before in Carreras & Owen (1996) using the Shapley value and its extension to $\mathcal{GI}$ and in Alonso-Meijide & Casas-Méndez (2007) using the Public good index (see Section 6.1) and its extension to $\mathcal{GI}$.

The Parliament of the Basque Country, one of Spain's seventeen regions, is constituted by 75 members. Since most decisions are taken by majority, the characteristic function of the game played by the parties with parliamentary representation is as follows, unity for any coalition adding up to 38 or more members, and zero for the rest. Since elections in 1986, the Parliament was composed by 19 members of the Spanish socialist party PSE, 17 members of the Basque nationalist conservative party PNV, 13 members of the Basque nationalist social democrat party EA, 13 members of the Basque nationalist left-wing party HB, 9 members of the Basque nationalist moderated left-wing party EE, 2 members of the Spanish conservative party CP, and 2 members of the Spanish centrist party CDS. In the papers mentioned above, taking into account the behavior of the parties and the declarations made by the representatives of the parties involved it is assumed the incompatibility graph is as described in Figure 5.1.

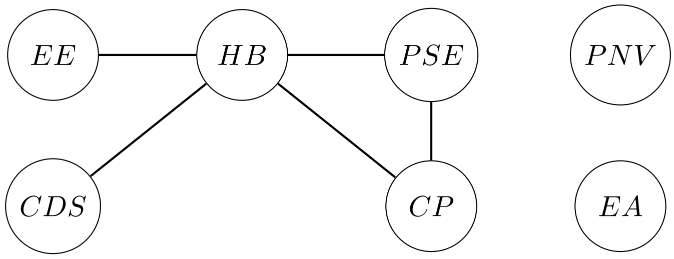

Figure 5.1: The incompatibility graph

For a more detailed description of each party and their political positions the reader is referenced to Carreras & Owen (1996).

In the original simple game there are twelve possible minimal winning coalitions, which are the following:

{PSE, PNV, EA}	{PSE, PNV, CDS}	{PNV, EA, HB}
{PSE, PNV, HB}	{PSE, EA, HB}	{PNV, EA, EE}
{PSE, PNV, EE}	{PSE, EA, EE}	{PNV, HB, EE}
{PSE, PNV, CP}	{PSE, HB, EE}	{EA, HB, EE, CP, CDS}

However when we consider the I-restricted game with the incompatibility graph only six minimal winning coalitions are feasible, which are:

{PSE, PNV, EA}	{PSE, EA, EE}	{PSE, PNV, CDS}
{PSE, PNV, EE}	{PNV, EA, HB}	{PNV, EA, EE}

In Table 5.2, we present the Banzhaf value and the incompatibility Banzhaf value for games with incompatibilities.

Party	Seats	Ba	Ba^{I}
PSE	19	.4688	.3750
PNV	17	.4688	.5000
EA	13	.2812	.3750
HB	13	.2812	.0625
EE	9	.2812	.2500
CP	2	.0312	.0000
CDS	2	.0312	.0625

Table 5.2: The Banzhaf and Incompatibility Banzhaf values

The depicted results are in line with those presented in Carreras & Owen (1996) and Alonso-Meijide & Casas-Méndez (2007). PNV ranks first, even though PSE has more seats. PNV and EA are the only parties which increase their power significantly. CP becomes a null player because his rejection to PSE and HB. In conclusion, when incompatibilities among players are considered the power distribution changes significantly.

6

Two new power indices based on null player free winning coalitions

In the last decades the measurement of power in decision making bodies has been a main topic in Political Sciences and many work has been done in order to attain an appropriate such a measure. However there is still a debate even on the definition of power. Most of the times, the power is understood as the ability of an agent to influence the outcome. But even when the definition of power is agreed the choice of an appropriate rule to represent it is still an open question. The modeling of decision making bodies and voting procedures has been tackled using simple games. In this chapter the simple games that were formally introduced in Section 1.2 are taken up again.

Among the most studied power indices in the literature one can find the Shapley-Shubik and Penrose-Banzhaf-Coleman indices (introduced back in Section 1.2) but also the Deegan-Packel (Deegan & Packel 1979) and the Public good indices (Holler 1982). The above mentioned power indices are evaluations of an agent's relative significance to each of the coalitions that might be formed. In this chapter, first of all we review some of the main results related to the Deegan-Packel and Public good indices. These power indices restrict their attention to some kind of coalitions that are particularly important. Indeed, they only take into account the minimal winning coalitions. More recently, other interesting power indices have been introduced. The Public help index (Bertini et al. 2008) is based on the set of all winning coalitions, more precisely, the power of each agent is proportional to the number of winning coalitions in which she participates. The Shift power index (Alonso-Meijide & Freixas 2010) considers only a subset of the minimal winning coalitions, the so-called minimal winning coalitions without any surplus, in a sense, these coalitions are the most efficient minimal winning coalitions. In this chapter we propose and characterize two new power indices, namely f^{np} and g^{np}. These power indices are also based on a particularly important set of coalitions, specifically on the winning coalitions that do not contain null players. Such set of coalitions contains the set of minimal winning coalitions and is contained in the set of winning coalitions. A first consequence of this fact is that f^{np} and g^{np} do not consider minimal winning coalitions as the only source of power. This is the case in many real situations, for instance many times the adopted decisions are more stable the greater the winning coalition supporting it is.

The results contained in this chapter are a joint work with my supervisor José M. Alonso-Meijide on the one hand, and professors Alberto Pinto from the University of Porto and Flávio Ferreira from the Polytechnic Institute of Porto, both in Portugal, on the other hand. The results contained in this chapter have already been published in the Journal of Difference Equations and Applications (Alonso-Meijide et al. 2011a) and as a chapter in the book Dynamics, Games, and Science II (Alonso-Meijide et al. 2011b). Moreover, another article has been recently submitted for its possible publication in an international journal (Álvarez-Mozos et al. submitted). The rest of the chapter is organized as follows. In Section 6.1, the Deegan-Packel and Public good power indices are presented together with the existing characterizations of them. Next, in Section 6.2 the main contributions are presented, that is, the new power indices f^{np} and g^{np} are defined and characterized by means of properties which are either standard in the literature or modifications of standard properties. Finally, in Section 6.3 the distribution of power in the IX term of office of the Portuguese Parliament is studied.

6.1 Power indices based on minimal winning coalitions

In this section the literature dealing with power indices based on minimal winning coalitions is summarized. In pursuing this objective, two power indices based on minimal winning coalitions are formally introduced and two parallel characterizations of them presented.

The Deegan-Packel power index (Deegan & Packel 1979) is based on the idea that when it comes to measure the power of an agent it should only be considered her participation in minimal winning coalitions. Moreover, it assumes the following three facts:

- Only minimal winning coalitions will emerge victorious.
- Each minimal winning coalition has an equal probability of forming.
- Players in minimal winning coalitions divide the spoils equally.

The conditions above seem reasonable in many situations. The first condition is a consequence of considering that players are rational in the sense that they seek for maximizing their power and, hence, they will only participate in minimal winning coalitions. In other words, if a winning coalition is not a minimal winning coalition it means that there is at least a player whose participation in the coalition is not needed. Hence, the remaining players will prefer to form the minimal winning coalition contained on the winning coalition since there will be less people to share the spoils with. The second condition states that all minimal winning coalitions are equally likely, which is very reasonable once the first condition is accepted. The last condition is a solidarity or equal treatment property. The requirements above lead to the following definition. Some of the concepts used throughout this chapter were defined in Section 1.2 (from page 15 to page 20).

Definition 6.1.1. (Deegan & Packel 1979). The *Deegan-Packel power index*, DP, is the power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$, by

$$\mathsf{DP}_i(N, v) = \frac{1}{|W^m(v)|} \sum_{S \in W_i^m(v)} \frac{1}{|S|}.$$

Deegan & Packel (1979) introduce the DP power index together with a probabilistic interpretation and a characterization by means of four properties. The characterization of SS presented in Section 1.2.1 shares three of them, namely, $\mathcal{SG}$:EFF, $\mathcal{SG}$:SYM, and $\mathcal{SG}$:NPP. Indeed, DP coincides with SS in the class of unanimity games. However, the DP power index does not satisfy $\mathcal{SG}$:TRP. Instead, it satisfied the so-called DP-mergeability property that is introduced next.

Two simple games (N, v) and (N, w) are *mergeable* if for all pairs of minimal winning coalitions $S \in W^m(v)$ and $T \in W^m(w)$, it holds that $S \not\subset T$ and $T \not\subset S$. If two games (N, v) and (N, w) are mergeable, the minimal winning coalitions in the maximum game $(N, v \vee w)$ are precisely by the union of the minimal winning coalitions in the two original games (N, v) and (N, w). Hence, the mergeability condition guarantees that $|W^m(v \vee w)| = |W^m(v)| + |W^m(w)|$. Recall from Section 1.1.2 (page 9) the definition of a *merged or maximum game*, $(N, v \vee w)$, given for every $S \subseteq N$ by, $(v \vee w)(S) = \max\{v(S), w(S)\}$.

$\mathcal{SG}$:DP-MER A power index f satisfies *DP-mergeability* if for every pair of mergeable simple games $(N, v), (N, w) \in \mathcal{SG}$,

$$\mathsf{f}(N, v \vee w) = \frac{\mathsf{f}(N, v)|W^m(v)| + \mathsf{f}(N, w)|W^m(w)|}{|W^m(v \vee w)|}.$$

The property above states that the power in a merged game is a weighted mean of the power in the two component games. The weights being the number of minimal winning coalitions in each component game, divided by the number of minimal winning coalitions in the merged game. Hence, it agrees with $\mathcal{SG}$:TRP in the sense that it assesses the power in a merged game in terms of the power in the two component games. At this point, the properties considered in the first characterization of the Deegan-Packel index have been introduced.

Theorem 6.1.2. (Deegan & Packel 1979) *The Deegan-Packel index,* DP*, is the unique power index satisfying* $\mathcal{SG}$:EFF*,* $\mathcal{SG}$:SYM*,* $\mathcal{SG}$:NPP*, and* $\mathcal{SG}$:DP-MER*.*

More recently, Lorenzo-Freire et al. (2007) propose a different characterization of the Deegan-Packel index. This characterization uses the so-called DP-minimal monotonicity property which is based on the strong monotonicity property (see $\mathcal{G}$:SMO in Section 1.1.2). The property is formally introduced next.

$\mathcal{SG}$:DP-MM A power index f satisfies *DP-minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v)|W^m(v)| \leq \mathsf{f}_i(N, w)|W^m(w)|.$$

The $\mathcal{SG}$:DP-MM property follows the same spirit as the $\mathcal{G}$:SMO property (see Section 1.1.2) used by Young (1985) to characterize the Shapley value. Indeed, $\mathcal{SG}$:DP-MM describes the behavior of a value in two simple games, (N, v) and (N, w), in which there is a player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$, in other words, $v(S \cup i) - v(S) \leq w(S \cup i) - w(S)$ for every $S \subseteq N \setminus i$. The difference between $\mathcal{SG}$:DP-MM and $\mathcal{G}$:SMO lies on the relation between the payoffs of player i in both games. The $\mathcal{G}$:SMO property states that player i's payoff in (N, w) is at least as big as in (N, v). Instead, $\mathcal{SG}$:DP-MM property states that the relation holds after multiplying the payoffs by the number of minimal winning coalitions. In other words, if player i improves her position in a game, her power times the number of minimal winning coalitions in the game increases. The characterization of DP power index proposed in Lorenzo-Freire et al. (2007) replaces the $\mathcal{SG}$:DP-MER property by the $\mathcal{SG}$:DP-MM property.

Theorem 6.1.3. (Lorenzo-Freire et al. 2007) *The Deegan-Packel index,* DP*, is the unique power index satisfying* $\mathcal{SG}$:EFF*,* $\mathcal{SG}$:SYM*,* $\mathcal{SG}$:NPP*, and* $\mathcal{SG}$:DP-MM*.*

In the literature related to power indices, one can find another power index that takes only minimal winning coalitions into account. Indeed, the so-called Public good index proposed in Holler (1982) considers that each player's power is proportional to the amount of minimal winning coalitions in which she participates.

Definition 6.1.4. (Holler 1982). The *Public good index*, PG, is the power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$, by

$$\mathsf{PG}_i(N, v) = \frac{|W_i^m(v)|}{\sum_{j \in N} |W_j^m(v)|}.$$

The first characterization of this power index by means of a set of properties is proposed in Holler & Packel (1983). The characterization follows the spirit of the characterization of the DP index presented in Theorem 6.1.2. Indeed, the property of $\mathcal{SG}$:DP-MER is modified to $\mathcal{SG}$:PG-MER. Next, this modification is formally introduced.

$\mathcal{SG}$:PG-MER A power index f satisfies *PG-mergeability* if for every pair of mergeable simple games $(N, v), (N, w) \in \mathcal{SG}$,

$$\mathsf{f}(N, v \vee w) = \frac{\mathsf{f}(N, v) \sum_{j \in N} |W_j^m(v)| + \mathsf{f}(N, w) \sum_{j \in N} |W_j^m(w)|}{\sum_{j \in N} |W_j^m(v \vee w)|}.$$

Hence, $\mathcal{SG}$:PG-MER describes the power in the merged game as a weighted mean of the power in the two component games as the $\mathcal{SG}$:DP-MER does. However, the weights used in $\mathcal{SG}$:PG-MER differ from the ones used in $\mathcal{SG}$:DP-MER. Note that for every $(N, v) \in \mathcal{SG}$, $\sum_{j \in N} |W_j^m(v)| = \sum_{S \in W^m} |S|$. Next, the counterpart of Theorem 6.1.2 for the PG index is presented.

Theorem 6.1.5. (Holler & Packel 1983) *The Public good index,* PG*, is the unique power index satisfying* $\mathcal{SG}$:EFF, $\mathcal{SG}$:SYM, $\mathcal{SG}$:NPP, *and* $\mathcal{SG}$:PG-MER.

More recently, Alonso-Meijide et al. (2008) propose a different characterization of the Public good index. This characterization is parallel to the one of the Deegan-Packel index presented in Theorem 6.1.3. It is based on the so called PG-minimal monotonicity property which is similar to the DP-minimal monotonicity property stated above. The property is formally introduced next.

$\mathcal{SG}$:PG-MM A power index f satisfies *PG-minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v) \sum_{j \in N} |W_j^m(v)| \leq \mathsf{f}_i(N, w) \sum_{j \in N} |W_j^m(w)|.$$

The $\mathcal{SG}$:PG-MM property keeps a close relation with the $\mathcal{SG}$:PG-MM property. Both properties describe the relation between the power of an agent in two different simple games when the minimal winning coalitions that contain the player in one game are minimal winning coalitions in the other game. The difference lies on the scalars that multiply the power in each of the simple games. Hence, using $\mathcal{SG}$:PG-MM property a counterpart of Theorem 6.1.3 is obtained for the PG index.

Theorem 6.1.6. (Alonso-Meijide et al. 2008) *The Public good index,* PG*, is the unique power index satisfying* $\mathcal{SG}$:EFF*,* $\mathcal{SG}$:SYM*,* $\mathcal{SG}$:NPP*, and* $\mathcal{SG}$:PG-MM*.*

The four characterization results presented in this section are summarized in Table 6.1. These parallel characterizations reveal the differences between DP and PG. First of all, it is worth to mention that both power indices are efficient, symmetric, and satisfy the null player property. These properties are also satisfied by SS while PBC satisfies only $\mathcal{SG}$:SYM and $\mathcal{SG}$:NPP. Hence, the differences among the characterizations of SS, DP, and PG are limited to a single property.

	DP	PG	
Deegan and Packel (1979)	$\mathcal{SG}$:DP-MER $\mathcal{SG}$:EFF $\mathcal{SG}$:SYM $\mathcal{SG}$:NPP	$\mathcal{SG}$:PG-MER $\mathcal{SG}$:EFF $\mathcal{SG}$:SYM $\mathcal{SG}$:NPP	Holler and Packel (1983)
Lorenzo-Freire et al. (2007)	$\mathcal{SG}$:DP-MM $\mathcal{SG}$:EFF $\mathcal{SG}$:SYM $\mathcal{SG}$:NPP	$\mathcal{SG}$:PG-MM $\mathcal{SG}$:EFF $\mathcal{SG}$:SYM $\mathcal{SG}$:NPP	Alonso-Meijide et al. (2008)

Table 6.1: Parallel characterizations of DP and PG

Taking into account the first pair of parallel characterizations (Theorem 6.1.2 and 6.1.5) the difference between DP and PG lies on the mergeability properties ($\mathcal{SG}$:DP-MER and $\mathcal{SG}$:PG-MER). Both properties state that the power in a merged game is a weighted sum of the power in the two component games and they only differ in the weights used. In the case of the $\mathcal{SG}$:DP-MER the weights represent the number of minimal winning coalitions whereas in the case of the $\mathcal{SG}$:PG-MER the weights represent the sum of the cardinalities of all minimal winning coalitions.

If we focus on the second pair of parallel characterizations (Theorem 6.1.3 and 6.1.6) the difference between DP and PG is the type of monotonicity. Moreover, note that since the SS power index is just the Shapley value restricted to simple games and Sh satisfies strong monotonicity (see $\mathcal{G}$:SMO on Section 1.1.2) the differences among SS, DP, and PG are restricted to the type of monotonicity that each of the power indices satisfies. All the monotonicity properties describe the way in which a player's power changes when a simple game is modified leaving this player better off. In such a situation, the SS index of this player increases whereas the DP and PG indices of the player increase but after multiplying them by a proper amount. In the case of DP by the number of minimal winning coalitions and in the case of PG by the sum of the number of minimal winning coalitions that contain each of the players. In conclusion, the differences between DP and PG are rather slight.

6.2 Two new power indices based on null player free winning coalitions

In this section two new power indices are introduced and characterized. These new power indices are based on a particular type of winning coalitions. Next, this new class of winning coalitions is introduced and its relation to $W(v)$ and $W^m(v)$ studied.

A winning coalition $S \in W(v)$ is said to be a *null player free winning coalition* if no null player (see Definition 1.1.2) belongs to S. The set of null player free winning coalitions is denoted by $W^{np}(v)$. As before, for every player $i \in N$, $W_i^{np}(v)$ denotes the set of null player free winning coalitions that contain player i, i.e., $W_i^{np}(v) = \{S \in W^{np}(v) : i \in S\}$. Note that for every $(N, v) \in \mathcal{SG}$, the following relation holds,

$$W^m(v) \subseteq W^{np}(v) \subseteq W(v).$$

Thus, the set of null player free winning coalitions can be seen either as a refinement of the set of winning coalitions or as an extension of the set of minimal winning coalitions.

Note that a simple game is determined by its set of null player free winning coalitions, $W^{np}(v)$. Recall from Section 1.2 that either $W(v)$ or $W^m(v)$ determine the simple game (N, v) and note that the set of winning coalitions can be easily obtained from $W^{np}(v)$, i.e.,

$$W(v) = \{T \subseteq N : \text{there is } S \in W^{np}(v) \text{ such that } S \subseteq T\}.$$

It is also easy to obtain the set of minimal winning coalitions given the set of null player free winning coalitions and vice versa, as follows,

$$W^m(v) = \{T \in W^{np}(v) : \text{for every } S \subsetneq T, \quad S \notin W^{np}(v)\}. \tag{6.1}$$

$$W^{np}(v) = \{S \subseteq N : \text{for every } i \in S, W_i^m(v) \neq \emptyset \text{ and there is } T \in W^m(v) \text{ such that } T \subseteq S\} \tag{6.2}$$

Next, the two new power indices based on null player free winning coalitions are introduced. The new power indices consider that only null player free winning coalitions should be taken into account when it comes to measure the power. The formal definitions are introduced next.

Definition 6.2.1. (Alonso-Meijide et al. 2011a). The f^{np} power index is the power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by

$$\mathsf{f}_i^{np}(N, v) = \frac{1}{|W^{np}(v)|} \sum_{S \in W_i^{np}(v)} \frac{1}{|S|}.$$

Definition 6.2.2. (Alonso-Meijide et al. 2011a). The g^{np} power index is the power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by

$$\mathsf{g}_i^{np}(N, v) = \frac{|W_i^{np}(v)|}{\sum_{j \in N} |W_j^{np}(v)|}.$$

The idea behind the power indices defined above is in line with the definitions of the Deegan-Packel and the Public good indices (see Definitions 6.1.1 and 6.1.4). The only difference is that f^{np} and g^{np} consider all winning coalitions that do not contain null players instead of only considering minimal winning coalitions. Consequently, f^{np} considers that all null player free winning coalitions are equally likely and that the players in a null player free winning coalition divide the spoils equally. The power index g^{np} assumes that the power of each player is proportional to the number of null player free winning coalitions in which she participates.

In order to characterize f^{np} and g^{np} the following monotonicity properties are introduced.

$\mathcal{SG}$:f^{np}-MM A power index f satisfies *f^{np}-minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v)|W^{np}(v)| \leq \mathsf{f}_i(N, w)|W^{np}(w)|.$$

$\mathcal{SG}$:g^{np}-MM A power index f, satisfies g^{np}-*minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v) \sum_{j \in N} |W_j^{np}(v)| \leq \mathsf{f}_i(N, w) \sum_{j \in N} |W_j^{np}(w)|.$$

The $\mathcal{SG}$:f^{np}-MM and $\mathcal{SG}$:g^{np}-MM properties are based on the $\mathcal{SG}$:DP-MM and $\mathcal{SG}$:DP-MM properties used to characterize the Deegan-Packel and Public good indices. The monotonicity properties describe the way in which the payoff of a player changes when the simple game is modified improving this players' possibilities.

Finally, f^{np} and g^{np} are characterized with a set of properties close to the ones used in Theorems 6.1.3 and 6.1.6 to characterize DP and PG, respectively.

Theorem 6.2.3. *The power index* f^{np} *is the unique power index satisfying* $\mathcal{SG}$:EFF, $\mathcal{SG}$:NPP, $\mathcal{SG}$:SYM, *and* $\mathcal{SG}$:f^{np}-MM.

Proof. (1) *Existence.* From Definition 6.2.1 it straightforward to check that f^{np} satisfies $\mathcal{SG}$:EFF, $\mathcal{SG}$:NPP, and $\mathcal{SG}$:SYM. For $\mathcal{SG}$:f^{np}-MM, note that by Eq. (6.2), $W_i^m(v) \subseteq W_i^m(w)$ implies $W_i^{np}(v) \subseteq W_i^{np}(w)$. Then,

$$\begin{aligned}&\mathsf{f}_i^{np}(N, w)\\ &= \frac{1}{|W^{np}(w)|} \sum_{S \in W_i^{np}(w)} \frac{1}{|S|} = \frac{1}{|W^{np}(w)|} \sum_{S \in W_i^{np}(v)} \frac{1}{|S|} + \frac{1}{|W^{np}(w)|} \sum_{S \in W_i^{np}(w) \setminus W_i^{np}(v)} \frac{1}{|S|},\end{aligned}$$

and hence,

$$\begin{aligned}&\mathsf{f}_i^{np}(N, w)|W^{np}(w)|\\ &\quad = \sum_{S \in W_i^{np}(v)} \frac{1}{|S|} + \sum_{S \in W_i^{np}(w) \setminus W_i^{np}(v)} \frac{1}{|S|} \geq \sum_{S \in W_i^{np}(v)} \frac{1}{|S|} = \mathsf{f}_i^{np}(N, v)|W^{np}(v)|.\end{aligned}$$

(2) *Uniqueness.* The uniqueness is proved by induction on the number of minimal winning coalitions. If $|W^m(v)| = 1$, then $v = u_S$ where $S \subseteq N$ is such that $W^m(v) = \{S\}$. If a power index, f satisfies $\mathcal{SG}$:EFF, $\mathcal{SG}$:NPP, and $\mathcal{SG}$:SYM, we have,

$$\mathsf{f}_i(N, v) = \begin{cases} \frac{1}{|S|} & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Hence, the uniqueness holds when $|W^m(v)| = 1$. Next, assume that a power index satisfying the properties is unique for every $(N, v) \in \mathcal{SG}$ with less than $m > 1$ minimal winning coalitions, i.e., f is unique for every $(N, v) \in \mathcal{SG}$ such that $|W^m(v)| < m$. Let $(N, v) \in \mathcal{SG}$ with $W^m(v) = \{S_1, \ldots, S_m\}$. Take $T = \cap_{k=1}^m S_k$. Then, for each $i \notin T$ let us define $(N, w) \in \mathcal{SG}$ by $W^m(w) = W_i^m(v)$. Then, since $W_i^m(v) = W_i^m(w)$, applying $\mathcal{SG}$:f^{np}-MM twice,

$$\mathsf{f}_i(N, v)|W^m(v)| = \mathsf{f}_i(N, w)|W^m(w)|.$$

Finally, note that $|W^m(w)| < m$ and hence by induction the right hand side of the equality above is unique. It remains to prove the uniqueness for $i \in T$. By $\mathcal{SG}$:SYM there is a constant $c \in \mathbb{R}$ such that for every $i \in T$, $\mathsf{f}_i(N, v) = c$. Moreover, by $\mathcal{SG}$:EFF and the uniqueness for every $i \notin T$, c is unique, which concludes the proof. □

Theorem 6.2.4. *The power index* g^{np} *is the unique power index satisfying* $\mathcal{SG}$:EFF, $\mathcal{SG}$:NPP, $\mathcal{SG}$:SYM, *and* $\mathcal{SG}$:g^{np}-MM.

Proof. The proof follows immediately from a reasoning similar to the one used in the proof of Theorem 6.2.3. □

Hence, Theorems 6.2.3 and 6.2.4 show that the differences between f^{np} and g^{np} are restricted to a monotonicity property. Moreover, the only difference among SS, DP, PG, f^{np}, and g^{np} is the type of monotonicity satisfied by each power index. Finally, the parallel characterizations of f^{np} and g^{np} are depicted in Table 6.2.

f^{np}	g^{np}
$\mathcal{SG}$:f^{np}-MER	$\mathcal{SG}$:g^{np}-MER
$\mathcal{SG}$:EFF	$\mathcal{SG}$:EFF
$\mathcal{SG}$:SYM	$\mathcal{SG}$:SYM
$\mathcal{SG}$:NPP	$\mathcal{SG}$:NPP

Table 6.2: Parallel characterizations of f^{np} and g^{np}

6.3 A political example

In this section the power indices considered so far are illustrated by means of a political example following Alonso-Meijide et al. (2011b). The Portuguese Parliament or Assembly of the Republic is constituted by 230 members. Since most of the decisions are taken by simple majority, the characteristic function of the game played by the parties with parliamentary representation is as follows, unity for any coalition adding up to 116 or more votes, and zero for the rest. Since elections in 2002, the Parliament was composed by 105 members of the center to the right liberal conservative party, PPD/PSD, 96 members of the social democratic party, PS, 14 members of the conservative, christian democratic party, CDS/PP, 10 members of the communist party, PCP, 3 members of the left-wing party BE, and 2 members of the green and eco-socialist party PEV.

Party	Seats	SS	PBC	DP	PG	f^{np}	g^{np}
PPD/PSD	105	.4667	.7188	.3333	.3077	.2473	.2288
PS	96	.1833	.2812	.1677	.1538	.1720	.1695
CDS/PP	14	.1833	.2812	.1677	.1538	.1720	.1695
PCP	10	.1333	.2188	.2000	.2308	.1559	.1610
BE	3	.0167	.0312	.0667	.0769	.1263	.1356
PEV	2	.0167	.0312	.0667	.0769	.1263	.1356

Table 6.3: The distribution of power in the IX term of office of Portuguese Parliament

In the simple game described above there are only 5 minimal winning coalition, which are the following:

{PPD/PSD, PS}
{PPD/PSD, CDS/PP}
{PPD/PSD, PCP, BE}
{PPD/PSD, PCP, PEV}
{PS, CDS/PP, PCP}

Note that there are not null players since each party is at least in one minimal winning coalition. Thus, the set of wining coalitions coincides with the set of null player free winning coalitions. There are up to 31 wining coalitions.

First of all, note that there are two pairs of symmetric players in this simple game. The two smallest parties on the one hand, BE and PEV, which have respectively two and three seats in the Parliament. On the other hand, and more surprisingly PS and CDS/PP are symmetric players even though PS has almost seven times more members in the parliament.

Second, note that the power indices considered may be classified in three different groups. The first group consist on SS and PBC and corresponds to the power indices that are based on swings. Note that the difference between the indices of the most and less powerful agents is the highest in this group, more precisely,

$$\mathsf{SS}_{\mathrm{PPD/PS}} - \mathsf{SS}_{\mathrm{PEV}} = 0.45, \qquad \mathsf{PBC}_{\mathrm{PPD/PS}} - \mathsf{PBC}_{\mathrm{PEV}} = 0.6876$$

This difference reveals the sensitivity of the power indices in the first group since the differences among the seats distribution is transferred to the power distribution. Note also that the ranking of the players with respect to their power is maintained by the power indices in this group. Moreover, the order is the one given by the seat distribution. The second group corresponds to the power indices based on minimal winning coalitions, i.e., DP and PG. If the range in which the power of the agents varies is computed as before,

$$\mathsf{DP}_{\mathrm{PPD/PS}} - \mathsf{DP}_{\mathrm{PEV}} = 0.2666, \qquad \mathsf{PG}_{\mathrm{PPD/PS}} - \mathsf{PG}_{\mathrm{PEV}} = 0.2308$$

Observe that the gap is significantly lower for these indices. This is due to the fact that there are smaller differences in the number of minimal winning coalitions in which each player participates. Hence, DP and PG are less sensitive to the differences in the seat distribution. On the other hand, note that the ranking of the players with respect to these two indices is the same, but different from the order with respect to the power indices in the first group. Indeed, PCP ranks second with respect to DP and PG. Many times parties with small representation participate in more minimal winning coalitions than parties with big representation. Finally, f^{np} and g^{np} form the third group of power indices. This group corresponds to the power indices based on null player free winning coalitions. In comparison with the second group of properties one may think that the indices in this group are more sensitive since they are based on a broader set of winning coalitions. However, the difference between the powers of the most and less powerful players is the smallest among the power indices considered here.

$$\mathsf{f}^{np}_{\mathrm{PPD/PS}} - \mathsf{f}^{np}_{\mathrm{PEV}} = 0.1210, \qquad \mathsf{g}^{np}_{\mathrm{PPD/PS}} - \mathsf{g}^{np}_{\mathrm{PEV}} = 0.0932$$

Lastly, note that the ranking of the players with respect to the power indices in this group coincides with the order given by the seat distribution. Hence, the ordering coincides also with the one proposed by SS and PBC.

Conclusions

In this chapter the main conclusions that can be obtained from this dissertation are summarized. Since the thesis is a collection of contributions, we will mention the main conclusions that can be obtained from each of the chapters. The first chapter is preliminary and hence, there is no relevant contribution to comment on.

In Chapter 2 we have studied games with levels structure of cooperation, which constitutes the natural generalization of games with a priori unions. The results obtained in this chapter imply, first of all, that the Banzhaf and Banzhaf-Owen values can be generalized to the framework of games with levels structure of cooperation in a natural way. Second, from the presented characterizations we can conclude that the differences between the Owen and Banzhaf-Owen values are transferred to games with levels structure of cooperation. More precisely, the differences between the Shapley and Banzhaf levels values can be summarized in the following two ideas. The first difference lies on the fact that the Shapley levels value is a sharing rule while the Banzhaf leveles value is not. The Shapley levels value shares the worth of the grand coalition efficiently. Besides, it shares the amount that a union obtains efficiently among the unions of the lower level that are contained on it. That is, it is efficient in each of the steps in which the sharing is carried out. Instead, the Banzhaf levels value generalizes the Banzhaf value and hence, it is not efficient. However, it satisfies the 2-efficiency property that indicates that the Banzhaf levels value is not manipulable against artificial merging or splitting of players. The second difference has to do with the consequences of an agent's isolation from the levels structure. Consider two agents that lie on the same union at every level, then, the fact that the first agent is isolated from the levels structure does not affect the second player's payoff when the Banzhaf levels value is used. Nevertheless, the Shapley levels value is sensitive to such changes, that is the second agent's payoff may change. What happens is that the second agent's payoff is affected in the same amount as the first agent's payoff would be affected if it were the second agent who got isolated from the levels structure.

Chapter 3 has dealt with share functions in the framework of monotone games, monotone games with a priori unions, and monotone games with levels structure of cooperation. In particular, the share functions associated with the values considered in Chapter 2 have been studied. Indeed, in Chapter 3 we have focused on the normalized versions of the values introduced in the previous chapter. The main consequence of the results in Chapter 3 is related to the multiplication property. Roughly speaking, the values that generalize the Shpapley value satisfy this property but the values that generalize the Banzhaf value do not. The multiplication property explains how is the sharing carried out when there is a structure with several levels if we know how to share when there is no level at all. In this chapter we have also introduced efficient generalizations of the Banzhaf value. In this way, if we assume that the sharing should satisfy the multiplication property, we have proposed a generalization of the Banzhaf value to each of the considered frameworks that does so. Finally, it is worth to mention that the approach used in Chapter 3 allows to build new sharing rules in each of the studied contexts in a quite easy way, we only need to take a real valued function on the set of monotone games that satisfies certain properties.

The main conclusion that can be obtained from Chapter 4 is that the restrictions to the cooperation arriving from a communication graph are compatible with the ones arriving from an a priori unions structure. Indeed, we can generalize the three values considered for games with a priori unions to this more general model in a natural way and hence, we can define the Owen, Banzhaf-Owen, and Symmetric coalitional Banzhaf graph values. Besides, the aforementioned generalizations are characterized by means of properties which are similar to the ones used to characterize the Owen, Banzhaf-Owen, and Symmetric coalitional Banzhaf values. In this framework, if we seek for a value that is efficient in each of the two stages in which the sharing is carried out, for each connected coalition in the graph, we should choose the Owen graph value. However, if we want the value not to be affected by possible desertions inside the unions, we should select the Banzhaf-Owen graph value. Finally, if we want a compromise between the two features explained above we should use the Symmetric coalitional Banzhaf graph value.

Taking into account the results presented in Chapter 5, we conclude that the model of games with incompatibilities is not as manageable as the one of games with graph restricted communication. As a consequence, the incompatibility Banzhaf value is not as easy to characterize as the Banzhaf graph value. I have only achieved characterizations of the incompatibility Banzhaf value by means of the total power property and I have not been able to characterize it by means of a 2-efficiency property, as is the case for the Banzhaf and Banzhaf graph values. Therefore, we can conclude that the differences between the incompatibility Shapley and Banzhaf values are restricted to the fact that the former is efficient while the latter is not.

To conclude, in Chapter 6, following the ideas behind the definitions of the Deegan-Packel and Public good indices, we have proposed two new power indices. These new power indices are based on null player free winning coalitions.

The motivation is that many times the minimal winning coalitions are not the only relevant coalitions. The new power indices are characterized by means of three standard properties and another one, which is a modification of the monotonicity property. In this way, we are able to spell out the distinguishing features of the new proposed indices.

Resumen en Castellano

La Teoría de Juegos es una rama de las Matemáticas que estudia modelos para la toma de decisiones en situaciones en las que hay varios agentes implicados y el resultado depende de la elección que realice cada uno de ellos. La importancia de esta disciplina radica principalmente en su aplicación a otros ámbitos, tales como la Economía, las Ciencias Políticas, la Sociología, la Filosofía, o incluso la Biología.

A pesar de que se conoce algún trabajo previo relacionado con la Teoría de Juegos, se puede decir que la Teoría de Juegos nace como disciplina científica en el año 1944 con la publicación del libro "Theory of Games and Economic Behavior", escrito por John von Newmann y Oskar Morgenstern. Posteriormente, en 1950, John Nash define el equilibrio que lleva su nombre que es considerado como uno de los conceptos más importantes dentro de la Teoría de Juegos. Desde ese momento, las contribuciones a la Teoría de Juegos experimentan un aumento considerable.

La importancia de la Teoría de Juegos queda demostrada al haberse otorgado en tres ocasiones el premio Nobel de Economía a teóricos de juegos. En el año 1994, a John Harsanyi, John Nash y Reinhard Selten, en el 2005, a Robert Aumann y Thomas Schelling, y en el 2007 a Leonid Hurwicz, Eric Maskin y Roger Myerson.

La Teoría de Juegos se divide en dos importantes áreas: los juegos no cooperativos y los juegos cooperativos. En el caso de los juegos no cooperativos, un juego es un modelo que describe todos los posibles movimientos de los jugadores. La investigación en este área trata fundamentalmente de encontrar las "mejores" estrategias que cada agente puede seguir desde un punto de vista egoísta. En cambio, en el caso de los juegos cooperativos, se asume que se puede llegar a acuerdos vinculantes entre jugadores y se describen únicamente los resultados que se obtienen en todas las posibles coaliciones de jugadores. Las principales líneas de investigación en este ámbito estudian reglas "justas" que puedan servir para repartir los beneficios generados por la cooperación

En esta tesis se realizan aportaciones al estudio de algunas clases de juegos cooperativos. Concretamente, se definen y caracterizan nuevos conceptos de solución para estas clases de juegos. También se presentan nuevas caracterizaciones de conceptos de solución existentes en la literatura científica. Este tipo de caracterizaciones constituyen una de las líneas de investigación más activas dentro de la Teoría de Juegos. De hecho, la caracterización de soluciones por medio de propiedades es una forma de resumir las principales características de cada concepto de solución. Las caracterizaciones que se presentan en esta memoria pueden considerarse paralelas ya que en un mismo contexto presentamos caracterizaciones de distintos conceptos de solución basadas en propiedades fácilmente comparables. Este tipo de caracterizaciones ayudan a comparar las distintas soluciones debido a que resaltan las similitudes y diferencias existentes, lo que es especialmente interesante cuando tratamos de aplicar alguna solución en un problema concreto.

En el Capítulo 1 se presenta el modelo básico de juegos cooperativos que servirá de base para los modelos estudiados en los siguientes capítulos. La mayor parte de esta tesis estudia juegos cooperativos en los que la cooperación está restringida. Existen muchas situaciones en las que no se puede aceptar que cualquier coalición de jugadores sea factible. Esto puede deberse a restricciones del entorno o a las preferencias de los individuos involucrados. Los modelos de juegos cooperativos con cooperación restringida intentan incorporar esta información adicional al modelo tratando así de ajustarse mejor a la situación que se quiere modelar. Los modelos considerados entre los Capítulos 2 y 5 son ejemplos de juegos con cooperación restringida. En el 2 se estudian los juegos con estructura de niveles. El 3 se centra en el estudio de las denominadas "share functions" en distintos contextos de juegos con cooperación restringida. En el 4 se consideran los juegos con comunicación restringida y uniones a priori. El 5 trata de juegos con incompatibilidades. Finalmente, el Capítulo 6 está dedicado al estudio de los juegos simples sin ningún tipo de restricción a la cooperación.

Resumen del Capítulo 1

En este capítulo se introduce el modelo básico de juegos cooperativos con utilidad transferible y se hace una revisión de algunos de los principales resultados existentes hasta el momento. Después de introducir el modelo formalmente, se enumeran ciertas propiedades que un juego cooperativo puede verificar. A continuación, se discuten los distintos enfoques existentes en la literatura en cuanto a conceptos de solución se refiere. Gran parte de este capítulo se centra en dos de los principales valores o conceptos de solución para juegos cooperativos como son los valores de Shapley (Shapley 1953) y Banzhaf (Owen 1975). En primer lugar se introducen los valores y se describe su interpretación probabilística. A continuación se hace una exhaustiva revisión de las distintas caracterizaciones propuestas hasta el momento. Finalmente, se introducen también los denominados juegos simples que se estudian con más profundidad en el Capítulo 6.

Resumen del Capítulo 2

Una de las primeras propuestas de juegos con cooperación restringida son los juegos con uniones a priori o estructura coalicional que introducen Aumann y Drèze (1974). En un juego con uniones a priori se asume que los jugadores se organizan en grupos y que la cooperación debe "respetar" esta estructura. Es decir, se supone que además de la información contenida en la función característica, se tiene información adicional que en este caso viene dada por medio de una partición del conjunto de jugadores. En primer lugar, se revisan algunos de los resultados más importantes existentes para esta clase de juegos. En concreto, se presentan tres generalizaciones de los valores de Shapley y Banzhaf para esta clase de juegos junto con una caracterización de cada una de estas generalizaciones. En segundo lugar, se introduce el modelo de juegos con estructura de niveles y se revisan los principales resultados existentes. Esta clase de juegos se introduce en Winter (1989), donde se propone y caracteriza una generalización del valor de Shapley: el denominado valor de niveles de Shapley. El valor de niveles de Shapley también generaliza el valor de Owen (Owen 1977) que es uno de los valores más estudiados para juegos cooperativos con uniones a priori. A continuación se presentan las aportaciones de este capítulo que consisten en proponer un nuevo valor que generaliza el valor de Banzhaf a este contexto, denominado valor de niveles de Banzhaf, y dos caracterizaciones, una para cada uno de los valores considerados en este contexto. El valor de niveles de Banzhaf generaliza a su vez el valor de Banzhaf-Owen (Owen 1982). Las caracterizaciones propuestas pueden considerarse paralelas y, por lo tanto, son fácilmente comparables. Las caracterizaciones ayudan a entender las diferencias entre ambos valores. Para terminar, se estudia un ejemplo en el que se pueden aplicar los valores estudiados en este capítulo y se obtienen las correspondientes conclusiones.

Resumen del Capítulo 3

En este capítulo se estudian las denominadas "share functions", que no son más que una forma alternativa de ver los valores, en distintas clases de juegos con cooperación restringida. Estas funciones se introducen en van der Laan y van den Brink (1998) permitiendo así el estudio conjunto de los valores de Shapley y Banzhaf. En este capítulo se estudian "share functions" y sus generalizaciones a los modelos de juegos considerados en el Capítulo 2. Es decir, juegos cooperativos, juegos cooperativos con uniones a priori y juegos cooperativos con estructura de niveles. En cada una de estas clases de juegos se propone una familia de "share functions" de forma que la mayoría de los valores estudiados en el capítulo anterior están incluidos en ellas. Por último, se proponen varias caracterizaciones para cada una de estas familias. De esta forma se completan los resultados de van der Laan y van den Brink (2002) y van den Brink y van der Laan (2005) y se extienden para el caso en el que exista una estructura formada por varios niveles de uniones a priori.

Resumen del Capítulo 4

En este capítulo se considera otro de los modelos de juegos con cooperación restringida que más interés ha despertado entre los teóricos de juegos. Se trata del modelo de juegos con comunicación restringida propuesto por Myerson (1977). Este modelo supone que los jugadores solo pueden cooperar a través de los cauces de comunicación bilateral existentes. Es decir, se supone la existencia de un grafo no dirigido que describe la forma en la que los agentes se comunican y la cooperación solo puede darse entre agentes que se comuniquen, bien directamente, o bien mediante otros agentes que también estén dispuestos a cooperar. El Capítulo 4 comienza repasando algunos de los resultados más importantes que existen para el modelo de juegos con comunicación restringida. En particular, se presentan las generalizaciones de los valores de Shapley y Banzhaf a este contexto junto con varias caracterizaciones que ayudan a identificar las características comunes y diferencias existentes entre ambas. A continuación se introducen los juegos con comunicación restringida y uniones a priori que consideran que la cooperación está restringida tanto por el grafo de comunicación como por la estructura de uniones a priori. Este modelo se propone en Vázquez-Brage et al. (1996). En este trabajo se propone y caracteriza una generalización del valor de Shapley a esta clase de juegos. La principal contribución de este capítulo consiste en proponer dos nuevos valores para juegos con comunicación restringida y uniones a priori y caracterizar estos dos valores y el propuesto por Vázquez-Brage et al. (1996) por medio de propiedades. Las tres caracterizaciones propuestas se basan en propiedades fácilmente comparables por lo que ayudan a identificar las diferencias existentes. El capítulo concluye con un ejemplo que ilustra el uso de los valores considerados. En concreto se estudia la distribución de poder en la VIII legislatura del Parlamento Vasco teniendo en cuenta las afinidades existentes entre los partidos políticos implicados.

Resumen del Capítulo 5

En este capítulo se consideran juegos cooperativos en los que existen jugadores incompatibles. En este caso, las restricciones a la cooperación vienen dadas por un grafo no dirigido en el que cada arco indica que los jugadores situados en cada extremo son incompatibles y, por tanto, no van a poder cooperar. El modelo que consideramos en este capítulo se propone en Carreras (1991) para la clase de juegos simples y se generaliza a la clase general de juegos cooperativos en Bergantiños (1993). En este último trabajo se propone y caracteriza una generalización del valor de Shapley. Este capítulo comienza revisando los principales resultados contenidos en los citados artículos. A continuación se propone y caracteriza una generalización del valor de Banzhaf a este contexto. Este capítulo también concluye estudiando la distribución de poder en el Parlamento Vasco pero en este caso en su III legislatura y teniendo en cuenta las incompatibilidades existentes en aquella época.

Resumen del Capítulo 6

En el Capítulo 6 se estudian juegos simples. Los juegos simples son una subclase importante de los juegos cooperativos. Es decir, en este caso se trabaja con un subconjunto de los juegos cooperativos en lugar de considerar modelos que generalizan los juegos cooperativos. Los juegos simples sirven de modelo para el estudio de órganos de toma de decisiones como pueden ser parlamentos o consejos de dirección. En este contexto, los índices de poder son medidas para cuantificar el poder que cada agente tiene en un órgano de toma de decisiones. El capítulo comienza revisando algunos de los más importantes índices de poder existentes y sus caracterizaciones por medio de propiedades. A continuación se proponen y caracterizan dos nuevos índices de poder. Las caracterizaciones propuestas utilizan propiedades similares a otras que podemos encontrar en la literatura pudiendo así comparar distintos índices de poder basándonos en las propiedades que estos satisfacen. El capítulo concluye ilustrando los distintos índices de poder con un ejemplo. En este caso, se estudia la distribución de poder en la IX legislatura del Parlamento Portugués.

Conclusiones

En el segundo y tercer capítulo de esta tesis hemos estudiado juegos cooperativos con estructura de niveles, que son la generalización natural de los juegos con uniones a priori. Los resultados obtenidos en el Capítulo 2 indican en primer lugar, que los valores de Banzhaf y Banzhaf-Owen pueden extenderse de forma natural a este contexto. En segundo lugar, de las caracterizaciones obtenidas concluimos que las diferencias existentes entre el valor de Owen y el valor de Banzhaf-Owen se transfieren al caso de juegos con niveles. En concreto, las diferencias entre el valor de niveles de Shapley y el valor de niveles de Banzhaf se pueden resumir en dos ideas. La primera es que el valor de niveles de Shapley es una regla de reparto y el valor de niveles de Banzhaf no. El valor de niveles de Shapley divide lo que la gran coalición puede obtener de forma eficiente, más aún, reparte de forma eficiente lo que obtiene cada unión entre las uniones del nivel inferior que la componen. En definitiva, el valor de niveles de Shapley es eficiente en cada etapa. En cambio el valor de niveles de Banzhaf generaliza el valor de Banzhaf y, por tanto, no es eficiente. La contrapartida es que satisface la propiedad de 2-eficiencia que indica que no es manipulable frente a uniones o divisiones artificiales de agentes. La segunda diferencia tiene relación con las consecuencias de que un agente se vea aislado de la estructura de niveles. Imaginemos a dos agentes que están en las mismas uniones de cada nivel, entonces el hecho de que el primer agente sea aislado de la estructura de niveles no afectará el pago del segundo agente si tomamos el valor de niveles de Banzhaf. El valor de niveles de Shapley en cambio sí que es sensible a este tipo de cambios, es decir el pago del segundo agente podría verse afectado. Lo que sucede es que el pago del segundo agente se ve afectado en la misma cantidad en la que se vería afectado el pago del primero si fuera el segundo quien quedase aislado de la estructura.

En el tercer capítulo se han estudiado las "share functions" para juegos con uniones a priori y juegos con estructura de niveles. En particular se han considerado las "share functions" asociadas con todos los valores estudiados en el capítulo anterior, es decir, en este se han estudiado las versiones normalizadas de los valores del Capítulo 2. La principal conclusión del Capítulo 3 tiene que ver con la propiedad de multiplicabilidad. Podríamos decir que los valores que generalizan Shapley sí la verifican mientras que los valores que generalizan Banzhaf no lo hacen. Esta propiedad explica cómo repartir cuando nos enfrentamos a una estructura con varios niveles si sabemos cómo se reparte cuando no hay estructura alguna. En este capítulo también se proponen generalizaciones eficientes del valor de Banzhaf para juegos con uniones a priori y estructura de niveles. De esta forma, si se asume que una regla de reparto debe satisfacer la propiedad de multiplicabilidad, tenemos una generalización de Banzhaf que la verifica para cada uno de los dos contextos estudiados. Finalmente, cabe destacar que el enfoque utilizado en el Capítulo 3 permite construir nuevas reglas de reparto en cada uno de los contextos considerados de forma bastante sencilla, solo hay que seleccionar una función con valores reales sobre el conjunto de juegos monótonos que cumpla ciertas propiedades.

La principal conclusión que podemos extraer del Capítulo 4 es que las restricciones a la cooperación dadas por un grafo de comunicación son compatibles con las dadas por una estructura de uniones a priori. Es decir, podemos extender los tres valores considerados para juegos con uniones a priori a este modelo más general de una forma natural, de esta manera se pueden definir el valor de comunicación de Owen, de Banzhaf-Owen y coalicional simétrico de Banzhaf. Además estas extensiones van a ser caracterizadas por propiedades similares a las que se utilizan para caracterizar los valores de Owen, Banzhaf-Owen y coalicional simétrico de Banzhaf. En este contexto, si se busca un valor que sea eficiente en cada una de las dos etapas en las que se hace el reparto para cada coalición conectada por el grafo, se debería escoger el valor de comunicación de Owen. En cambio si lo que se quiere es que la solución no se vea afectada por posibles deserciones dentro de una unión se debería elegir el valor de comunicación de Banzhaf-Owen. Por último, si se busca un valor que tenga características de los dos valores mencionados se debería considerar el valor de comunicación coalicional simétrico de Banzhaf.

En cuanto al Capítulo 5, se puede decir que el modelo de juegos cooperativos con incompatibilidades no es tan manejable como el modelo de juegos con comunicación restringida. En consecuencia, el valor de Banzhaf con incompatibilidades no se caracteriza tan fácilmente como el valor de Banzhaf o el valor de comunicación de Banzhaf. Solo se han obtenido dos caracterizaciones basadas en el poder total y no se ha podido encontrar ninguna basada en la propiedad de 2-eficiencia que tanto Banzhaf como Banzhaf de comunicación satisfacen. Por lo tanto, se puede concluir que las diferencias entre los valores de Shapley con incompatibilidades y Banzhaf con incompatibilidades se restringen a que el primero es eficiente en cada coalición conectada por el grafo dual mientras que el segundo no.

Para terminar, en el Capítulo 6, siguiendo las ideas de los índices de poder de Deegan-Packel y Public good se han definido dos nuevos índices de poder. Estos índices de poder se basan en las coaliciones ganadoras que no contienen jugadores nulos. Para definirlos se ha tenido en cuenta que existen situaciones en las que las coaliciones minimales no son las únicas que juegan un papel importante. Los nuevos índices de poder propuestos se pueden caracterizar con tres propiedades estandares en la literatura y una más, que es una modificación de la propiedad de monotonía. Lo cual permite desgranar las características diferenciadoras de los nuevos índices de poder propuestos.

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