UNIVERSIDADE DE SANTIAGO DE COMPOSTELA
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Essays on Operations Research Games and Cautious Behavior

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Realizado el acto público de defensa y mantenimiento de esta tesis doctoral el día 2 de febrero de 2007, en la Facultad de Matemáticas de la Universidad de Santiago de Compostela, ante el tribunal formado por:

Presidente: Dr. D. José Manuel Zarzuelo Zarzosa

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siendo directores la Dr.ª D.ª María Gloria Fiestras Janeiro y el Dr. D. Ignacio García Jurado, obtuvo la máxima calificación de SOBRESALIENTE CUM LAUDE. Además, esta tesis ha cumplido los requisitos necesarios para la obtención del DOCTORADO EUROPEO.

A Yoli, mis padres y mi hermano.

Gracias.

Preface

Some years ago, I had to decide which bachelor I should take. The Bachelor of Mathematics was not my first option, but because of several reasons I had to choose it. The first year was not fully pleasant because I had in mind to change my studies. However, during that year, I started to discover the world of Mathematics and decided to continue once year more. Year after year, I was more and more interested by the wonderful universe of Mathematics.

In my fifth Bachelor's year, I followed a course in game theory by Ignacio García Jurado. It was there where my interest in the field of game theory was born. The next year I became a PhD student in the Department of Statistics and Operations Research of the University of Santiago de Compostela. Ignacio García Jurado and Gloria Fiestras Janeiro became my advisors. They helped me a lot during my years as a PhD student, not only in the academic field, but also in making decisions about my future. I am indebted to them for accepting me as their PhD student and for being patient with my stubbornness. Thanks Ignacio and Gloria!

Also Ignacio and Gloria have been my link with prestigious researchers in game theory. I had the good luck of working with some of them and I have learned of them how to make good research. I am indebted to all of my co-authors, Arantza, Gloria, Herbert, Ignacio, José Manuel, Mark, and Peter. Also, I want to thank to my (ex-)officemate Julio for the fruitful comments and *discussions* on some topics than we have had.

Through my years as a PhD student I could visit a lot of prestigious research centers like Tilburg University, Basque Country University, Stockholm School of Economics and Universitat Autònoma de Barcelona. There, I knew a lot of people and I could experiment different ways of researching; these two aspects have helped me to have a wide perspective on game theory and related fields. I want to thank to Peter for hosting me in the Department of Econometrics and Operations Research of University of Tilburg (once was not enough, I was there three times), and to Arantza, Herbert, Mark, Henk, Ruud, Baris, Emiliya, Corrado, ... and others at CentER for the great atmosphere that I found there. An special mention is deserved to Arantza. She was my *Spanish link* with the people in Tilburg; without her help my visits had not been so fruitful. I am indebted to Mark for hosting me at Stockholm School of Economics and for working with me also during my visits to Tilburg University. I am grateful to José Manuel for having invited me to the Department of Applied Economics IV of the Basque Country University, and to other members of his department and of the Department of Foundations of Economic Analysis I for their reception (Josune, Miguel, Fede, Luis, Elena, Javier, . . .). Finally, I am also indebted to Jordi for being my host during my stay at Universitat Autònoma de Barcelona.

I have to write some lines about my officemates, Rosa and Julio. They have been a great support with my problems with the *literature*, and they have always been there for any request. We have spent a lot of time in our *"warm"* office and it would not have been the same without their friendship.

I do not want to forget the people from the group of Galician game theorists and from the Department of Statistics and Operations Research at University of Santiago de Compostela. They have been an important support during my PhD years. I also want to thank all the other PhD students of the Faculty of Mathematics for our funny *"pachangas"* and for the pleasant coffee breaks and dinners. Thanks to Tere, Marco, Bea, Carlitos, Miguel, Cris, Juanjo, . . .

I also want to recall all my friends, Silvia, Bea, Mirian, Sol, Rosa, Trini, Martín, . . . thank you for your friendship and for keeping me into the real world.

One important person who was close to me from the beginning of my Bachelor until now, is Yoli. I am deeply indebted to her for an uncountable set of reasons. Without her I would not be in this point of my life. This thesis is also yours, Yoli.

Finally, the most important support comes from my parents and from my brother. They taught me the most important lessons of the life, things that nobody could teach me better than them. *Muchas gracias por haber estado a mi lado todo este tiempo.*

Manuel Alfredo Mosquera Rodríguez
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Contents

Notations

This thesis consists of independent chapters and for this reason all of them are self-contained. It is possible that some of the notation is introduced in more than one chapter. Anyway, the following symbols and notation are common for all the chapters.

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integer numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of non-negative real numbers
$\mathbb{R}_{++}$	The set of positive real numbers
$\emptyset$	The empty set
2^N	The set of all subsets of N
$T \subseteq S$	T is a subset of S
$T \subset S$	T is a subset of S and T is not equal to S
$T \times S$	The cartesian product of T and S
$\lvert S \rvert$	The number of elements of S
$\square$	The end mark of a proof
$\diamond$	The end mark of an example
$\triangleleft$	The end mark of a remark

Let $u, v \in \mathbb{R}^n$:

$u \geq v$	For each $i \in \{1, \ldots, n\}$, $u_i \geq v_i$
$u > v$	For each $i \in \{1, \ldots, n\}$, $u_i > v_i$

Part I

Operations Research Games

Introduction to operations research games

Part I of the dissertation is devoted to cooperative behavior in operations research situations. It is organized in three independent chapters. Each of them takes a well-known operations research model and studies the sharing costs/benefits issue that arises from possible cooperation among agents involved in the model.

Chapter 1 deals with *inventory* situations. Meca et al. (2003) and Meca et al. (2004) define *inventory games* associated with the multi-agent deterministic continuous review models. In such models there exists the possibility of cooperation among agents through making joint orders. That cooperation gives rise to cost savings and the problem is how to share them among the agents. They also define a way of sharing the benefits originated by cooperation, the so-called Sharing Ordering Costs rule (SOC-rule). In this chapter we introduce the property of *immunity to coalitional manipulation* and it is shown that the SOC- rule is the unique allocation rule for inventory games which satisfies this property. This chapter is based on the paper Mosquera et al. (2006).

The problem of sharing costs in a common project used by a set of agents is studied in Chapter 2. A project, throughout this chapter, has a particular structure: it is componed by several ordered subprojects, each of them has an associated cost and each agent needs some consecutive subprojects. We focus on how the total cost has to be shared among the agents. Villarreal-Cavazos and García-Díaz (1985) apply game theoretical models to a similar problem where the project is a national highway and the total cost has to be shared via taxes for the different classes of vehicles. Moreover, our problem is closely related to *airport games* defined by Littlechild and Owen (1973) and to *realization games* studied by Koster et al. (2003). Here, we define a cost TU game associated with the problem and we study three well-known allocation rules in the setting of cooperative games: the *Shapley value*, the *compromise value*, and the *nucleolus*. It is shown that the two first ones have simple formulas, independent of the game. For the nucleolus we provide an easy procedure to compute it. This chapter is based on Mosquera and Zarzuelo (2006).

Finally, Chapter 3 deals with a *scheduling problem*. The first paper studying scheduling problems from a game theoretical point of view was Curiel et al. (1989). There, they study the class of one-machine sequencing situations. Here we study *proportionate flow shop* (PFS) problems with game theory tools. In a PFS problem several jobs have to be processed through a fixed sequence of machines and the processing time of each job is equal on all machines. By identifying jobs with agents, whose costs linearly depend on the completion time of their jobs, and assuming an initial processing order on the jobs, we face an additional problem: how to allocate the cost savings obtained by ordering the jobs optimally? In this chapter, *PFS games* are defined as cooperative games associated to PFS problems. It is seen that PFS games have a nonempty core. Moreover, it is shown that PFS games are convex if the jobs are initially ordered in decreasing urgency. For this case an explicit expression for the Shapley value and a specific type of equal gain splitting rule which leads to core elements of the PFS game are proposed. The latter rule follows from the algorithm proposed in Shakhlevich et al. (1998) for obtaining an optimal schedule for PFS problems. This chapter is based on the paper Estévez-Fernández et al. (2006).

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Chapter 1

A note on coalitional manipulation and centralized inventory management

Contents

1.1 Introduction

Inventory centralization is known to reduce costs in several models of multi-agent inventory cost optimization. An important problem which arises in these models is how the central management should allocate the costs among the agents.

This problem has been tackled by several authors in the last years. For instance, Hartman and Dror (1996) consider some properties that an allocation rule should satisfy in the context of multi-agent stochastic continuous review models, and propose a rule satisfying these properties. Hartman et al. (2000) and Müller et al. (2002) study the core of a class of games arising in multi-agent stochastic single-period models. Meca et al. (2003) and Meca et al. (2004) introduce the so-called *inventory games*, which model cost allocation problems in multi-agent deterministic continuous review models, and propose and characterize the *SOC-rule* (Share the Ordering Cost) for these games, which always proposes core allocations. More recently, van den Heuvel et al. (2007) and Guardiola et al. (To appear) analyze new classes of games connected with cost allocation in deterministic periodic review models, more precisely in economic lot-sizing problems. The main difference with Meca et al.'s setting is that in economic lot-sizing the orders can only be made in a collection of fixed time instants, and that the demand and the holding costs depend on the time period considered.

In this chapter we consider the model studied in Meca et al. (2003). So, although there are several classes of games arising in inventory centralization, when we write *inventory games* throughout this chapter, we mean the class of games described in Meca et al. (2003). In this context we look for an allocation rule which is immune to possible manipulations of the agents involved in the problem via artificial merging or splitting. We prove that the unique allocation rule for inventory games which is immune to coalitional manipulation is the SOC-rule. Although coalitional manipulation had never been studied in the context of centralized inventory models, it is an interesting property which has deserved a wide attention in the economical literature. Ju (2003), Bergantiños and Sánchez (2002) and de Frutos (1999) are three recent examples in which coalitional manipulation is considered in various allocation problems.

The organization of this chapter is as follows. In the next section we summarize the main features concerning inventory games and introduce a property of immunity to coalitional manipulation for allocation rules in this context. In Section 1.3 we state and prove the main result.

1.2 Coalitional manipulation in inventory games

A *cost TU game* is a pair (N, c) where $N = \{1, \ldots, n\}$, with $n \in \mathbb{N}$, is the set of agents and $c : 2^N \to \mathbb{R}$ is the characteristic function of the game which assigns to each subset $S \subseteq N$ a cost $c(S)$ that has to be paid if players in S cooperate. By convention, $c(\emptyset) = 0$.

In this chapter, as in Meca et al. (2003), we deal with inventory games. An *inventory game* is a cost TU game arising from a centralized multi-agent inventory cost situation, in which every agent faces a deterministic continuous review inventory problem which can be modeled as an Economic Production Quantity (EPQ) with shortages problem[1]. In such a situation, a finite group of agents N agrees to make jointly the orders of a certain good which all them need, so that they spend a instead of $|N|a$ ($a > 0$ being the fixed cost of an order) every time an order is placed. We denote by m_i ($m_i > 0$) the optimal number of orders per time unit for agent $i \in N$ if ordering alone. In Meca et al. (2003) it is proved that the triplet (N, a, m), with $m = (m_1, \ldots, m_n)$, characterizes such a centralized multi-agent inventory cost situation in the sense that, for every coalition $S \subseteq N$, $m_S = \sqrt{\sum_{i \in S} m_i^2}$ is the *optimal number of orders per time unit* for the agents in S and $c(S) = 2am_S$ is the *optimal average inventory cost per time unit* if they place their orders jointly; (N, c) is the inventory game associated with (N, a, m). Note that, according to this expression of the costs, the agents of N make savings if ordering together.

We denote by I^N the class of inventory games with player set N, by I the class of all inventory games, and by G the class of all cost TU games. Clearly,

$$I = \left\{ (N, c) \in G \;\middle|\; c(S) > 0 \text{ for all non-empty } S \subseteq N \text{ and } \; c(S)^2 = \sum_{i \in S} c(i)^2 \right\}.$$

From now on, we sometimes identify a TU game (N, c) with its characteristic function c.

An important issue for these inventory games is how to allocate the total costs when the agents in N cooperate. Given an inventory game $(N, c) \in I$, an *allocation* of the total cost is a vector $x \in \mathbb{R}^n_+$ such that $x(N) = c(N)$, where, for each coalition $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$. A well-known set of allocations is the *core*, defined by

$$Core(N, c) = \{ x \in \mathbb{R}^n_+ \mid x(N) = c(N), x(S) \leq c(S) \text{ for each } S \subset N \}.$$

The core consists of all allocations of the total cost that are coalitionally rational. An *allocation rule* is a map ψ which assigns to every inventory game $(N, c) \in I$ an allocation of the total cost, i.e. a vector $\psi(c) = (\psi_i(c))_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} \psi_i(c) = c(N)$. An allocation rule ψ satisfies the *null player property* if for each agent $i \in N$ such that $c(i) = 0$, then $\psi_i(c) = 0$. Let $c, c' \in I^N$, an allocation rule ψ is said to be *monotone* if for each agent $i \in N$ such that $c(i) \geq c'(i)$ it holds that $c(i)\psi_i(c) \geq c'(i)\psi_i(c')$.

An important allocation rule in the context of cost TU games is the *Shapley value*. Given a cost TU game (N, c) and an agent $i \in N$, it is defined by

$$\Phi(c) = \sum_{S \subseteq N, S \ni i} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} \left(c(S) - c(S \setminus \{i\}) \right).$$

Meca et al. (2003) propose a context-specific rule for inventory games, the *SOC-rule*. It is defined,

[1]The EPQ with shortages problem is a rather general deterministic inventory model; see, for instance, Tersine (1994) for details.

for every $(N, c) \in I$, and every $i \in N$, in the following way:

$$\sigma_i(c) := \frac{c(i)^2}{c(N)}.$$

Note that

$$\frac{c(i)^2}{c(N)} = \frac{c(i)^2}{\sum_{j \in N} c(j)^2} c(N),$$

so the SOC-rule is a proportional allocation rule. In Meca et al. (2003) some good properties of this rule are proved: it provides core allocations (i.e. $\sigma(c)$ belongs to the core of c for every $c \in I$), it can be characterized using convenient sets of properties (for instance, using efficiency, the null player property and a monotonicity property), it can be easily implemented in practice (according to the SOC-rule each agent pays his holding costs and the fixed order costs are paid proportionally to the squared m_i parameters of the agents). In this chapter we demonstrate that it also behaves in an excellent way from the point of view of immunity to coalitional manipulation.

Immunity to coalitional manipulation in this context means the following. We want to find allocation rules which are immune to manipulations whereby a group of agents artificially merges to represent a single agent, or a single agent artificially splits to represent several agents. Immunity to these manipulations is relevant in practice because in many inventory situations it is feasible for the agents to merge, simply forming an a priori centralized inventory unit, or to split, by presenting the different sections of a unique firm which can only manage its inventory in a centralized way as if they were different inventory units. Let us formally introduce this property.

Definition 1.1. *Let $m, n \in \mathbb{N}$ and write $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$. Take $c \in I^N$, $d \in I^M$ and a non-empty $S \subseteq N$. We say that d is the S-manipulation of c if:*

- $M = (N \setminus S) \cup \{i_S\}$,
- $d(T) = c(T)$ *for all* $T \subseteq M$ *with* $i_S \notin T$,
- $d(T) = c((T \setminus \{i_S\}) \cup S)$ *for all* $T \subseteq M$ *with* $i_S \in T$.

Definition 1.2. *An allocation rule ψ is said to be* immune to coalitional manipulation *if, for every $c, d \in I$ such that d is the S-manipulation of c for a certain S, it holds that*

$$\psi_{i_S}(d) = \sum_{i \in S} \psi_i(c). \tag{1.1}$$

Let us make some comments in relation with this property. Note first that it is the aggregation of a property of *no advantageous splitting* (corresponding to the "less than or equal to" in (1.1)) and a property of *no advantageous merging* (corresponding to the "greater than or equal to" in (1.1)).

Observe that the effect of merging or splitting is well modeled in Definition 1.1. Let us discuss this a bit. When the group of agents S merge in a centralized multi-agent inventory cost situation

(N, a, m) with corresponding inventory game c, they have a new optimal number of orders per time unit. We have already mentioned that Meca et al. (2003) prove that this number is $m_S = \sqrt{\sum_{i\in S} m_i^2}$. Hence, it is easy to check that the inventory game associated with the resulting centralized multi-agent inventory cost situation is precisely d as defined in Definition 1.1.

So, immunity to coalitional manipulation seems to be an interesting property in this context. The main result of this chapter is that the unique rule which satisfies this property for the class of inventory games is the SOC-rule. The next section includes a proof of this result. We finish this section with an example which illustrates that the Shapley value is not immune to coalitional manipulation.

Example 1.1. Consider the inventory game c, associated with the multi-agent inventory cost situation (N, a, m) given by $N = \{1, 2, 3\}$, $a = 2$, $m = (1, 2, 2)$. So, $c(1) = 4$, $c(2) = c(3) = 8$, $c(12) = c(13) = 8.94$, $c(23) = 11.31$, $c(N) = 12$. If $S = \{2, 3\}$, then the S-manipulation of c is the inventory game d given by $d(1) = 4$, $d(i_S) = 11.31$, $d(M) = 12$. The Shapley value of c and d is, respectively, $\Phi(c) = (1.88, 5.06, 5.06)$ and $\Phi(d) = (2.34, 9.66)$. Notice that $\Phi_{i_S}(d) = 9.66 \neq \Phi_2(c) + \Phi_3(c) = 10.12$, so Φ is not immune to coalitional manipulation. $\diamond$

1.3 The Main Result

Theorem 1.1. *The unique allocation rule for inventory games which satisfies immunity to coalitional manipulation is the SOC-rule.*

Proof. Let us see first that the SOC-rule σ satisfies immunity to coalitional manipulation. Take c, d and S as in Definition 1.1. Then,

$$\sigma_{i_S}(d) = \frac{d(i_S)^2}{d(M)} = \frac{c(S)^2}{c(N)} = \sum_{i\in S} \frac{c(i)^2}{c(N)} = \sum_{i\in S} \sigma_i(c).$$

Take now ψ an allocation rule satisfying immunity to coalitional manipulation. Let us check that, for any $c \in I^N$ and any $j \in N$, $\psi_j(c)$ only depends on $c(N)^2$ and on $c(j)^2$. This is obviously true if $|N| \leq 2$. In any other case take $\bar{d}$, the $N \setminus \{j\}$-manipulation of c. Since $\bar{d}$ is a two-player game, $\psi_j(\bar{d})$ only depends on $\bar{d}(M)^2$ and $\bar{d}(j)^2$. Note that $\bar{d}(M) = c(N)$, $\bar{d}(j) = c(j)$ and, since ψ satisfies immunity to coalitional manipulation,

$$\psi_j(c) = c(N) - \sum_{k\in N\setminus\{j\}} \psi_k(c) = \bar{d}(M) - \psi_{i_{N\setminus\{j\}}}(\bar{d}) = \psi_j(\bar{d}).$$

Hence, $\psi_j(c)$ only depends on $c(N)^2$ and on $c(j)^2$, which means that there exists a function f such that $\psi_j(c) = f(c(N)^2, c(j)^2)$ for all $c \in I^N$ and all $j \in N$. Assume now that f is linear in the second component (we demonstrate that this is true at the end of this proof). Then, $\psi_j(c) =$

$g(c(N)^2)c(j)^2$ for all $c \in I^N$ and all $j \in N$. Thus,

$$c(N) = \sum_{k \in N} \psi_k(c) = g(c(N)^2) \sum_{k \in N} c(k)^2 = g(c(N)^2)c(N)^2,$$

and so

$$g(c(N)^2) = \frac{1}{c(N)}$$

which means that

$$\psi_j(c) = \frac{c(j)^2}{c(N)} = \sigma_j(c).$$

So, to finish the proof we only need to check that f is linear in the second component. To prove it, take into account that we have a collection of functions

$$\{f(\alpha, .) \mid \alpha \in (0, +\infty)\}$$

satisfying that $f(\alpha, .) : (0, \alpha] \to [0, \alpha]$, for all $\alpha \in (0, +\infty)$. Take now $\alpha, x, y \in (0, +\infty)$ with $x + y \leq \alpha$. Then, there exists $(N, \hat{c}) \in I$ such that $\alpha = \hat{c}(N)^2$, $x = \hat{c}(1)^2$, $y = \hat{c}(2)^2$. Define $S = \{1, 2\}$ and take $\hat{d}$, the S-manipulation of $\hat{c}$. Then,

$$\begin{aligned} f(\alpha, x + y) &= f(\hat{c}(N)^2, \sum_{k \in S} \hat{c}(k)^2) = f(\hat{c}(N)^2, \hat{c}(S)^2) = f(\hat{d}(M)^2, \hat{d}(i_S)^2) \\ &= \psi_{i_S}(\hat{d}) = \sum_{k \in S} \psi_k(\hat{c}) = \sum_{k \in S} f(\hat{c}(N)^2, \hat{c}(k)^2) \\ &= f(\alpha, x) + f(\alpha, y). \end{aligned}$$

So, for every $\alpha \in (0, +\infty)$, $f(\alpha, .)$ is additive. Then, since $f(\alpha, .)$ is also non-negative, it is clear that it is moreover increasing[2]. It is an easy exercise to prove that every increasing additive function $h : (0, \alpha] \to [0, \alpha]$ is also linear. This completes the proof. □

We finish this chapter with a comment. The class of p-additive cost games A^p (for every non-zero real number p) can be defined in the following way:

$$A^p = \{(N, c) \in G \mid c(S) > 0 \text{ for all non-empty } S \subseteq N \text{ and } \ c(S)^p = \sum_{i \in S} c(i)^p\}.$$

Notice that $I = A^2$. If we define allocation rule and immunity to coalitional manipulation for A^p in an analogous way as we did for I, Theorem 1.1 can be immediately extended to A^p. So, the unique allocation rule for p-additive games which satisfies immunity to coalitional manipulation is the modified SOC-rule σ^p, which is defined, for every $(N, c) \in A^p$ and every $i \in N$, by:

$$\sigma_i^p(c) = \frac{c(i)^p}{\sum_{j \in N} c(j)^p} c(N) = \frac{c(i)^p}{c(N)^{p-1}}.$$

[2]We do not mean strictly increasing, but just increasing, i.e. $x \leq y \Rightarrow f(\alpha, x) \leq f(\alpha, y)$ for each $x, y \in (0, \alpha]$.

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Chapter 2

Sharing costs in highways: a game theoretical approach

Contents

2.1 Introduction

In this chapter we address the question of road pricing. A large amount of the related literature has been concerned with the congestion issue. Instead, we are to pay attention to the problem of sharing the cost of construction and maintenance among the users, according to the principles of equity and efficiency.

There are some papers that deal with the problem of assigning taxes to different classes of vehicles (cars, trucks, motorbikes, etc.) in a road. Villarreal-Cavazos and García-Díaz (1985) study such a problem. They provide four assigning methods based on the different characteristics of the classes of vehicles. Only one of those methods make use of game theory tools: the nucleolus. They call this method the generalized method.

In this chapter we want to go a bit further. We will study a resource more general than roads. Note that a road can be seen as a public resource with a particular structure. Therefore, we will refer to public resources instead of roads. The cost function in this case can be stated very easily. The resource (highway, projects, etc..) can be split up in several segments or sections, and each agent uses a subset of adjoining segments. The cost of each segment depends mostly on its length and the number of potential users. The simplicity of the cost function makes the application of the game theory particularly suitable. To apply the concepts and techniques coming from this theory is the main purpose of the present work.

A natural way for sharing costs is to divide the total cost proportionally to the individual costs. This rule is used in a lot of environments and it has very good properties. In our setting, the proportional rule coincide with the compromise value (τ- value) for cost TU games. Another well-known rule for sharing costs is the equalitarian rule. In our setting, the equalitarian rule that share the cost of each section equally among the agents who want/use such section coincides with the Shapley value.

In this chapter we also study the nucleolus. Even though this solution concept is not so easy to compute, we know that it is used in real situations (see Villarreal-Cavazos and García-Díaz, 1985). Moreover, we find a simple way of computing it in our particular setting.

Let us recall some basic definitions from game theory we will use through this chapter. A *cooperative cost game with transferable utility*, or *cost TU game*, is a pair (N, c), where $N = \{1, \ldots, n\}$, with $n \in \mathbb{N}$, is the finite set of players and $c : 2^N \longrightarrow \mathbb{R}$ is the characteristic function which assigns to each subset $S \subseteq N$ of players a cost $c(S)$ that has to be paid if players in S cooperate. By convention, $c(\emptyset) = 0$. Nonempty subsets of N are called *coalitions*. A cost TU game is called *subadditive* if, for each $S, T \subset N$ such that $S \cap T = \emptyset$, it holds that $c(S) + c(T) \geq c(S \cup T)$. A cost TU game is said to be *monotone* if $c(S) \leq c(T)$ for each $S \subseteq T \subseteq N$. For each $T \subseteq N$, the *unanimity game* $(N, \mathbf{1}_T)$ is defined by

$$\mathbf{1}_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} \qquad \text{for each } S \subseteq N.$$

An *allocation* is a vector $x \in \mathbb{R}^n$ assigning cost $x_i \in \mathbb{R}$ to player $i \in N$. Let $x \in \mathbb{R}^n$ be an allocation and let $S \subseteq N$ be such that $S \neq \emptyset$. We write x_S for the restriction of x to $\mathbb{R}^s$, and $x(S) = \sum_{i \in S} x_i$. An allocation $x \in \mathbb{R}^n$ is said to be *efficient* if $x(N) = c(N)$, and it is said to be *individually rational* for (N, c) if $x_i \leq c(\{i\})$ for each $i \in N$. The set of efficient and individually rational allocations for (N, c) is called the *imputation set* of (N, c) and it is denoted by

$$\mathcal{I}(N, c) = \{x \in \mathbb{R}^n \mid x(N) = c(N), x_i \leq c(\{i\}) \text{ for each } i \in N\}.$$

Moreover, x is *stable* for (N, c) if $x(S) \leq c(S)$ for each $S \subseteq N$, i.e., no coalition can improve by not joining to the grand coalition. The set of stable imputations for (N, c) is called the *core* of (N, c) and it is denoted by

$$Core(N, c) = \{x \in \mathbb{R}^n \mid x(N) = c(N), x(S) \leq c(S) \text{ for each } S \subset N\}.$$

A cost TU game (N, c) is said to be *balanced* if its core is nonempty (Bondareva, 1963; Shapley, 1967). An important subclass of balanced games is the class of concave games. A cost TU game (N, c) is said to be *concave* if $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$ for each $S, T \subseteq N$. Two cost TU games (N, c) and $(N, \bar{c})$ are called *strategically equivalent* if there exist $k \in \mathbb{R}_{++}$ and $a \in \mathbb{R}^n$ such that $\bar{c}(S) = kc(S) + a(S)$ for each $S \subseteq N$.

The structure of the chapter is as follows. First, in Section 2.2, we introduce a cooperative game, that we call *highway game*. A movement along a subset of adjoining resource sections is a player in this game. The characteristic function will be given by the cost function. It turns out that airport games (Littlechild and Owen, 1973) are a special class of highway games. On the other hand, this class of games is in its turn a subclass of realization games defined in Koster et al. (2003), which arise from a realization problem. In this paper, the authors show that realization games are convex games and they prove a relation among the core of the game and the set of strong Nash equilibria of a noncooperative game arising from the realization problem. However, they do not study particular allocation rules for such games. In Section 2.3, taking advantage of the simple structure of the highway games, we derive simple formulas for the Shapley value and for the compromise value. Finally, in Section 2.4, we focus on the nucleolus. We present an algorithm finding the nucleolus in at most n steps, each one requiring at most $O(n)$ elementary operations, where n is the cardinality of the player set.

2.2 Highway problems

In many real situations several agents want to use the same public resource. Sometimes, the public resource can be decomposed in a finite number of indivisible and ordered sections where each section has an associated cost (construction, maintenance, ...). Each agent needs some consecutive sections of the resource and all of them have to bear the total cost of the resource. An easy example of that situation is a linear highway, where the indivisible sections are delimited by the entry/exit points and each agent only needs the highway sections in between his entry and his exit points.

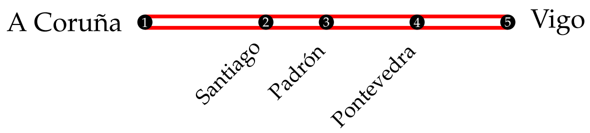

Figure 2.1: Example of a linear highway.

We call a *highway problem* to a 4-tuple (N, M, C, T), where $N = \{1, \ldots, n\}$, with $n \in \mathbb{N}$, represents the set of *agents*, $M = \{t_1, \ldots, t_m\}$, with $m \in \mathbb{N}$, is the completely ordered set of indivisible resource sections, $C : M \to \mathbb{R}_{++}$ is the cost function which represents the cost of each resource section, and $T : N \to 2^M$ is such that

a) for each $i \in N$ there exist $a_i, b_i \in M$ with $a_i \leq b_i$ and $T(i) = \{t \in M \mid a_i \leq t \leq b_i\}$,

b) $\bigcup_{i \in N} T(i) = M$

where each $T(i)$ represents the resource sections agent i needs. Let us note that the second condition on T means that all sections of the resource are used.

Remark 2.1. Let us note that highway problems are generalizations of airport problems, since an airport problem is a highway problem with $a_i = t_1$ for all $i \in N$. $\lhd$

Associated with each highway problem one can define a cost TU game. Let (N, M, C, T) be a highway problem. The cost of attending agent i is denoted by $c(i) = \sum_{t \in T(i)} C(t)$. Given a coalition $S \subseteq N$, we denote by $T(S) = \bigcup_{i \in S} T(i)$ the set of sections needed by coalition S and by

$$c(S) = \sum_{t \in T(S)} C(t)$$

the cost of attending the members of S. By convention, $c(\emptyset) = 0$. The pair (N, c) is the cost TU game called the associated *highway game*.

Example 2.1. Take the linear highway represented in Figure 2.1 (names are places in Spain). The highway connects "A Coruña" and "Vigo" and numbered black points represent entries and exits of the highway. This points split up the highway into 4 sections: A Coruña $\leftrightarrow$ Santiago (t_1); Santiago $\leftrightarrow$ Padrón (t_2); Padrón $\leftrightarrow$ Pontevedra (t_3); and Pontevedra $\leftrightarrow$ Vigo (t_4). Then, $M = \{t_1, t_2, t_3, t_4\}$. Suppose that the cost of those sections are: $C(t_1) = 8$, $C(t_2) = 4$, $C(t_3) = 6$, and $C(t_4) = 6$. Let us assume that 4 agents are the users of this highway. Agent 1 wants to go from A Coruña to Santiago; agent 2 from Padrón to Vigo; agent 3 from Coruña to Pontevedra; and agent 4 wants to go from Santiago to Vigo. For clearness, we denote each agent by a pair ij,

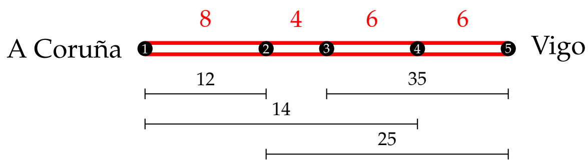

Figure 2.2: Linear highway of Example 2.1.

$i < j$, where i is his entry point and j is his exit point, so, for instance, agent 3 will be represented by the pair 14. This situation is represented in Figure 2.2.

The set of agents is $N = \{12, 35, 14, 25\}$ and the set of highway sections needed for each agent are: $T(12) = \{t_1\}$, $T(35) = \{t_3, t_4\}$, $T(14) = \{t_1, t_2, t_3\}$, and $T(25) = \{t_2, t_3, t_4\}$. Note that, with our notation, $T(ij) = \{t_i, \ldots, t_{j-1}\}$.

Hence, we have the highway problem represented by (N, M, C, T). The associated highway game is

S	$\emptyset$	$\{12\}$	$\{35\}$	$\{14\}$	$\{25\}$	$\{12,35\}$	$\{12,14\}$	$\{12,25\}$	$\{35,14\}$	$\{35,25\}$
$c(S)$	0	8	12	18	16	20	18	24	24	16

S	$\{14,25\}$	$\{12,35,14\}$	$\{12,35,25\}$	$\{12,14,25\}$	$\{35,14,25\}$	N
$c(S)$	24	24	24	24	24	24

It is easy to check that this game is monotone, subadditive and concave. $\diamond$

The next proposition states some properties of the highway games.

Proposition 2.1. *Let (N, M, C, T) be a highway problem. Then the associated game (N, c) is monotone and concave.*

Proof. Let $S \subseteq R \subseteq N$. Since $T(S) \subseteq T(R)$ and $C(t) > 0$ for each $t \in M$, we have that $c(S) \leq c(R)$. Therefore, (N, c) is monotone.

Let $S, R \subseteq N$. Then,

$$\begin{aligned} c(S) + c(R) &= \sum_{t \in T(S)} C(t) + \sum_{t \in T(R)} C(t) \\ &= \sum_{t \in T(S) \cup T(R)} C(t) + \sum_{t \in T(S) \cap T(R)} C(t) \\ &\geq \sum_{t \in T(S \cup R)} C(t) + \sum_{t \in T(S \cap R)} C(t) \\ &= c(S \cup R) + c(S \cap R), \end{aligned}$$

where the inequality follows from that $C(t) > 0$ for each $t \in M$, $T(S \cup R) = T(S) \cup T(R)$ and $T(S \cap T) \subseteq T(S) \cap T(R)$. Therefore, (N, c) is concave. $\square$

Remark 2.2. Let (N, M, C, T) be a highway problem. Let $t \in M$. It is easy to check that:

$$\text{if } \{i \in N \mid t \in T(i)\} = \{j\} \text{ then } c = c_{|M\setminus\{t\}} + c(t)\mathbf{1}_{\{j\}}$$

where $(N, c_{|M\setminus\{t\}})$ is the highway game arising from (N, c) when the section t is dropped out, and $(N, \mathbf{1}_S)$ represents the unanimity game associated with coalition $S \subseteq N$. Then (N, c) and $(N, c_{|M\setminus\{t\}})$ are strategically equivalent. $\triangleleft$

By Remark 2.2, we can make the following assumption:

Assumption 2.1. $\big|\{i \in N \mid t \in T(i)\}\big| > 1$ for each $t \in M$.

Note that Assumption 2.1 is not so restrictive. Our aim is to calculate the Shapley value, the compromise value and the nucleolus, and these solution concepts satisfy covariance under strategic equivalence. Recall that a solution concept ψ is said to be *covariant under strategic equivalence* if, for each pair of strategically equivalent games (N, c) and $(N, \bar{c})$, we have $\psi(N, \bar{c}) = k\psi(N, c) + a$.

2.3 Natural sharing cost rules for highway games

Let (N, M, C, T) be a highway problem. A natural way of sharing costs in a highway problem can be to share the cost of each resource section equally among the agents who use it. Then each agent $i \in N$ would pay

$$\zeta_i(N, M, C, T) = \sum_{t \in T(i)} \frac{C(t)}{\big|\{j \in N \mid t \in T(j)\}\big|}.$$

Another possible and natural way of sharing cost can be to share the total cost proportionally to the individual costs, i.e., proportionally to $c(\{i\})$, and so, each agent $i \in N$ would pay

$$\pi_i(N, M, C, T) = \frac{\sum\limits_{t \in T(i)} C(t)}{\sum\limits_{j \in N} \sum\limits_{t \in T(j)} C(t)} \sum_{t \in M} C(t) = \frac{c(\{i\})}{\sum\limits_{j \in N} c(\{j\})} c(N).$$

In this section, we prove that $\zeta(N, M, C, T)$ and $\pi(N, M, C, T)$ coincide with two well-known solution concepts for cost TU games: the Shapley value and the compromise value, respectively.

An *allocation rule*, or *sharing cost rule*, is a function ψ which, for each cost TU game (N, c), selects an allocation in $\mathbb{R}^n$. It is said that ψ is *efficient* if it always selects efficient allocations; ψ is *symmetric* if for each pair $i, j \in N$ such that $c(S \cup \{i\}) = c(S \cup \{j\})$ for each $S \subseteq N \setminus \{i, j\}$, we have $\psi_i(N, c) = \psi_j(N, c)$; ψ is *additive* if for each pair of cost TU games (N, c) and (N, d), $\psi(N, c + d) = \psi(N, c) + \psi(N, d)$ where $(N, c + d)$ represents the sum game of (N, c) and (N, d). The *Shapley value* (Shapley, 1953) is an allocation rule which assigns, to each cost TU game (N, c), an allocation $\Phi(N,c)$ given by

$$\Phi_i(N,c) = \sum_{S\subseteq N, S\ni i} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \left(c(S) - c(S\setminus\{i\})\right)$$

for each player $i \in N$. Recall that the Shapley value satisfies efficiency, symmetry and additivity.

Proposition 2.2. *Let (N,M,C,T) be a highway problem and let (N,c) be its associated highway game. Then*

$$\Phi(N,c) = \zeta(N,M,C,T).$$

Proof. Let $t \in M$. Define the characteristic game (N,c_t), where

$$c_t(S) = \begin{cases} C(t) & \text{if } t \in T(S) \\ 0 & \text{otherwise} \end{cases}$$

for all $S \subseteq N$, and $c_t(\emptyset) = 0$. By symmetry and efficiency of Shapley value,

$$\Phi_i(N,c_t) = \begin{cases} \dfrac{C(t)}{|\{j\in N \mid t \in T(j)\}|} & \text{if } t \in T(i) \\ 0 & \text{otherwise} \end{cases}$$

for each $i \in N$.

Moreover, $c = \sum_{t\in M} c_t$. Then, by additivity of Shapley value,

$$\Phi_i(N,c) = \sum_{t\in M} \Phi_i(N,c_t) = \sum_{t\in T(i)} \frac{C(t)}{|\{j\in N \mid t\in T(j)\}|} = \zeta(N,M,C,T).$$

for each $i \in N$. $\square$

Remark 2.3. Proposition 2.2 is also valid for the class of *cooperative realization games* defined in Koster et al. (2003), which is a generalization of the class of highway games. $\triangleleft$

Another well-known allocation rule for cost TU games is the compromise value. In its definition the following concepts appear. Let (N,c) be a cost TU game. The *utopia vector* $U(N,c) \in \mathbb{R}^n$ for (N,c) is given by $U_i(N,c) = c(N) - c(N\setminus\{i\})$ for each $i \in N$. The *minimum right vector* $m(N,c) \in \mathbb{R}^n$ is given, for each $i \in N$, by

$$m_i(N,c) = \min_{S\subseteq N: S\ni i} \left\{ c(S) - \sum_{j\in S\setminus\{i\}} U_j(N,c) \right\}.$$

The *core cover* of (N, c) is defined as[1]

$$CC(N, c) = \{x \in \mathbb{R}^n \mid x(N) = c(N), m(N, c) \geq x \geq U(N, c)\}.$$

A cost TU game (N, c) is called *compromise admissible* if $CC(N, c) \neq \emptyset$.

The *compromise value*, or *τ-value* (Tijs, 1981), is the allocation rule on the class of compromise admissible games defined as the point on the line segment joining $m(N, c)$ and $M(N, c)$ that is efficient, i.e.

$$\tau(N, c) = \alpha U(N, c) + (1 - \alpha) m(N, c)$$

with $\alpha \in [0, 1]$ such that $\sum_{i \in N} \tau_i(N, c) = c(N)$.

Proposition 2.3. *Let (N, c) be a highway game. Then,*

(A) $U_i(N, c) = 0$ *for each* $i \in N$.

(B) $m_i(N, c) = c(\{i\})$ *for each* $i \in N$.

Proof.

(A) It easily follows from Assumption 2.1.

(B) Since (N, c) is monotone and by (A) in this proposition, $m_i(N, c) = c(\{i\})$ for each $i \in N$. □

By Proposition 2.3, the following corollary can be established.

Corollary 2.1. *Let (N, M, C, T) be a highway problem and let (N, c) be its associated highway game. Then (N, c) is compromise admissible and*

$$\tau(N, c) = \pi(N, M, C, T).$$

2.4 The nucleolus of highway games

The nucleolus is another well-known solution concept for cost TU games. Roughly speaking, the nucleolus tries to maximize the "happiness" of the less happy coalitions with the proposed imputation (or to maximize the minimum satisfaction). The main goal of this section is to find an easy procedure for computing the nucleolus for highway games.

Let (N, c) be a cost TU game, let $S \subseteq N$ and let $x \in \mathbb{R}^n$ be an allocation. The *excess $e(S, x)$ of coalition S at allocation x* is given by $e(S, x) = c(S) - x(S)$. The greater $e(S, x)$, the happier S will be with x. Let $\theta(x) \in \mathbb{R}^{2^n}$ be a vector obtained by arranging the excess of all coalitions of N at x in non decreasing order. Let $x, y \in \mathbb{R}^{2^n}$. It is said that x is *lexicographically greater* than (or equal

[1] Recall that we are working with cost games, then "utopia" means the minimum cost that one has to carry out and "minimum right" means the maximum cost that one has to pay.

to) y, and it is denoted by $x \geq_L y$, if $x = y$ or if there exists an $s \in \{1, \ldots, 2^n\}$ such that $x_k = y_k$ for each $k \in \{1, \ldots, s-1\}$ and $x_s > y_s$.

If $\mathcal{I}(N, c) \neq \emptyset$, the *nucleolus* $\nu(N, c)$ of (N, c) is the unique imputation that lexicographically maximizes $\theta(x)$ over the imputation set (Schmeidler, 1969), i.e.

$$\nu(N, c) = \{x \in \mathcal{I}(N, c) \mid \theta(x) \geq_L \theta(y) \text{ for all } y \in \mathcal{I}(N, c)\}.$$

If no confusion is possible, we write ν for the nucleolus of a cost TU game (N, c).

If an allocation $x \in \mathbb{R}^n$ has been proposed in the game (N, c), player i can compare his position with that of player j by considering the *minimum surplus* $s_{ij}(x)$ of i against j with respect to x, which is defined by

$$s_{ij}(x) = \min_{S \in \Gamma_{ij}} e(S, x)$$

where $\Gamma_{ij} = \{S \subset N \mid i \in S, j \notin S\}$. The *prekernel* of (N, c) (Maschler et al., 1972) consists of those efficient allocations x such that $s_{ij}(x) = s_{ji}(x)$ for each $i, j \in N$. Let us note that, for concave games, the prekernel coincides with the nucleolus (Maschler et al., 1972).

Next, in Subsection 2.4.1, through several results, we identify some coalitions with minimal excess at the nucleolus. We provide an easy way to calculate the excess at the nucleolus of those coalitions. Indeed, we only need to know the parameters of the highway problem and not the full highway game for computing them. Finally, in Subsection 2.4.2, we provide the procedure for computing the nucleolus.

2.4.1 The minimal excess at the nucleolus of a highway game

In this subsection we study the structure of the collection of coalitions with minimal excess at the nucleolus for a highway game. This collection plays a key role in the procedure to be presented in the next subsection. For the rest of this subsection (N, M, C, T) is a fixed highway problem.

First of all, we want to point out that $\nu_i = e(N \setminus \{i\}, \nu)$ for each highway game (N, c) and each $i \in N$, because of Assumption 2.1. Then, in order to compute the value of the nucleolus for a highway game, we only have to compute the value of $e(N \setminus \{i\}, \nu)$ for each $i \in N$. We will see that there exists an easy procedure for computing $e(N \setminus \{i\}, \nu)$ without knowing the value of the nucleolus ν.

A highway problem (N, M, C, T) is said to be *completely separable* if there exists a finite collection of highway subproblems $\{(N_\ell, M_\ell, C_\ell, T_\ell)\}_{\ell=1}^k$, with $k > 1$, such that

a) $\{N_1, \ldots, N_k\}$ is a partition of N such that $T(N_j) \cap T(N_\ell) = \emptyset$ for each pair $j, \ell \in \{1, \ldots, k\}$,

b) $M_\ell = T(N_\ell)$, $C_\ell = C_{|M_\ell}$ and $T_\ell = T_{|N_\ell}$ for each $\ell \in \{1, \ldots, k\}$.

Notice that, for each $S \subseteq N$, $c(S) = \sum_{\ell=1}^k c_\ell(S \cap N_\ell)$ and that, for each $\ell = \{1, \ldots, k\}$, the game (N_ℓ, c_ℓ) corresponds with a highway game.

Lemma 2.1. *Let (N, M, C, T) be a completely separable highway problem. Let (N, c) be its associated highway game and let (N_ℓ, c_ℓ) be the highway game associated with $(N_\ell, M_\ell, C_\ell, T_\ell)$ for each $\ell \in \{1, \dots, k\}$. Then, for each $\ell \in \{1, \dots, k\}$,*

$$\nu(N, c)_{|N_\ell} = \nu(N_\ell, c_\ell).$$

Proof. By simplicity in the proof, assume that $k = 2$ and that $i \geq j$ for each $i \in N_1$ and each $j \in N_2$. Let also $\nu = \nu(N, c)$, $\mu = (\nu(N_1, c_1), \nu(N_2, c_2))$, $\nu^\ell = \nu(N, c)_{|N_\ell}$, and $\mu^\ell = \nu(N_\ell, c_\ell)$ for each $\ell \in \{1, 2\}$. Note that, by definition of a completely separable highway problem, $c(S) = c(S \cap N_1) + c(S \cap N_2)$. Then,

(i) $e(S, x) = e(S \cap N_1, x_{|N_1}) + e(S \cap N_2, x_{|N_2})$ for each $S \subseteq N$ and $x \in \mathbb{R}^n$,

(ii) $\mu \in Core(N, c)$, $\nu^1 \in Core(N_1, c_1)$, and $\nu^2 \in Core(N_2, c_2)$,

(iii) (N_1, c_1) and (N_2, c_2) are concave since (N, c) is concave.

(iv) $e(S, x) \geq \max\{e(S \cap N_1, x_{|N_1}), e(S \cap N_2, x_{|N_2})\}$, for each $S \subseteq N$ and $x \in Core(N, c)$.

Let $x \in \mathbb{R}^n$. By (i), $\theta(x)$ is obtained from $\theta(x_{|N_1})$ and $\theta(x_{|N_2})$ by adding up one component of $\theta(x_{|N_1})$ and one component of $\theta(x_{|N_2})$. Namely, for each $\ell \in \{1, \dots, 2^n\}$, there exist $\ell_1 \in \{1, \dots, 2^{n_1}\}$ and $\ell_2 \in \{1, \dots, 2^{n_2}\}$ such that $\theta_\ell(x) = \theta_{\ell_1}(x_{|N_1}) + \theta_{\ell_2}(x_{|N_2})$.

We will show that $\theta(\mu^1) = \theta(\nu^1)$ and $\theta(\mu^2) = \theta(\nu^2)$. Assume that one of the former equalities is not true. For instance, $\theta(\mu^1) \neq \theta(\nu^1)$ (analogously for $\theta(\mu^2) \neq \theta(\nu^2)$).

First we recall some definitions and implications for keeping in mind through this proof. Since μ^1 is the nucleolus for (N_1, c_1) and $\theta(\mu^1) \neq \theta(\nu^1)$, there exists $i_0 \in \{1, \dots, 2^{n_1}\}$ such that $\theta_\ell(\mu^1) = \theta_\ell(\nu^1)$ for each $\ell < i_0$ and $\theta_{i_0}(\mu^1) > \theta_{i_0}(\nu^1)$. Moreover, since μ^2 is the nucleolus for (N_2, c_2), then $\theta(\mu^2) = \theta(\nu^2)$ or, there exists $j_0 \in \{1, \dots, 2^{n_2}\}$ such that $\theta_\ell(\mu^2) = \theta_\ell(\nu^2)$ for each $\ell < j_0$ and $\theta_{j_0}(\mu^2) > \theta_{j_0}(\nu^2)$. If there exists such j_0, we assume, w.l.g., that $\theta_{i_0}(\mu^1) \leq \theta_{j_0}(\mu^2)$.

We want to compare vectors $\theta(\mu)$ and $\theta(\nu)$. Let then $k_0 \in \{1, \dots, 2^n\}$ be such that $\theta_{i_0}(\mu^1) = \theta_{k_0}(\mu)$ and $\theta_{k_0}(\mu) > \theta_{k_0-1}(\mu)$, i.e, k_0 is the smallest index in $\{1, \dots, 2^n\}$ such that $\theta_{i_0}(\mu^1) = \theta_{k_0}(\mu)$. Let $j_1 \in \{1, \dots, 2^{n_2}\}$ be such that $\theta_{i_0}(\mu^1) \leq \theta_{j_1}(\mu^2)$ and $\theta_{i_0}(\mu^1) > \theta_{j_1-1}(\mu^2)$. Let us note that, j_1 always exists because the vector $\theta(\mu^2)$ has at least two different components, and if there exists j_0, then $j_1 \leq j_0$. Taking into account that, for each $\ell < k_0$, $\theta_\ell(\mu) < \theta_{k_0}(\mu) = \theta_{i_0}(\mu^1)$ and that $\theta_{i_0}(\mu^1) \leq \theta_{j_1}(\mu^2)$, then there exist $\ell_1 < i_0$ and $\ell_2 < j_1$ such that $\theta_\ell(\mu) = \theta_{\ell_1}(\mu^1) + \theta_{\ell_2}(\mu^2)$. Then,

$$\begin{aligned} \theta_\ell(\mu) &= \theta_{\ell_1}(\mu^1) + \theta_{\ell_2}(\mu^2) \\ &= \theta_{\ell_1}(\nu^1) + \theta_{\ell_2}(\nu^2) \\ &= \theta_\ell(\nu), \end{aligned} \tag{2.1}$$

for each $\ell < k_0$, where the second equality follows from the choice of i_0 and j_1.

By the above reasoning, we also know that for each $\ell < k_0$ there exist $\ell_1 < i_0$ and $\ell_2 < j_1$ such that $\theta_\ell(\nu) = \theta_{\ell_1}(\nu^1) + \theta_{\ell_2}(\nu^2)$. Then, there exists $\ell \geq k_0$ such that $\theta_\ell(\nu) = \theta_{i_0}(\nu^1)$, and so $\theta_{k_0}(\nu) \leq \theta_{i_0}(\nu^1)$. Moreover, $\theta_{i_0}(\nu^1) < \theta_{i_0}(\mu^1)$ by the choice of i_0. Thus,

$$\theta_{k_0}(\nu) \leq \theta_{i_0}(\nu^1) < \theta_{i_0}(\mu^1) = \theta_{k_0}(\mu).$$

Therefore, $\theta(\mu) >_L \theta(\nu)$ which is a contradiction because ν is the nucleolus for (N, c).

In conclusion, we have showed that $\theta(\mu^1)$ has to equal $\theta(\nu^1)$ and then, by the uniqueness of the nucleolus, $\nu(N_1, c_1) = \mu^1 = \nu^1 = \nu(N, c)_{|N_1}$. □

Note that we could state an analogous version of Lemma 2.1 for the following more general class of cost TU games. A cost TU game (N, c) belongs to this class if the game is concave and, there exists a partition $\{N_1, \ldots, N_k\}$ of N such that $c(S) = c(S \cap N_1) + \ldots + c(S \cap N_k)$ for each $S \subseteq N$. The proof would be analogous. From now on we will work with highway problems that are not completely separable.

Let (N, c) be a cost TU game and let $x \in \mathbb{R}^n$. We denote by

$$\mathcal{D}_1(x) = \{S \subset N \mid e(S, x) \leq e(T, x), \text{ for all } T \subset N,\ S, T \neq \emptyset\}.$$

to the set of *proper coalitions of N with minimal excess at x.*

For each coalition $S \subseteq N$ we denote by $S^c = N \setminus S$ to the complement of S in N. A family of coalitions $\{S_1, \ldots, S_k\}$ is said to be an *antipartition* of N if the family $\{S_1^c, \ldots, S_k^c\}$ formed by their complements is a partition of N. We shall use the following result due to Arin and Iñarra (1998)[2] which relates partitions and antipartitions with the nucleolus.

Theorem 2.1. *[Arin and Iñarra, 1998] If (N, c) is concave, and ν its nucleolus, then $\mathcal{D}_1(\nu)$ contains a partition or an antipartition of N.*

Let (N, c) be a cost TU game, let $x \in \mathbb{R}^n$, and let $\mathcal{B}$ a family of coalitions of N. Define

$$e(\mathcal{B}, x) = \frac{\sum_{S \in \mathcal{B}} e(S, x)}{|\mathcal{B}|}$$

as the average excess of coalitions in $\mathcal{B}$ at x. For each $\mathcal{B} \subseteq \mathcal{D}_1(x)$ and each $S \in \mathcal{D}_1(x)$, it holds

$$e(\mathcal{B}, x) = e(S, x) = e(\mathcal{D}_1(x), x).$$

If $\mathcal{B}$ is a partition or an antipartition, and $x(N) = c(N)$, then $e(\mathcal{B}, x)$ is independent of x. Indeed, it is easy to check that

- if $\mathcal{P}$ is a partition: $e(\mathcal{P}, x) = \dfrac{\sum_{S \in \mathcal{P}} c(S) - c(N)}{|\mathcal{P}|}$,

[2] They showed Theorem 2.1 for convex games.

- if $\mathcal{A}$ is an antipartition: $e(\mathcal{A},x) = \dfrac{\sum_{S\in\mathcal{A}} c(S) - \big(|\mathcal{A}|-1\big)c(N)}{|\mathcal{A}|}$.

A coalition $S \subset N$ is said to be *relevant* if there exist $a_S, b_S \in M$ such that:

(i) $T(S) = \{t \in M \mid a_S \le t \le b_S\}$,

(ii) if $T(i) \subseteq T(S)$, then $i \in S$.

Note that S is relevant if and only if $S = \{i \in N \mid a_S \le t \le b_S \text{ for each } t \in T(i)\}$. We denote by $\mathcal{RC}(N,c)$ the set of relevant coalitions for the highway game (N,c). If no confusion is possible, we write $\mathcal{RC}$.

Next proposition identifies some partitions or antipartitions contained in $\mathcal{D}_1(v)$ according to Theorem 2.1.

Proposition 2.4. *Let (N,c) be a highway game. At least one of the following statements is true.*

(A) There exists a partition of N, $\{S_1,S_2\} \subseteq \mathcal{D}_1(v)$ with $S_1,S_2 \in \mathcal{RC}$.

(B) There exists an antipartition of N, $\{S_1,S_2\} \cup \big\{N\setminus\{i\} \bigm| i \in S_1 \cap S_2\big\} \subseteq \mathcal{D}_1(v)$ with $S_1,S_2 \in \mathcal{RC}$.

(C) There exists an antipartition of N, $\{S\} \cup \big\{N\setminus\{i\} \bigm| i \in S\big\} \subseteq \mathcal{D}_1(v)$ with $S \in \mathcal{RC}$.

(D) The antipartition of N, $\big\{N\setminus\{i\} \bigm| i \in N\big\} \subseteq \mathcal{D}_1(v)$.

Proposition 2.4 simplifies the search for coalitions with minimal excess, since one only has to seek among the combination of partitions or antipartition described in such proposition. We need some extra lemmas to prove that proposition.

Lemma 2.2. *Let (N,c) be a highway game. For each partition or antipartition $\mathcal{B} \neq \{\emptyset, N\}$ of N, it holds $e(\mathcal{B},v) > 0$.*

Proof. Let $\mathcal{B}$ be a partition of N. Then, $e(\mathcal{B},v) = \frac{\sum_{S\in\mathcal{B}} c(S) - c(N)}{|\mathcal{B}|}$. By subadditivity of (N,c), $e(\mathcal{B},v) \ge 0$. If $e(\mathcal{B},v) = 0$, then $T(S) \cap T(R) = \emptyset$ for each pair $S,R \in \mathcal{B}$. Hence, (N,M,C,T) is completely separable, which is a contradiction.

Let $\mathcal{B}$ be an antipartition of N. Then, $e(\mathcal{B},v) = \frac{\sum_{S\in\mathcal{B}} c(S) - \big(|\mathcal{B}|-1\big)c(N)}{|\mathcal{B}|}$. For each $t \in M$, we will prove that t is needed at least for all but one coalitions in $\mathcal{B}$. Suppose that it is not true. Then, there exists $S,R \in \mathcal{B}$ such that $t \notin T(S) \cup T(R)$. Moreover, $S \cup R = N$ since $\mathcal{B}$ is an antipartition of N. Then, $T(S) \cup T(R) = M$ which is a contradiction with $t \notin T(S) \cup T(R)$.

Then,

$$\sum_{S\in\mathcal{B}} c(S) = \sum_{S\in\mathcal{B}} \sum_{t\in T(S)} C(t) = \big(|\mathcal{B}|-1\big) \sum_{t\in M} C(t) + \sum_{t\in\bigcap_{S\in\mathcal{B}} T(S)} C(t). \tag{2.2}$$

Suppose that $\bigcap_{S\in\mathcal{B}} T(S) = \emptyset$. Let $\mathcal{B}' = \big\{R, \bigcap_{S\in\mathcal{B}:S\neq R} S\big\}$ with $R \in \mathcal{B}$. It is easy to check that $\mathcal{B}'$ is a partition of N. Moreover, by assumption, $T(\bigcap_{S\in\mathcal{B}:S\neq R} S) \cap T(R) = \emptyset$. Then, (N,M,C,T) can be decomposed in $(R, T(R), C_{|R}, T_{|R})$ and $(\bigcap_{S\in\mathcal{B}:S\neq R} S, T(\bigcap_{S\in\mathcal{B}:S\neq R} S), C_{|\bigcap_{S\in\mathcal{B}:S\neq R} S}, T_{|\bigcap_{S\in\mathcal{B}:S\neq R} S})$, and so, it is completely separable, which is a contradiction. Then, $\bigcap_{S\in\mathcal{B}} T(S) \neq \emptyset$, and $\sum_{t\in\bigcap_{S\in\mathcal{B}} T(S)} C(t) > 0$.

Using the above result and equation (2.2), we conclude that $e(\mathcal{B}, \nu) > 0$. □

Lemma 2.3. *Let (N, c) be a concave TU game.*

(A) Let $S, T \in \mathcal{D}_1(\nu)$. If $S \cap T \neq \emptyset$, then $S \cap T \in \mathcal{D}_1(\nu)$. If $S \cup T \neq N$, then also $S \cup T \in \mathcal{D}_1(\nu)$.

(B) Let $S_1, \ldots, S_k \in \mathcal{D}_1(\nu)$. If $\bigcap_{i=1}^k S_i \neq \emptyset$, then $\bigcap_{i=1}^k S_i \in \mathcal{D}_1(\nu)$. If $\bigcup_{i=1}^k S_i \neq N$, then also $\bigcup_{i=1}^k S_i \in \mathcal{D}_1(\nu)$.

Proof.

(A) See Maschler et al. (1972).[3]

(B) By induction. □

Lemma 2.4. *For all $i \in N$, $\nu_i > 0$.*

Proof. The proof will be done by contradiction. Suppose that there exists $i \in N$ such that $\nu_i = 0$. In that case, $e(N \setminus \{i\}, \nu) = 0$. By concavity, $e(S, \nu) \geq 0$ for all $S \subseteq N$. Therefore, $N \setminus \{i\} \in \mathcal{D}_1(\nu)$. Moreover, $s_{ji}(\nu) = 0$ for all $j \in N \setminus \{i\}$, where $s_{ji}(\nu)$ denotes the minimum surplus of j against i with respect to ν.

Let $j \in N \setminus \{i\}$. Since ν belongs to the prekernel of (N, c), it holds $s_{ij}(\nu) = s_{ji}(\nu)$ $(= 0)$. Then, there exists a coalition $S(j) \subseteq N \setminus \{j\}$ such that $i \in S(j)$ and $e(S(j), \nu) = 0$. Then, $S(j) \in \mathcal{D}_1(\nu)$.

By (B) of Lemma 2.3 it follows $\{i\} = \bigcap_{j\neq i} S(j) \in \mathcal{D}_1(\nu)$. Therefore, $e(\{i\}, \nu) = 0$ which is a contradiction with $e(\{i\}, \nu) = c(\{i\}) > 0$. □

Lemma 2.5. *Let $S \subset N$. If $S \in \mathcal{D}_1(\nu)$, then*

(A) $S \in \mathcal{RC}$ or

(B) $|S| = n - 1$.

Proof. The proof will be done by contradiction. Suppose that there exists $S \in \mathcal{D}_1(\nu)$ such that $S \notin \mathcal{RC}$ and $|S| < n - 1$. Two cases are considered.

Case I. $c(S) = c(N)$.

Let $i \in N \setminus S$ and consider $\bar{S} = S \cup \{i\}$ $(\neq N$, since $|S| < n - 1)$. Then,

$$e(\bar{S}, \nu) = c(\bar{S}) - \nu(\bar{S}) = c(S) - \nu(S) - \nu_i = e(S, \nu) - \nu_i < e(S, \nu),$$

where the inequality follows from Lemma 2.4. Therefore, $S \notin \mathcal{D}_1(\nu)$, which is a contradiction.

[3] They showed it for convex games.

Case II. $c(S) \neq c(N)$.

Since $T(S) \neq M$ and $S \notin \mathcal{RC}$, there exists a finite collection $\{\{a^1, b^1\}, \ldots, \{a^k, b^k\}\} \subset M$ such that,

(a) if $M_\ell = \left\{t \in M \;\middle|\; a^\ell \leq t \leq b^\ell\right\}$ for each $\ell \in \{1, \ldots, k\}$, then $\bigcup_{\ell=1}^k M_\ell = T(S)$,

(b) for each $\ell \in \{1, \ldots, k\}$, there exists $S_\ell \subseteq S$ such that $T(S_\ell) = M_\ell$ and, for each $R \subseteq S$ with $T(R) = M_\ell$, it holds that $R \subseteq S_\ell$,

(c) for each $\ell \in \{1, \ldots, k-1\}$, there exists $t \in M$ such that $b^\ell < t < a^{\ell+1}$.

Let $\ell \in \{1, \ldots, k\}$ and $R_\ell = \{i \in N \mid T(i) \subseteq M_\ell\}$. By definition, $T(R_\ell) = M_\ell$ and, for each $i \in N$ such that $T(i) \subseteq T(R_\ell)$ it holds that $i \in R_\ell$. Then, $R_\ell \in \mathcal{RC}$. Since $c(\bigcup_{\ell=1}^k R_\ell) = c(S) \neq c(N)$, we have that $\bigcup_{\ell=1}^k R_\ell \neq N$.

Suppose that $\bigcup_{\ell=1}^k R_\ell = S$. Let us check that this case is not possible. Since $S \notin \mathcal{RC}$ and $R_\ell \in \mathcal{RC}$ for each $\ell \in \{1, \ldots, k\}$, then $k \geq 2$. Moreover, $c(S) = \sum_{\ell=1}^k c(R_\ell)$ since $T(R_\ell) \cap T(R_p) = \emptyset$ for each $\ell, p \in \{1, \ldots, k\}$ and then $e(S, v) = \sum_{\ell=1}^k e(R_\ell, v)$. Since $S \in \mathcal{D}_1(v)$ and $e(R, v) \geq 0$ for each $R \subset N$, then $e(S, v) = e(R_\ell, v) = 0$. Moreover, we obtain that $e(R, v) = 0$ for each $R \in \mathcal{D}_1(v)$. However, by Theorem 2.1, there exists a partition or an antipartition $\mathcal{B}$ in $\mathcal{D}_1(v)$ and by Lemma 2.2, $e(\mathcal{B}, v) > 0$, which is a contradiction.

Let then $\bigcup_{\ell=1}^k R_\ell \neq S$. Then,

$$
\begin{aligned}
e\left(\bigcup_{\ell=1}^k R_\ell, v\right) &= c(S) - v(S) - v\left(\left(\bigcup_{\ell=1}^k R_\ell\right) \setminus S\right) \\
&= e(S, v) - v\left(\left(\bigcup_{\ell=1}^k R_\ell\right) \setminus S\right) \\
&< e(S, v),
\end{aligned}
$$

where the inequality follows from Lemma 2.4. Therefore, $S \notin \mathcal{D}_1(v)$ which is a contradiction. □

Now, we are in conditions to prove Proposition 2.4.

Proof of Proposition 2.4. By Theorem 2.1, $\mathcal{D}_1(v)$ contains a partition or an antipartition of N so, two cases are considered.

Case I. There exists a partition $\mathcal{P} = \{P_1, \ldots, P_k\} \subseteq \mathcal{D}_1(v)$.

By Lemma 2.5 and by definition of partition, either $\mathcal{P} = \{\{i\}, N \setminus \{i\}\}$ for some $\{i\} \in \mathcal{RC}$, or $P_\ell \in \mathcal{RC}$ for all $\ell \in \{1, \ldots, k\}$. In the former case, $\mathcal{P}$ is also an antipartition of N and the case (C) is true.

Let then $\mathcal{P} = \{P_1, \ldots, P_k\}$ be such that $P_\ell \in \mathcal{RC}$ for all $\ell \in \{1, \ldots, k\}$. Assume, w. l. g., that $t_1 \in T(P_1)$ (recall that M is an ordered set and t_1 represents its first element). By (B) of Lemma 2.3, $P_1^c = \bigcup_{\ell \neq 1} P_\ell \in \mathcal{D}_1(v)$. Moreover, by Lemma 2.5, $|P_1^c| = n-1$ or $P_1^c \in \mathcal{RC}$. If the latter case is true, then $\mathcal{P}' = \{P_1, P_1^c\}$ is a partition of N and $\mathcal{P}' \subseteq \mathcal{D}_1(v)$. Therefore, case (A) is true.

Let see that $|P_1^c| = n-1$ is not possible. Suppose that $|P_1^c| = n-1$, i.e., there exists $i \in N$ such that $P_1 = \{i\}$. By Assumption 2.1, $t_1 \in T(P_1^c)$, then there exists $\ell_0 \neq 1$ such that $T(i) \subseteq T(P_{\ell_0})$. Since $i \notin P_{\ell_0}$, then $P_{\ell_0} \notin \mathcal{RC}$ which is a contradiction.

Case II. There exists an antipartition $\mathcal{A} = \{A_1, \ldots, A_k\} \subseteq \mathcal{D}_1(v)$.

By Lemma 2.5, for each $\ell \in \{1, \ldots, k\}$, $A_\ell \in \mathcal{RC}$ or $|A_\ell| = n-1$. Let r be the number of relevant coalitions in $\mathcal{A}$. The cases (B), (C) and (D) correspond with $r = 2$, $r = 1$, and $r = 0$, respectively. Then, we have to prove that $r \leq 2$.

It will be proven by contradiction. Suppose that $r \geq 3$, consider, w. l. g., that $A_1, A_2, A_3 \in \mathcal{RC}$ and suppose that $t_1 \in T(A_1)$. By definition of antipartition $A_3^c \subseteq A_1 \cap A_2 \neq \emptyset$ and then, $T(A_3^c) \subseteq T(A_1 \cap A_2) \subseteq T(A_1) \cap T(A_2)$. Moreover, $A_1 \cup A_2 = N$ and then $T(A_1) \cup T(A_2) = M$. Since $A_1, A_2 \in \mathcal{RC}$, it follows that $T(A_1) \neq M$, $T(A_2) \neq M$ and $T(A_1) \cap T(A_2) = \{t \in M \mid a_{A_2} \leq t \leq b_{A_1}\}$ with $a_{A_2} \neq t_1$ and $b_{A_1} \neq t_m$, i.e. $T(A_1) \cap T(A_2)$ is in the middle of the resource. Therefore, $T(A_3^c) \subseteq T(A_1) \cap T(A_2)$ is in the middle of the resource.

On the other hand, $M = T(A_3) \cup T(A_3^c)$ and, since $A_3 \in \mathcal{RC}$, $T(A_3) \neq M$. Then, there exist no $a_{A_3}, b_{A_3} \in M$ such that $T(A_3) = \{t \in M \mid a_{A_3} \leq t \leq b_{A_3}\}$. Otherwise,

$$\begin{aligned} M &= T(A_3) \cup T(A_3^c) \subseteq T(A_3) \cup (T(A_1) \cap T(A_2)) \\ &= \{t \in M \mid a_{A_3} \leq t \leq b_{A_3}\} \cup \{t \in M \mid a_{A_2} \leq t \leq b_{A_1}\} \subset M, \end{aligned}$$

where the last strict set inclusion follows from $a_{A_2} \neq t_1$, $b_{A_1} \neq t_m$ and $a_{A_3} \neq t_1$ or $b_{A_3} \neq t_m$ since $T(A_3) \neq M$.

Therefore, $A_3 \notin \mathcal{RC}$ which is a contradiction. □

Proposition 2.4 provides some families of coalitions with minimal excess at v. Then, in order to identify some coalitions in $\mathcal{D}_1(v)$, we have to calculate the excesses at v for such families and to identify the minimum ones. Next proposition provides an easy way to calculate such excesses.

Proposition 2.5. *Let (N, c) be a highway game.*

(A) Let $\mathcal{P} = \{S_1, S_2\}$ be a partition of N with $S_1, S_2 \in \mathcal{RC}$. Then,

$$e(\mathcal{P}, v) = \frac{\sum_{a_{S_2} \leq t \leq b_{S_1}} C(t)}{2}.$$

(B) Let $\mathcal{A} = \{S_1, S_2\} \cup \big\{N \setminus \{i\} \;\big|\; i \in S_1 \cap S_2\big\}$ be an antipartition of N with $S_1, S_2 \in \mathcal{RC}$. Then,

$$e(\mathcal{A}, v) = \frac{\sum_{a_{S_2} \le t \le b_{S_1}} C(t)}{|S_1 \cap S_2| + 2}.$$

(C) Let $\mathcal{A} = \{S\} \cup \big\{N \setminus \{i\} \;\big|\; i \in S\big\}$ be an antipartition of N with $S \in \mathcal{RC}$. Then,

$$e(\mathcal{A}, v) = \frac{\sum_{a_S \le t \le b_S} C(t)}{|S| + 1}.$$

(D) Let $\mathcal{A} = \big\{N \setminus \{i\} \;\big|\; i \in N\big\}$ be an antipartition of N. Then,

$$e(\mathcal{A}, v) = \frac{\sum_{t \in M} C(t)}{|N|}.$$

Proof. (A) Let $\mathcal{P} = \{S_1, S_2\}$ be a partition of N with $S_1, S_2 \in \mathcal{RC}$. By definition of $e(\mathcal{P}, v)$,

$$\begin{aligned}
e(\mathcal{P}, v) &= \frac{\sum_{S \in \mathcal{P}} c(S) - c(N)}{2} \\
&= \frac{c(S_1) + c(S_2) - c(N)}{2} \\
&= \frac{\sum_{a_{S_1} \le t \le b_{S_1}} C(t) + \sum_{a_{S_2} \le t \le b_{S_2}} C(t) - \sum_{t_1 \le t \le t_m} C(t)}{2} \\
&= \frac{\sum_{a_{S_2} \le t \le b_{S_1}} C(t)}{2},
\end{aligned}$$

where the third equality is consequence of $S_1, S_2 \in \mathcal{RC}$, and the fourth one is a consequence of $S_1 \cup S_2 = N$.

(B) Let $\mathcal{A} = \{S_1, S_2\} \cup \big\{N \setminus \{i\} \;\big|\; i \in S_1 \cap S_2\big\}$ be an antipartition of N with $S_1, S_2 \in \mathcal{RC}$. By definition of $e(\mathcal{A}, v)$,

$$\begin{aligned}
e(\mathcal{A}, v) &= \frac{\sum_{S \in \mathcal{A}} c(S) - (|\mathcal{A}| - 1)\, c(N)}{|\mathcal{A}|} \\
&= \frac{c(S_1) + c(S_2) + |S_1 \cap S_2| c(N) - (|S_1 \cap S_2| + 1)\, c(N)}{|S_1 \cap S_2| + 2} \\
&= \frac{c(S_1) + c(S_2) - c(N)}{|S_1 \cap S_2| + 2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sum_{a_{S_1} \leq t \leq b_{S_1}} C(t) + \sum_{a_{S_2} \leq t \leq b_{S_2}} C(t) - \sum_{t_1 \leq t \leq t_m} C(t)}{|S_1 \cap S_2| + 2} \\
&= \frac{\sum_{a_{S_2} \leq t \leq b_{S_1}} C(t)}{|S_1 \cap S_2| + 2},
\end{aligned}$$

where the second equality follows from $c(N \setminus \{i\}) = c(N)$ for each $i \in N$, the fourth one is a consequence of $S_1, S_2 \in \mathcal{RC}$, and the last one follows from $S_1 \cup S_2 = N$.

(C) Let $\mathcal{A} = \{S\} \cup \{N \setminus \{i\} \mid i \in S\}$ be an antipartition of N with $S \in \mathcal{RC}$. By definition of $e(\mathcal{A}, v)$,

$$\begin{aligned}
e(\mathcal{A}, v) &= \frac{\sum_{S \in \mathcal{A}} c(S) - (|\mathcal{A}| - 1)\, c(N)}{|\mathcal{A}|} \\
&= \frac{c(S) + |S|c(N) - |S|c(N)}{|S| + 1} \\
&= \frac{\sum_{a_S \leq t \leq b_S} C(t)}{|S| + 1},
\end{aligned}$$

where the second equality follows from $c(N \setminus \{i\}) = c(N)$ for each $i \in N$, and the third equality is a consequence of $S \in \mathcal{RC}$.

(D) Let $\mathcal{A} = \{N \setminus \{i\} \mid i \in N\}$ be an antipartition of N. By definition of $e(\mathcal{A}, v)$,

$$\begin{aligned}
e(\mathcal{A}, v) &= \frac{\sum_{S \in \mathcal{A}} c(S) - (|\mathcal{A}| - 1)\, c(N)}{|\mathcal{A}|} \\
&= \frac{|N|c(N) - (|N| - 1)\, c(N)}{|N|} \\
&= \frac{\sum_{t \in M} C(t)}{|N|},
\end{aligned}$$

where the second equality follows from $c(N \setminus \{i\}) = c(N)$ for each $i \in N$. □

The numbers given in Proposition 2.5 are the basic tool for the procedure that will be presented in the following subsection. For that reason, we denote:

- for each $S_1, S_2 \in \mathcal{RC}$ s. t. $\{S_1, S_2\}$ is a partition of N,

$$\alpha(S_1, S_2) = \frac{\sum_{t : a_{S_2} \leq t \leq b_{S_1}} C(t)}{2},$$

- for each $S_1, S_2 \in \mathcal{RC}$ s. t. $\{S_1, S_2\} \cup \{N \setminus \{i\} \mid i \in S_1 \cap S_2\}$ is an antipartition of N,

$$\beta(S_1, S_2) = \frac{\sum_{t: a_{S_2} \leq t \leq b_{S_1}} C(t)}{|S_1 \cap S_2| + 2},$$

- for each $S \in \mathcal{RC}$ s. t. $\{S\} \cup \{N \setminus \{i\} \mid i \in S\}$ is an antipartition of N,

$$\gamma(S) = \frac{\sum_{t: a_S \leq t \leq b_S} C(t)}{|S| + 1},$$

- for $\{N \setminus \{i\} \mid i \in N\}$, antipartition of N,

$$\delta = \frac{\sum_{t \in M} C(t)}{|N|}.$$

Denote also the minima of each group of above numbers as

$$\begin{aligned}
\alpha &= \min \{\alpha(S_1, S_2) \mid \{S_1, S_2\} \subseteq \mathcal{RC} \text{ partition}\}, \\
\beta &= \min \left\{ \beta(S_1, S_2) \;\middle|\; \begin{array}{l} S_1, S_2 \in \mathcal{RC}, \\ \{S_1, S_2\} \cup \{N \setminus \{i\} \mid i \in S_1 \cap S_2\} \text{ antipartition} \end{array} \right\}, \\
\gamma &= \min \left\{ \gamma(S) \;\middle|\; \begin{array}{l} S \in \mathcal{RC}, \\ \{S\} \cup \{N \setminus \{i\} \mid i \in S\} \text{ antipartition} \end{array} \right\}.
\end{aligned}$$

By propositions 2.4 and 2.5, the nucleolus for some (or all) agents in N can be easily obtained. Next corollary shows it.

Corollary 2.2. *Let $\lambda = \min\{\alpha, \beta, \gamma, \delta\}$.*

(A) If $\lambda = \beta$, then $\nu_i = \lambda$, for each $i \in \bigcup_{\{S_1, S_2\} \in \mathcal{A}_2(\beta)} (S_1 \cap S_2)$ where

$$\mathcal{A}_2(\beta) = \left\{ \{S_1, S_2\} \subseteq \mathcal{RC} \;\middle|\; \begin{array}{l} \{S_1, S_2\} \cup \{N \setminus \{i\} \mid i \in S_1 \cap S_2\} \textit{ antipartition,} \\ \beta(S_1, S_2) = \beta \end{array} \right\}.$$

(B) If $\lambda = \gamma$, then $\nu_i = \lambda$, for each $i \in \bigcup_{S \in \mathcal{A}_1(\gamma)} S$ where

$$\mathcal{A}_1(\gamma) = \left\{ S \in \mathcal{RC} \;\middle|\; \begin{array}{l} \{S\} \cup \{N \setminus \{i\} \mid i \in S\} \textit{ antipartition,} \\ \gamma(S) = \gamma \end{array} \right\}.$$

(C) If $\lambda = \delta$, then $\nu_i = \lambda$ for each $i \in N$.

Proof. By propositions 2.4, and 2.5, it follows that $e(S, \nu) = \lambda$ for all $S \in \mathcal{D}_1(\nu)$.

(A) Let $\lambda = \beta$ and let $\{S_1, S_2\} \in \mathcal{A}_2(\beta)$. Then, for each $i \in S_1 \cap S_2$, $N \setminus \{i\} \in \mathcal{D}_1(\nu)$ and

$$\lambda = e(N \setminus \{i\}, \nu) = c(N) - \nu(N \setminus \{i\}) = \nu_i.$$

(B) Let $\lambda = \gamma$ and let $\{S\} \in \mathcal{A}_1(\gamma)$. Then, for each $i \in S$, $N \setminus \{i\} \in \mathcal{D}_1(\nu)$ and

$$\lambda = e(N \setminus \{i\}, \nu) = c(N) - \nu(N \setminus \{i\}) = \nu_i.$$

(C) Let $\lambda = \delta$. Then, for each $i \in N$, $N \setminus \{i\} \in \mathcal{D}_1(\nu)$ and

$$\lambda = e(N \setminus \{i\}, \nu) = c(N) - \nu(N \setminus \{i\}) = \nu_i. \qquad \square$$

2.4.2 Algorithm for computing the nucleolus of a highway game

Finally, we propose a procedure to calculate the nucleolus for highway games. We follow the Kopelowitz's algorithm (Kopelowitz, 1967) and we use the above results.

Let (N, M, C, T) be a highway problem, with $|N| \geq 3$, otherwise the problem is straightforward. Kopelowitz's algorithm starts with the following linear program,

$$\begin{aligned} \max \quad & r \\ \text{s. t.} \quad & \\ & r + y(S) \leq c(S) \quad \forall\, S \subset N \\ & y(N) = c(N). \end{aligned} \tag{P1}$$

Using Assumption 2.1 and by Lemma 2.5, the initial problem can be reduced to

$$\begin{aligned} \max \quad & r \\ \text{s. t.} \quad & \\ & r + y(S) \leq c(S) \quad \forall\, S \in \mathcal{RC} \\ & y_i \geq r \quad \forall\, i \in N \\ & y(N) = c(N). \end{aligned} \tag{P2}$$

By propositions 2.4 and 2.5 it is easy to check that the value for r in an optimal solution to (P2) is $r_1 = \lambda = \min\{\alpha, \beta, \gamma, \delta\}$. If $r_1 = \delta$, then $y_i = \nu_i = \delta$ for each $i \in N$ by Corollary 2.2. Hence, the procedure finishes. Otherwise, we have two possibilities: $r_1 = \alpha$ or $r_1 \in \{\beta, \gamma\}$.

(a) Let $r_1 = \alpha$.

In such situation we know that there exists a partition of N, $\{S_1, S_2\}$, such that

$$\begin{aligned} \alpha + y(S_1) &= c(S_1), \\ \alpha + y(S_2) &= c(S_2). \end{aligned}$$

Then, following Kopelowitz's procedure, we reduce Problem (P2) as follows: a new linear program is formed by adding to (P2) the above two constrains, i.e,

$$\begin{aligned} \max \quad & r \\ \text{s. t.} \quad & \\ & r + y(S) \le c(S) \qquad \forall\, S \in \mathcal{RC} \setminus \{S_1, S_2\} \\ & y_i \ge r \qquad \forall\, i \in N \\ & y(S_1) = c(S_1) - \alpha \\ & y(S_2) = c(S_2) - \alpha \\ & y(N) = c(N). \end{aligned} \tag{P3}$$

Moreover, solving (P3) is equivalent to solve the following two disjoint problems.

$$\begin{aligned} \max \quad & r \\ \text{s. t.} \quad & \\ & r + \sum_{i \in S} y_i \le \min\{c(S), c(S \cup S_2) - c(S_2) + \alpha\} \qquad \forall\, S \in \mathcal{RC} \text{ s. t. } S \subset S_1 \\ & y_i \ge r \qquad \forall\, i \in S_1 \\ & \sum_{i \in S_1} y_i = c(S_1) - \alpha \end{aligned} \tag{P4}$$

and

$$\begin{aligned} \max \quad & r \\ \text{s. t.} \quad & \\ & r + \sum_{i \in S} y_i \le \min\{c(S), c(S \cup S_1) - c(S_1) + \alpha\} \qquad \forall\, S \in \mathcal{RC} \text{ s. t. } S \subset S_2 \\ & y_i \ge r \qquad \forall\, i \in S_2 \\ & \sum_{i \in S_2} y_i = c(S_2) - \alpha. \end{aligned} \tag{P5}$$

Let check it. First, we will see how restrictions in Problem (P3) are obtained from restrictions in problems (P4) and (P5). Let $R \subset \mathcal{RC} \setminus \{S_1, S_2\}$. If $R \subset S_1$ (resp. $R \subset S_2$), then the restriction in Problem (P3) for R immediately follows from restriction in Problem (P4) (resp. (P5)) for R. Let then $R \subset \mathcal{RC} \setminus \{S_1, S_2\}$ be such that $R_1 = R \cap S_1 \neq \emptyset$ and

$R_2 = R \cap S_2 \neq \emptyset$. It is straightforward to prove that $R_1, R_2 \in \mathcal{RC}$. Then,

$$\begin{aligned} r + y(R) &\leq 2r + y(R_1) + y(R_2) \\ &\leq \min\{c(R_1), c(R_1 \cup S_2) - c(S_2) + \alpha\} + \min\{c(R_2), c(R_2 \cup S_1) - c(S_1) + \alpha\} \\ &= \sum_{t \in T(R_1) \setminus T(S_2)} C(t) + \sum_{t \in T(R_2) \setminus T(S_1)} C(t) + \min\left\{ \sum_{t \in T(R_1) \cap T(S_2)} C(t), \alpha \right\} \\ &\quad + \min\left\{ \sum_{t \in T(R_2) \cap T(S_1)} C(t), \alpha \right\} \\ &\leq \sum_{t \in T(R_1) \setminus T(S_2)} C(t) + \sum_{t \in T(R_2) \setminus T(S_1)} C(t) + 2\alpha \\ &= c(R) \end{aligned}$$

where the first inequality follows from $r \geq 0$ and $\{S_1, S_2\}$ is a partition of N, and the second one follows from restrictions in problems (P4) and (P5) for R_1 and R_2, respectively.

Next, we will see how restrictions in problems (P4) and (P5) are obtained from restrictions in Problem (P3). Several cases are distinguished.

(a.1) Let $S \in \mathcal{RC}$ be such that $S \subset S_1$ and $c(S) \leq c(S \cup S_2) - c(S_2) + \alpha$. Then,

$$\min\{c(S), c(S \cup S_2) - c(S_2) + \alpha\} = c(S).$$

Therefore, $r + \sum_{i \in S} y_i \leq c(S)$, as in Problem (P3).

Analogously for $S \in \mathcal{RC}$ such that $S \subset S_2$ and $c(S) \leq c(S \cup S_1) - c(S_1) + \alpha$.

(a.2) Let $S \in \mathcal{RC}$ be such that $S \subset S_1$ and $c(S) > c(S \cup S_2) - c(S_2) + \alpha$. Notice that $T(S) \cap T(S_2) \neq \emptyset$. Moreover, by Problem (P3), we obtain that $y(S_2) = c(S_2) - \alpha$ and $y_i \geq r \geq 0$ for each $i \in N$. We also know that either $c(S \cup S_2) = c(N)$ or $c(S \cup S_2) \neq c(N)$.

- Let $c(S \cup S_2) = c(N)$. Since $\{S_1, S_2\}$ is a partition of N, $y(N) = y(S_1) + y(S_2)$ and then, $y(S_1) = c(S \cup S_2) - y(S_2) = c(S \cup S_2) - c(S_2) + \alpha$. Moreover $y(S_1) = y(S) + y(S_1 \setminus S) \geq y(S) + y_i \geq y(S) + r$ for each $i \in S_1 \setminus S$ because $S_1 \setminus S \neq \emptyset$. Thus,

$$y(S) + r \leq y(S_1) = c(S \cup S_2) - c(S_2) + \alpha = \min\{c(S), c(S \cup S_2) - c(S_2) + \alpha\}.$$

- Let $c(S \cup S_2) \neq c(N)$. Then, there exists $S' \in \mathcal{RC}$ such that $S \subseteq S' \subset S_1$, $S' \cup S_2 \in \mathcal{RC}$ and $c(S' \cup S_2) = c(S \cup S_2)$. By Problem (P3), $r + y(S' \cup S_2) \leq c(S' \cup S_2)$. Since

$\{S_1, S_2\}$ is a partition of N, $y(S' \cup S_2) = y(S') + y(S_2)$. Then,

$$\begin{aligned} r + y(S') &\leq c(S' \cup S_2) - y(S_2) \\ &= c(S' \cup S_2) - c(S_2) + \alpha \\ &= c(S \cup S_2) - c(S_2) + \alpha \\ &= \min\{c(S), c(S \cup S_2) - c(S_2) + \alpha\}. \end{aligned}$$

If $S' = S$, then we obtain the inequality in Problem (P4). Otherwise, $S \subset S'$ and $y(S) \leq y(S')$. Therefore,

$$r + y(S) \leq r + y(S') \leq \min\{c(S), c(S \cup S_2) - c(S_2) + \alpha\}$$

and we again obtain the inequality in Problem (P4).

Analogously, for $S \in \mathcal{RC}$ such that $S \subset S_2$ and $c(S) > c(S \cup S_1) - c(S_1) + \alpha$.

(b) Let $r_1 \in \{\beta, \gamma\}$.
By Corollary 2.2 and $r_1 \neq \delta$, we can calculate the nucleolus for a subgroup of players $Z \subset N$.

(b.1) If $r_1 = \beta$ then $y_i = v_i = r_1$ for every $i \in Z_1$ where

$$Z_1 = \bigcup_{\{S_1, S_2\} \in \mathcal{A}_2(\beta)} (S_1 \cap S_2).$$

(b.2) If $r_1 = \gamma$, then $y_i = v_i = r_1$, for every $i \in Z_2$ where

$$Z_2 = \bigcup_{S \in \mathcal{A}_1(\gamma)} S.$$

Then, we can reduce Problem (P1) following Kopelowitz's algorithm as follows. Take $Z = Z_1 \cup Z_2$ as the set of players for which we have already obtained the value of the nucleolus. The new problem is

$$\begin{aligned} \max \quad & r \qquad\qquad \text{(P6)} \\ \text{s. t.} \quad & \\ & r + y(S) \leq \min_{S' \subseteq Z} \left(c(S' \cup S) - r_1|S'|\right) \quad \forall\, S \subset N \setminus Z \\ & y(N \setminus Z) = c(N) - r_1|Z|. \end{aligned}$$

Remark 2.4. Notice that there exists the options where $r_1 = \alpha = \beta$ or $r_1 = \alpha = \gamma$. In such cases it is not important which procedure ((a) or (b)) we apply first, because we will obtain identical results. $\triangleleft$

In both cases, (a) and (b), we have reduced Problem (P2). Next we check that these reductions correspond to new highway problems.

In the case (a), we reduce the original highway problem in two smaller highway problems in the following way. Let $b \in T(S_1) \cap T(S_2)$ be such that $\sum_{t \in T(S_1) \cap T(S_2): t<b} C(t) < \alpha$ and $\sum_{t \in T(S_1) \cap T(S_2): t \leq b} C(t) \geq \alpha$. The highway problem associated with Problem (P4) is (N_1, M_1, C_1, T_1) where $N_1 = S_1$, $M_1 = \{t \in M \mid t \leq b\}$, $C_1 : M_1 \longrightarrow \mathbb{R}_{++}$ is such that

$$C_1(t) = \begin{cases} \alpha - \displaystyle\sum_{s \in T(S_1) \cap T(S_2): s<b} C(s) & \text{if } t = b, \\ C(t) & \text{otherwise,} \end{cases}$$

and $T_1 : N_1 \longrightarrow 2^{M_1}$ is such that $T_1(i) = \{t \in M_1 \mid a_i \leq t \leq \min\{b_i, b\}\}$ for each $i \in N_1$.

Let $a \in T(S_1) \cap T(S_2)$ be such that $\sum_{t \in T(S_1) \cap T(S_2): t<a} C(t) \leq \alpha$ and $\sum_{t \in T(S_1) \cap T(S_2): t \leq a} C(t) > \alpha$. The highway problem associated with Problem (P5) is (N_2, M_2, C_2, T_2) where $N_2 = S_2$, $M_2 = \{t \in M \mid t \geq a\}$, $C_2 : M_2 \longrightarrow \mathbb{R}_{++}$ is such that

$$C_2(t) = \begin{cases} \displaystyle\sum_{s \in T(S_1) \cap T(S_2): s \leq a} C(s) - \alpha & \text{if } t = a, \\ C(t) & \text{otherwise,} \end{cases}$$

and $T_2 : N_2 \longrightarrow 2^{M_2}$ is such that $T_2(i) = \{t \in M_2 \mid \max\{a_i, a\} \leq t \leq b_i\}$ for each $i \in N_2$.

Remark 2.5. Notice that, in above situations, Assumption 2.1 can be not fulfilled. Then, the new problems are not highway problems. Nevertheless, this lack can be easily skipped by taking the following change of variable in Problems (P4) (resp. (P5)): $y'_i = y_i - (c_1(N_1) - c_1(N_1 \setminus \{i\}))$ for each $i \in N_1$ (resp. N_2) where Assumption 2.1 is not fulfilled. Then, the associated highway problems are the above problems without the sections that are used by only one agent. $\triangleleft$

For the case (b), a similar idea is applied: to reduce the sections in $T(Z)$ in function of the payments that are done by players in Z. First, we check that $Z \in \mathcal{RC}$. By Lemma 2.3, $Z \in \mathcal{D}_1(v)$. Using Lemma 2.5, $Z \in \mathcal{RC}$ or $|Z| = n - 1$. But the latter one is not possible. Otherwise, there exists $i \in N$ such that $Z = N \setminus \{i\}$ and

$$r_1 = e(N \setminus \{i\}, v) = c(N \setminus \{i\}) - v(N \setminus \{i\}) = c(N) - (n-1)r_1.$$

Then, $r_1 = \frac{c(N)}{n} = \delta$, which is not possible.

Let then, $Z \in \mathcal{RC}$. By Corollary 2.2, $v_i = \lambda$ for each $i \in Z$. We need some notation for defining the reduced highway problem. Recall that $T(Z) = \{t \in M \mid a_Z \leq t \leq b_Z\}$. Let $Z^c_\ell = \{i \in Z^c \mid a_i < a_Z\}$ be the set of agents in Z^c whose first segment is before the segments in $T(Z)$. Let $Z^c_r = \{i \in Z^c \mid b_i > b_Z\}$ be the set of players in Z^c whose last segment is after the segments in $T(Z)$. Let us note that $Z_\ell^c \neq \emptyset$ or $Z_r^c \neq \emptyset$ since $Z \neq N$. Moreover, $T(Z_\ell^c) \cap T(Z) \neq \emptyset$ or $T(Z_r^c) \cap T(Z) \neq \emptyset$, by the same reason.

Agents in Z leave the problem paying the cost $|Z| \cdot \lambda > 0$ of segments in $T(Z)$. Then, the remainder cost of segments in $T(Z)$ is $C^\lambda(Z) = c(Z) - |Z|\lambda = \lambda > 0$ and it has to be paid by agents in Z^c. We will take the whole resource in $T(Z)$ and we will divide it in a new set of sections. For agents $i \in Z_\ell^c$, we define $\beta_i = \sum_{t \in T(i) \cap T(Z)} C(t)$, and for agents $j \in Z_r^c$ we define $\alpha_j = C^\lambda(Z) - \sum_{t \in T(j) \cap T(Z)} C(t)$. Notice that $Z_\ell^c \cap Z_r^c \neq \emptyset$ can be possible. In such a case, it is easy to check that $\beta_i > C^\lambda(Z)$ and $\alpha_i < 0$. Let $V^\lambda(Z) = \{\beta_i \mid i \in Z_\ell^c, 0 < \beta_i < C^\lambda(Z)\} \cup \{\alpha_j \mid j \in Z_r^c, 0 < \alpha_j < C^\lambda(Z)\} \cup \{C^\lambda(Z)\}$ and let $v = |V^\lambda(Z)|$. If an agent $i \in Z_\ell^c$ is such that $\beta_i \notin V^\lambda(Z)$, then agent i has no sections in $T(Z)$ or the cost of the sections in $T(i) \cap T(Z)$ is greater or equal to $C^\lambda(Z)$ (the same for agents in Z_r^c). Now, take the vector $\xi \in \mathbb{R}_{++}^v$ of all values in $V^\lambda(Z)$ arranged in strict increasing order, i.e, $\xi = (\xi_1, \ldots, \xi_v)$ is such that $\xi_1 < \ldots < \xi_v = C^\lambda(Z)$ and $\xi_i \in V^\lambda(Z)$ for each $i \in N$. Finally, let $NS^\lambda = \{t_1^\lambda, \ldots, t_v^\lambda\}$ be a set of new sections in which we divide $T(Z)$, whose costs are

$$\begin{aligned} C_{Z^c}(t_1^\lambda) &= \xi_1, \\ C_{Z^c}(t_k^\lambda) &= \xi_k - \xi_{k-1} \qquad \text{for each } k \in \{2, \ldots, v\}. \end{aligned}$$

The reduced highway problem associated with Problem (P6) is $(N_{Z^c}, M_{Z^c}, C_{Z^c}, T_{Z^c})$ where $N_{Z^c} = Z^c$, $M_{Z^c} = \big(M \setminus T(Z)\big) \cup NS^\lambda$, $C_{Z^c} : M_{Z^c} \longrightarrow \mathbb{R}_{++}$ is such that

$$C_{Z^c}(t) = \begin{cases} C(t) & \text{if } t \in M \setminus T(Z), \\ \xi_1 & \text{if } t = t_1^\lambda, \\ \xi_k - \xi_{k-1} & \text{if } 1 < k \leq v. \end{cases}$$

and $T_{Z^c} : N_{Z^c} \longrightarrow 2^{M_{Z^c}}$ is such that, for each $i \in Z^c$,

$$T_{Z^c}(i) = \begin{cases} T(i) & \text{if } T(i) \cap T(Z) = \emptyset, \\ \big(T(i) \setminus T(Z)\big) \cup NS^\lambda & \text{if } \begin{array}{l} i \in Z_\ell^c,\ \beta_i \geq C^\lambda(Z) \text{ or} \\ i \in Z_r^c,\ \alpha_i \leq 0, \end{array} \\ \big(T(i) \setminus T(Z)\big) \cup \{t_1^\lambda, \ldots, t_{\ell_i}^\lambda\} & \text{if } i \in Z_\ell^c,\ 0 < \beta_i < C^\lambda(Z), \\ \big(T(i) \setminus T(Z)\big) \cup \{t_{\ell_i+1}^\lambda, \ldots, t_v^\lambda\} & \text{if } i \in Z_r^c,\ 0 < \alpha_i < C^\lambda(Z), \end{cases}$$

with $\ell_i \in \{1, \ldots, v\}$ is such that $\xi_{\ell_i} = \begin{cases} \beta_i \text{ if } i \in Z_\ell^c \\ \alpha_i \text{ if } i \in Z_r^c \end{cases}$.

Notice that Remark 2.5 can be applied to this situation, but it is solved in a similar way.

Summarizing, from a highway problem (N, M, C, T) we obtain one or two reduced highway problems. Then, we again apply the procedure to each new problem until we obtain the nucleolus for all agents in N. The procedure to compute the nucleolus for highway games is represented in Figure 2.3.

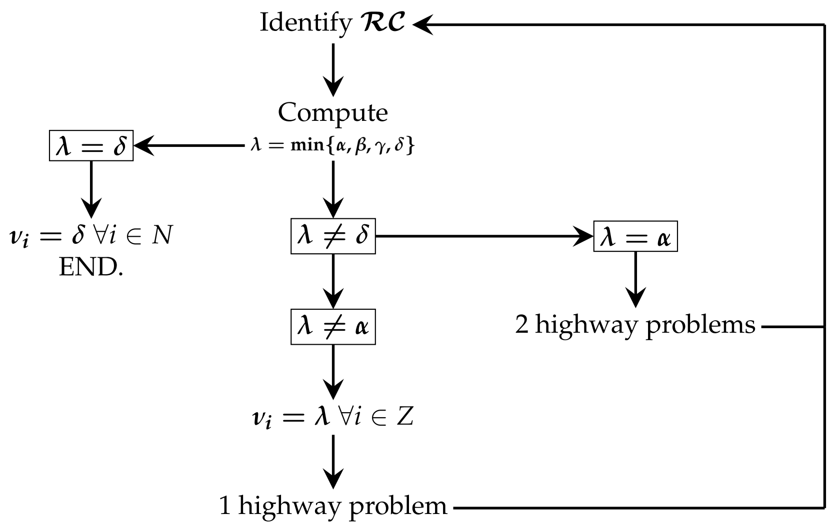

Figure 2.3: Procedure for computing the nucleolus of a highway game.

Finally, we will see with a numerical example how the procedure works.

Example 2.2. Let (N, M, C, T) be the highway problem represented in Figure 2.4, with $N = \{13, 24, 35, 15\}$, $M = \{t_1, t_2, t_3, t_4\}$, $C : M \longrightarrow \mathbb{R}_{++}$ is such that $C(t_1) = 8$, $C(t_2) = 4$ and $C(t_3) = C(t_4) = 6$, and $T : N \longrightarrow 2^M$ is such that $T(13) = \{t_1, t_2\}$, $T(24) = \{t_2, t_3\}$, $T(35) = \{t_3, t_4\}$ and $T(15) = M$. The cost TU game associated with this problem is

S	$\emptyset$	$\{13\}$	$\{24\}$	$\{35\}$	$\{15\}$	$\{13,24\}$	$\{13,35\}$	$\{13,15\}$	$\{24,35\}$	$\{24,15\}$
$c(S)$	0	12	10	12	24	18	24	24	16	24

S	$\{35,15\}$	$\{13,24,35\}$	$\{13,24,15\}$	$\{13,35,15\}$	$\{24,35,15\}$	N
$c(S)$	24	24	24	24	24	24

First, we identify relevant coalitions:

$$\mathcal{RC}(N, c) = \big\{\{13\}, \{24\}, \{35\}, \{13, 24\}, \{24, 35\}\big\}.$$

Next, we seek for partitions and antipartition in conditions of Proposition 2.4 and we compute λ. In this example, there are neither partition in condition (A) of Proposition 2.4 nor antipartition in condition (B) of Proposition 2.4. Then, we only have to compute γ and δ. We have,

$$\gamma(\{13\}) = \frac{12}{2},\ \gamma(\{24\}) = \frac{10}{2},\ \gamma(\{35\}) = \frac{12}{2},\ \gamma(\{13, 24\}) = \frac{18}{3} \text{ and } \gamma(\{24, 35\}) = \frac{16}{3}.$$

Then,

$$\gamma = \min\left\{6, 5, 6, \tfrac{16}{3}\right\} = 5 \quad \text{and} \quad \mathcal{A}_1(5) = \{24\}.$$

On the other hand, $\delta = \frac{24}{4}$. Then,

$$\lambda = \min\{\gamma, \delta\} = 5 \quad \text{and} \quad Z = \mathcal{A}_1(5) = \{24\}.$$

Therefore, $\nu_{24} = 5$.

Now, we proceed to reduce (N, M, C, T). Agents in Z^c has to pay the cost of $T(Z)$ minus the cost that was paid by agent 24, i.e,

$$C^5(\{24\}) = c(\{24\}) - \nu_{24} = 10 - 5 = 5.$$

The set of agents to the left of $T(Z)$ is $Z^c_\ell = \{13, 15\}$ and the set of agents to the right of $T(Z)$ is $Z^c_r = \{35, 15\}$. Then,

$$\begin{aligned}
\beta_{13} &= \sum_{t \in T(\{13\}) \cap T(\{24\})} C(t) = 4, \\
\beta_{15} &= \sum_{t \in T(\{15\}) \cap T(\{24\})} C(t) = 10, \\
\alpha_{35} &= C^5(\{24\}) - \sum_{t \in T(\{35\}) \cap T(\{24\})} C(t) = 5 - 6 = -1, \\
\alpha_{15} &= C^5(\{24\}) - \sum_{t \in T(\{15\}) \cap T(\{24\})} C(t) = 5 - 10 = -5,
\end{aligned}$$

and

$$V^5(\{24\}) = \left\{\beta_i \,\middle|\, i \in Z^c_\ell, 0 < \beta_i < C^5(\{24\})\right\} \cup \left\{\alpha_j \,\middle|\, j \in Z^c_r, 0 < \alpha_j < C^5(\{24\})\right\} \cup \left\{C^5(\{24\})\right\} = \{4, 5\}.$$

The vector of elements in $V^5(\{24\})$ arranged in strict increasing order is $\xi = (4, 5)$. Then, the reduced highway problem is $(N_{Z^c}, M_{Z^c}, C_{Z^c}, T_{Z^c})$ such that $N_{Z^c} = \{13, 35, 15\}$, $M_{Z^c} = \{t_1, t_1^5, t_2^5, t_4\}$, $C_{Z^c} : M_{Z^c} \longrightarrow \mathbb{R}_{++}$ is such that $C_{Z^c}(t_1) = C(t_1) = 8$, $C_{Z^c}(t_1^5) = \xi_1 = 4$, $C_{Z^c}(t_2^5) = \xi_2 - \xi_1 = 1$ and $C_{Z^c}(t_4) = C(t_4) = 6$, and $T_{Z^c} : N_{Z^c} \longrightarrow 2^{M_{Z^c}}$ is such that

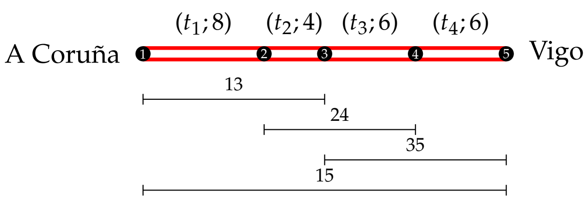

Figure 2.4: Linear highway of Example 2.2.

$T_{Z^c}(13) = \{t_1, t_1^5\}$, $T_{Z^c}(35) = \{t_1^5, t_2^5, t_4\}$ and $T_{Z^c}(15) = M_{Z^c}$. The problem is represented in Figure 2.5 and the cost TU game associated with this problem is

S	$\emptyset$	$\{13\}$	$\{35\}$	$\{15\}$	$\{13,35\}$	$\{13,15\}$	$\{35,15\}$	N_{Z^c}
$c_{Z^c}(S)$	0	12	11	19	19	19	19	19

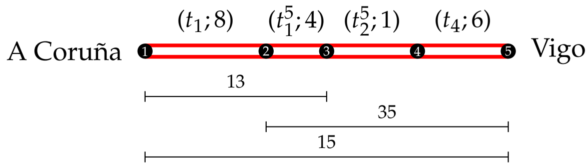

Figure 2.5: Second linear highway of Example 2.2.

Note that, the new problem is different from the original one. A sign of it is that agents 13 and 35, in the new problem, have sections in common, but in the original problem they did not have them.

For this new problem we follow the same procedure as before for computing the nucleolus. The set of relevant coalitions is

$$\mathcal{RC}(N_{Z^c}, c_{Z^c}) = \{\{13\}, \{35\}\}.$$

Then,

$$\gamma = \min\left\{6, \frac{11}{2}\right\} = 5.5,\ \mathcal{A}_1(\gamma) = \{35\} \text{ and } \delta = \frac{19}{3}.$$

Then, $\lambda = 5.5$, $\overline{Z} = \{35\}$ and $\nu_{35} = 5.5$.

The reduced problem that one can obtain from this problem is the following. From the procedure we obtain,

$$C^{5.5}(\overline{Z}) = 5.5,$$
$$\overline{Z}^c_\ell = \{13, 15\} \text{ and } \overline{Z}^c_r = \emptyset,$$
$$\beta_{13} = 4 \text{ and } \beta_{15} = 11,$$
$$V^{5.5}(\overline{Z}) = \{4, 5.5\} \text{ and } \xi = (4, 5.5).$$

Then, the reduced highway problem is $(N_{\overline{Z}^c}, M_{\overline{Z}^c}, C_{\overline{Z}^c}, T_{\overline{Z}^c})$ such that $N_{\overline{Z}^c} = \{13, 15\}$, $M_{\overline{Z}^c} = \{t_1, t_1^{5.5}, t_2^{5.5}\}$, $C_{\overline{Z}^c} : M_{\overline{Z}^c} \longrightarrow \mathbb{R}_{++}$ is such that $C_{\overline{Z}^c}(t_1) = C_{Z^c}(t_1) = 8$, $C_{\overline{Z}^c}(t_1^{5.5}) = \xi_1 = 4$ and $C_{\overline{Z}^c}(t_2^{5.5}) = \xi_2 - \xi_1 = 1.5$, and $T_{\overline{Z}^c} : N_{\overline{Z}^c} \longrightarrow 2^{M_{\overline{Z}^c}}$ is such that $T_{\overline{Z}^c}(13) = \{t_1, t_1^{5.5}\}$ and $T_{\overline{Z}^c}(15) = M_{\overline{Z}^c}$. The problem is represented in Figure 2.6 and the cost TU game associated with this problem is

S	$\emptyset$	$\{13\}$	$\{15\}$	$N_{\overline{Z}^c}$
$c_{\overline{Z}^c}(S)$	0	12	13.5	13.5

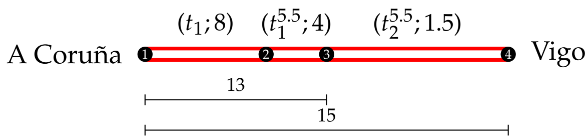

Figure 2.6: Third linear highway of Example 2.2.

Notice that, $(N_{\overline{Z}^c}, M_{\overline{Z}^c}, C_{\overline{Z}^c}, T_{\overline{Z}^c})$ does not correspond to a highway problem because Assumption 2.1 is not fulfilled. However, $(N_{\overline{Z}^c}, M_{\overline{Z}^c}, C_{\overline{Z}^c}, T_{\overline{Z}^c})$ corresponds to an airport problem and we know that $\nu_{13} = 6$ and $\nu_{15} = 7.5$.

Therefore, the nucleolus for (N, c) is

$$\nu = (6, 5, 5.5, 7.5).$$

If the latter problem would neither correspond to an airport problem, we could drop out all the segments for which Assumption 2.1 is not fulfilled, taking into account that, for each one of these segments, there exists only one agent who use it and this agent will pay the cost of such segment. Thus, we would obtain a highway problem in our setting.

We can compare the nucleolus with the Shapley value and the compromise value. Following the formulas given in Section 2.3, the Shapley value is

$$\Phi(N,c) = \left(\frac{16}{3}, \frac{10}{3}, 5, \frac{31}{3}\right),$$

and the compromise value is

$$\tau(N,c) = \left(\frac{144}{29}, \frac{120}{29}, \frac{144}{29}, \frac{288}{29}\right). \qquad \diamond$$

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Chapter 3

Proportionate flow shop games

Contents

3.1 Introduction

In a flow shop problem a group of jobs has to be processed through a fixed number of machines and the order of the machines in which the jobs have to be processed is the same for all jobs. To each job a cost is associated dependent on its completion time. In this chapter we will consider proportionate flow shop problems. A proportionate flow shop problem is a flow shop problem in which additionally every job has the same processing time on each machine (e.g. a wooden door needs several layers of paint, each with a different product, but the painting time is always the same). Proportionate flow shop problems have gained considerable attention lately and various papers have been devoted to this topic. In Shakhlevich et al. (1998) an algorithm is provided to obtain an optimal schedule for this kind of problems. Shiau and Huang (2004) generalize this type of problems by considering multiple identical machines at any stage. In Allahverdi (1996) and Allahverdi and Savsar (2001) proportionate flow shop problems with breakdowns and setup times are studied, respectively. Cheng and Shakhlevich (1999) propose algorithms for proportionate flow shop problems where the processing times can be controlled by incurring extra costs.

By associating jobs to clients, a proportionate flow shop problem gives rise to an interactive decision making problem. Each client incurs costs, which we assume to depend linearly on the completion time of its job. By assuming an initial order on the jobs, the first problem the clients jointly face is an optimization problem: the problem of finding an optimal reordering of all jobs, i.e., a schedule that maximizes joint cost savings. The subsequent problem is of a game theoretic nature: how to reallocate these cost savings in a fair way. By defining the value of a coalition of clients as the maximal attainable costs savings by means of an optimal admissible reordering, we obtain a cooperative proportionate flow shop game (a PFS game) related to the proportionate flow shop problem. The core of this game provides insight in the allocation problem at hand since core elements lead to a stable reallocation of the joint cost savings. A game is said to be balanced if it has a non-empty core.

The above game-theoretic approach to sequencing situations has been initiated by Curiel et al. (1989) for the class of one-machine sequencing situations. Generalizations to e.g. ready times, due dates, multiple ownership and more machines have been studied in Hamers et al. (1995); Borm et al. (2002); Calleja et al. (2006); Estévez-Fernández et al. (2004); Hamers et al. (1999); Calleja et al. (2002). A recent review on sequencing games can be found in Curiel et al. (2002). Finally, within the context of flow shop problems, van den Nouweland et al. (1990) and van den Nouweland (1993) have studied the specific case of a dominant machine.

This chapter analyzes proportionate flow shop problems and related PFS games. It is shown that PFS games are balanced. Moreover, PFS games turn out to be convex if the initial order is the urgency order, in which case the Shapley value is in the core of the game. We provide an explicit expression for the Shapley value. Under this assumption, we also provide a context-specific allocation rule (the γ-rule) in the same spirit as the equal gain splitting (EGS) rule introduced in

Curiel et al. (1989). This allocation rule follows the algorithm in Shakhlevich et al. (1998). In this way, the optimization problem of determining the optimal order of the grand coalition and the allocation problem of how to share joint savings can be solved in an integrated way.

The remainder of the chapter is organized as follows. Section 3.2 provides the basic definitions and terminology of proportionate flow shop problems. Moreover, two useful results in Shakhlevich et al. (1998) are recalled. Section 3.3 deals with cooperation within proportionate flow shop problems. The γ-rule is introduced as a specific allocation rule of the maximal joint cost savings. In Section 3.4 PFS games are defined. It is shown that these games are convex provided that the initial order is an urgency order and an expression of the Shapley value is provided. Moreover, it is seen that in this case also the γ-rule will provide a core element.

3.2 Proportionate flow shop problems

A flow shop situation consists of a fixed sequence of m machines, and a finite set of jobs N that have to be processed on all machines. A proportionate flow shop (PFS) situation is a flow shop situation where the processing time of every job is the same on each machine. Hence, a PFS situation can be described by a 3-tuple (M, N, p) with $M = \{M_1, \ldots, M_m\}$ the set of machines, $N = \{1, \ldots, n\}$ the set of jobs, and $p \in \mathbb{R}^n_{++}$ the vector of processing times of the jobs.

A *schedule* fixes for every job i and every machine r a time interval of length p_i in which job i will be processed in such a way that neither a job is processed on two different machines at the same time, nor a machine processes two different jobs at the same time. Given a PFS situation (M, N, p) we denote a schedule of the jobs in the machines as $\sigma = (\sigma^1, \ldots, \sigma^m)$ with $\sigma^r : N \to \{1, \ldots, |N|\}$ a bijection describing the processing order in machine M_r. We will denote by $\Pi(N, M)$ the set of all schedules of the jobs in the machines. Given $\sigma \in \Pi(N, M)$, $i \in N$, and $M_r \in M$, we denote by $P(\sigma^r, i)$ the *set of predecessors of job i in machine* M_r, i.e., $P(\sigma^r, i) = \{j \in N | \sigma^r(j) < \sigma^r(i)\}$. Further, we define $\bar{P}(\sigma^r, i) := P(\sigma^r, i) \cup \{i\}$. We denote by $p(\sigma^r, i)$ the *immediate predecessor of job i in machine* M_r, i.e., $p(\sigma^r, i) \in N$ such that $\bar{P}(\sigma^r, p(\sigma^r, i)) = P(\sigma^r, i)$. Note that in principle the order in machines need not be the same. A schedule $\sigma = (\sigma^1, \ldots, \sigma^m)$ with $\sigma^1 = \ldots = \sigma^m$ is called a *permutation schedule* or *order*. With minor abuse of notation, σ will then denote the order in each machine. We will denote by $\Pi(N)$ the set of all permutations schedules of the jobs.

Assuming that processing starts at time 0 and that there are no unnecessary delays, the *completion time of job i* in machine M_r *with respect to* an arbitrary *schedule* σ, $C_i^\sigma(r)$, can be recursively determined by

$$C_i^\sigma(1) = \sum_{j \in \bar{P}(\sigma^1, i)} p_j$$

and for $r = 2, \ldots, m$,

$$C_i^\sigma(r) = \begin{cases} C_i^\sigma(r-1) + p_i & \text{if } P(\sigma^r, i) = \emptyset, \\ \max\{C^\sigma_{p(\sigma^r,i)}(r), C_i^\sigma(r-1)\} + p_i & \text{otherwise.} \end{cases}$$

It is assumed that each job $i \in N$ incurs costs, c_i, which are linear with respect to the time in which the job leaves the system according to the schedule σ. Hence, there exist positive numbers α_i, $i \in N$, such that $c_i(\sigma) = \alpha_i C_i^\sigma(m)$. From now on we will denote the overall completion time $C_i^\sigma(m)$ by C_i^σ.

Given a PFS situation (M, N, p) and a linear cost associated to each job, which will be represented by $\alpha \in \mathbb{R}^n$, the associated *PFS problem*, (M, N, p, α) has as objective to find a schedule that minimizes the total cost originated in the system, i.e., find $\hat{\sigma}$ such that

$$c_N(\hat{\sigma}) = \min_{\sigma \in \Pi(N,M)} c_N(\sigma)$$

with $c_N(\sigma) = \sum_{i \in N} c_i(\sigma) = \sum_{i \in N} \alpha_i C_i^\sigma$. Note that $\Pi(N, M)$ is finite and therefore there exists at least one optimal solution.

Next, we will recall three lemmas from Shakhlevich et al. (1998) that will be used throughout the chapter.

Lemma 3.1 (Shakhlevich et al. (1998))**.** *Let (M, N, p, α) be a PFS problem. Then,*

(i) Every optimal schedule is a permutation schedule.

(ii) For a permutation schedule σ and $i \in N$, the completion time C_i^σ is given by

$$C_i^\sigma = \sum_{j \in \bar{P}(\sigma,i)} p_j + (m-1) \max_{j \in \bar{P}(\sigma,i)} \{p_j\}.$$

Since every optimal schedule is a permutation schedule, we will restrict our study to permutation schedules from now on.

Example 3.1. Let (M, N, p, α) be a PFS problem with machines $M = \{M_1, M_2\}$, jobs $N = \{1, 2, 3, 4\}$, vector of processing times $p = (4, 5, 6, 1)$, and vector of cost coefficients $\alpha = (32.5, 32, 32, 5)$. Let $\sigma = (1\ 2\ 3\ 4)$ be a permutation schedule. This situation is represented in Figure 3.1.

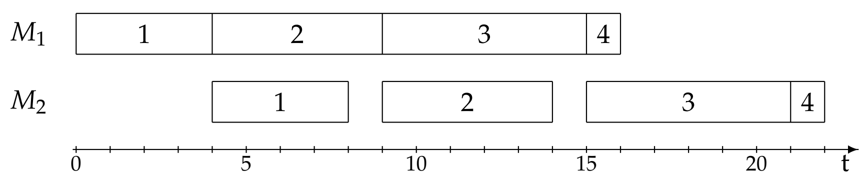

Figure 3.1: Gantt Chart of the PFS situation in Example 3.1

Here, $C_1^\sigma = 8$, $C_2^\sigma = 14$, $C_3^\sigma = 21$ and $C_4^\sigma = 22$. We illustrate how to calculate C_3^σ below.

$$\begin{aligned} C_3^\sigma &= p_1 + p_2 + p_3 + (m-1)\max\{p_1, p_2, p_3\} \\ &= 4 + 5 + 6 + (2-1)\max\{4, 5, 6\} = 21. \end{aligned}$$

Hence, the total weighted completion time according to σ is $c_N(\sigma) = 1490$. $\diamond$

Since the processing time of a job is the same in all machines, we can define an *urgency* (index) of job $i \in N$ as $u_i = \frac{\alpha_i}{p_i}$. The next lemma states that if a job has higher urgency than another with larger processing time, then the one with higher urgency will be processed first in an optimal order.

Lemma 3.2 (Shakhlevich et al., 1998)**.** *Let (M, N, p, α) be a PFS problem and σ an optimal order. If $i, j \in N$ are such that $u_i \geq u_j$ and $p_i < p_j$ or $u_i > u_j$ and $p_i \leq p_j$, then $\sigma(i) < \sigma(j)$.*

Let (M, N, p, α) be a PFS problem and let $\sigma \in \Pi(N)$. We say that job $i \in N$ is a *new-max job according to σ* if $p_i > \max_{j \in P(\sigma,i)}\{p_j\}$. Let $a_1^\sigma, \ldots, a_s^\sigma$ be the new-max jobs according to σ, with $\sigma(a_1^\sigma) < \ldots < \sigma(a_s^\sigma)$. Then, N can be partitioned into s so-called *segments* $A_1^\sigma, \ldots, A_s^\sigma$ in the following way

$$A_r^\sigma := \begin{cases} P(\sigma, a_{r+1}^\sigma) \setminus P(\sigma, a_r^\sigma) & \text{if } 1 \leq r < s, \\ N \setminus P(\sigma, a_r^\sigma) & \text{if } r = s. \end{cases}$$

Note that, since $\sigma(a_1^\sigma) = 1$, $P(\sigma, a_1^\sigma) = \emptyset$. The above partition into segments is denoted by $Seg(\sigma)$.

The lemma below states that in any optimal order the jobs in a segment are processed in decreasing urgency order.

Lemma 3.3 (Shakhlevich et al., 1998)**.** *Let (M, N, p, α) be a PFS problem and σ an optimal order. Let A_r^σ be a segment corresponding to σ and $i, j \in A_r^\sigma$. If $\sigma(i) < \sigma(j)$, then $u_i \geq u_j$.*

3.3 Cooperation in proportionate flow shops

In this section we will recall the algorithm to find an optimal schedule for PFS problems given in Shakhlevich et al. (1998) and propose an allocation rule to share the costs savings obtained by reordering the jobs into an optimal order if the initial order is in decreasing urgency order.

We first describe the algorithm in Shakhlevich et al. (1998). Let (M, N, p, α) be a PFS problem. We define the *urgency order*, σ_u, as the order in which the jobs are ordered in decreasing urgency. Since the starting point of the algorithm is σ_u, we can assume without loss of generality that $\sigma_u = (1 \ \ldots \ n)$. To find the optimal order we will generate orders $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$ where $\hat{\sigma}_1 := \sigma_u$ and $\hat{\sigma}_n$ is optimal. Note that associated to the order $\hat{\sigma}_{i-1}$ we have the segments $A_1^{\hat{\sigma}_{i-1}}, \ldots, A_s^{\hat{\sigma}_{i-1}}$ which give a partition of N. Now, we explain how to obtain $\hat{\sigma}_i$ from $\hat{\sigma}_{i-1}$. Let $s_i \in \{1, \ldots, s\}$ be such that $A_{s_i}^{\hat{\sigma}_{i-1}} \cap \{1,\ldots,i-1\} \neq \emptyset$ and $A_{s_i+1}^{\hat{\sigma}_{i-1}} \cap \{1,\ldots,i-1\} = \emptyset$. We define $A(i,1),\ldots,A(i,s_i)$ as $A(i,1) = A_{s_i}^{\hat{\sigma}_{i-1}} \cap \{1,\ldots,i-1\}$ and $A(i,r) = A_{s_i-r+1}^{\hat{\sigma}_{i-1}}$ for $r = 2,\ldots,s_i$.

Here, we have numbered the segments from right to left (instead from left to right) for convenience of the description of the rule that we will give later on. Subsequently, $\hat{\sigma}_i$ is obtained from $\hat{\sigma}_{i-1}$ by placing i in first position or in between two consecutive segments or remain in its initial position. The decision will be taken in such a way that $c_N(\hat{\sigma}_i)$ is minimal and $\max_{k\in\bar{P}(\hat{\sigma}_i,i)}\{p_k\}$ is maximal.

Now we turn to interactive proportionate flow shop situations and assume that each job belongs to a player. We define the γ-rule which allocates the gains $\sum_{i\in N}(c_N(\hat{\sigma}_{i-1}) - c_N(\hat{\sigma}_i))$. Here, we will decompose the gain $c_N(\hat{\sigma}_{i-1}) - c_N(\hat{\sigma}_i)$ into "positive jumps" and the associated "positive gains" will be shared among the jobs involved. For this, we will need some additional notation. We define

$$g_{A(i,r)i} := \sum_{j\in A(i,r)} (\alpha_i p_j - \alpha_j p_i) + \alpha_i(m-1)\big(\max_{j\in A(i,r)\cup\{i\}}\{p_j\} - \max_{j\in A(i,r+1)\cup\{i\}}\{p_j\}\big), \tag{3.1}$$

for $r = 1,\ldots,s_i$, with $A(i,s_i+1) := \emptyset$.

Hence, $g_{A(i,r)i}$ represents the cost savings obtained when job i goes from the tale of $A(i,r)$ to its front. Note that $g_{A(i,r)i}$ can be negative. Define $N(i,1) := A(i,1)$, $g_{N(i,1)} := g_{A(i,1)i}$, and $h_{N(i,1)} := (g_{N(i,1)})_+$. For $r = 2,\ldots,s_i$ we define recursively

$$N(i,r) := \begin{cases} N(i,r-1)\cup A(i,r) & \text{if } h_{N(i,r-1)} = 0, \\ A(i,r) & \text{otherwise,} \end{cases}$$

$$g_{N(i,r)} := \begin{cases} g_{N(i,r-1)} + g_{A(i,r)i} & \text{if } h_{N(i,r-1)} = 0, \\ g_{A(i,r)i} & \text{otherwise,} \end{cases}$$

and

$$h_{N(i,r)} := (g_{N(i,r)})_+.$$

Easily, $c(\hat{\sigma}_{i-1}) - c(\hat{\sigma}_i) = \sum_{r=1}^{s_i} h_{N(i,r)}$ and therefore $\sum_{i\in N}\sum_{r=1}^{s_i} h_{N(i,r)}$ gives the total cost savings gained by means of cooperation. The γ-rule simply gives half of $h_{N(i,r)}$ to i while the other half is shared equally among the jobs in $N(i,r)$ for each $i\in N$ and $1\le r\le s_i$. Formally, we define

$$\gamma(M,N,p,\alpha) = \sum_{i\in N}\sum_{r=1}^{s_i}\left(\frac{h_{N(i,r)}}{2}e^{\{i\}} + \frac{h_{N(i,r)}}{2|N(i,r)|}e^{N(i,r)}\right)$$

with $e^R \in \mathbb{R}^n$ a vector of zeros and ones with $e_i^R = 1$ if $i\in R$ and $e_i^R = 0$ otherwise, for $R\subseteq N$.

The following example illustrates the computation of the γ-rule.

Example 3.2. Let (M,N,p,α) be a PFS problem with machines $M = \{M_1,M_2,M_3\}$, jobs $N = \{1,2,3,4,5,6,7,8,9\}$, vector of processing times $p = (20,30,10,30,10,30,20,10,40)$ and vector of cost coefficients $\alpha = (200,270,80,210,69,180,130,59,200)$. Hence, the urgency order is $\sigma_u =$

$(1\,2\,3\,4\,5\,6\,7\,8\,9)$. Suppose that initially the jobs are processed according to the urgency order. Then, $c_N(\sigma_u) = 224320$. The situation is represented in Figure 3.2.

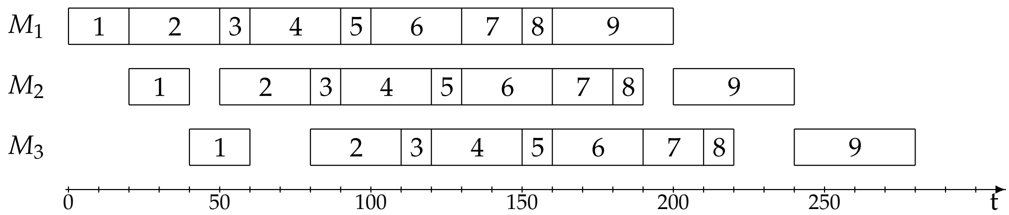

Figure 3.2: Gantt Chart of the PFS situation in Example 3.2.

The allocation of the total cost savings after reordering the jobs in the optimal order is summarized in Table 3.1.

	1	2	3	4	5	6	7	8	9
$i=1$	0	0	0	0	0	0	0	0	0
$i=2$	0	0	0	0	0	0	0	0	0
$i=3$	600	650	$650+600$	0	0	0	0	0	0
$i=4$	0	0	0	0	0	0	0	0	0
$i=5$	380	180	0	180	$360+380$	0	0	0	0
$i=6$	0	0	0	0	0	0	0	0	0
$i=7$	0	$\frac{550}{3}$	0	$\frac{550}{3}$	0	$\frac{550}{3}$	550	0	0
$i=8$	13	13	0	13	0	13	13	65	0
$i=9$	0	0	0	0	0	0	0	0	0
	993	$1026\frac{1}{3}$	1250	$376\frac{1}{3}$	740	$196\frac{1}{3}$	563	65	0

Table 3.1: Allocation of the cost savings in Example 3.2.

We explain below how the cost savings are shared when jobs 5 and 8 are reordered.

First, we will study the case in which job 5 is reordered. We leave it to the reader to verify that the order obtained after reordering jobs 1, 2, 3 and 4 is $\hat{\sigma}_4 = (3\,1\,2\,4\,5\,6\,7\,8\,9)$ and $Seg(\hat{\sigma}_4) = \{\{3\},\{1\},\{2,4,5,6,7,8\},\{9\}\}$. Take $i = 5$ and as previous order $\hat{\sigma}_4$. Hence, $A(5,1) = \{2,4\}$, $A(5,2) = \{1\}$, and $A(5,3) = \{3\}$. Moreover,

$$
\begin{aligned}
g_{A(5,1)5} &= c_N(\hat{\sigma}_4) - c_N(\tau_5^1) \\
&= (\alpha_2 C_2^{\hat{\sigma}_4} + \alpha_4 C_4^{\hat{\sigma}_4} + \alpha_5 C_5^{\hat{\sigma}_4}) - (\alpha_2 C_2^{\tau_5^1} + \alpha_4 C_4^{\tau_5^1} + \alpha_5 C_5^{\tau_5^1})
\end{aligned}
$$

$$= \quad 720,$$

with $\tau_5^1 = (3\ 1\ 5\ 2\ 4\ 6\ 7\ 8\ 9)$,

$$\begin{aligned} g_{A(5,2)5} &= c_N(\tau_5^1) - c_N(\tau_5^2) \\ &= (\alpha_1 C_1^{\tau_5^1} + \alpha_5 C_5^{\tau_5^1}) - (\alpha_1 C_1^{\tau_5^2} + \alpha_5 C_5^{\tau_5^2}) \\ &= 760, \end{aligned}$$

with $\tau_5^2 = (3\ 5\ 1\ 2\ 4\ 6\ 7\ 8\ 9)$, and

$$\begin{aligned} g_{A(5,3)5} &= c_N(\tau_5^2) - c_N(\tau_5^3) \\ &= (\alpha_3 C_3^{\tau_5^2} + \alpha_5 C_5^{\tau_5^2}) - (\alpha_3 C_3^{\tau_5^3} + \alpha_5 C_5^{\tau_5^3}) \\ &= -110, \end{aligned}$$

with $\tau_5^3 = (5\ 3\ 1\ 2\ 4\ 6\ 7\ 8\ 9)$.
Hence, $N(5,1) := A(5,1) = \{2,4\}$, $g_{N(5,1)} = 720$, $h_{N(5,1)} = 720$, $N(5,2) = \{1\}$, $g_{N(5,2)} = 760$, $h_{N(5,2)} = 760$, $N(5,3) = \{3\}$, $g_{N(5,3)} = -110$, $h_{N(5,3)} = 0$. In this step, the owner of job 5 gets 360 from $h_{N(5,1)}$ and the owners of jobs in $N(5,1)$ share equally 360, i.e., 2 and 4 get 180 each. Similarly, the owner of job 5 gets 380 from $h_{N(5,2)}$ and the owner of the job in $N(5,2)$ gets 380, i.e., 1 gets 380.

Hence, an optimal order after reallocating 5 is $\hat{\sigma}_5 = \tau_5^2 = (3\ 5\ 1\ 2\ 4\ 6\ 7\ 8\ 9)$ and the cost savings obtained after this reorder are $h_{N(5,1)} + h_{N(5,2)} + h_{N(5,3)} = 1480$.

Next, we will study the case in which job 8 is reordered. In this case, $\hat{\sigma}_7 = (3\ 5\ 1\ 7\ 2\ 4\ 6\ 8\ 9)$ and $Seg(\hat{\sigma}_7) = \{\{3,5\},\{1,7\},\{2,4,6,8\},\{9\}\}$. Take $i = 8$ and as previous order $\hat{\sigma}_7$. Here, $A(8,1) = \{2,4,6\}$, $A(8,2) = \{1,7\}$, and $A(8,3) = \{3,5\}$. Moreover,

$$\begin{aligned} g_{A(8,1)8} &= c_N(\hat{\sigma}_7) - c_N(\tau_8^1) \\ &= (\alpha_2 C_2^{\hat{\sigma}_7} + \alpha_4 C_4^{\hat{\sigma}_7} + \alpha_6 C_6^{\hat{\sigma}_7} + \alpha_8 C_8^{\hat{\sigma}_7}) - (\alpha_2 C_2^{\tau_8^1} + \alpha_4 C_4^{\tau_8^1} + \alpha_6 C_6^{\tau_8^1} + \alpha_8 C_8^{\tau_8^1}) \\ &= -110, \end{aligned}$$

with $\tau_8^1 = (3\ 5\ 1\ 7\ 8\ 2\ 4\ 6\ 9)$,

$$\begin{aligned} g_{A(8,2)8} &= c_N(\tau_8^1) - c_N(\tau_8^2) \\ &= (\alpha_1 C_1^{\tau_8^1} + \alpha_7 C_7^{\tau_8^1} + \alpha_8 C_8^{\tau_8^1}) - (\alpha_1 C_1^{\tau_8^2} + \alpha_7 C_7^{\tau_8^2} + \alpha_8 C_8^{\tau_8^2}) \\ &= 240, \end{aligned}$$

with $\tau_8^2 = (3\ 5\ 8\ 1\ 7\ 2\ 4\ 6\ 9)$.

Note that job 8 can not be reallocated in an earlier position since it would violate Lemma 3.2.

Hence, $N(8,1) := A(8,1) = \{2,4,6\}$, $g_{N(8,1)} = -110$, $h_{N(8,1)} = 0$, $N(8,2) = \{1,2,4,6,7\}$, $g_{N(8,2)} = -110 + 240 = 130$, $h_{N(8,2)} = 130$, $N(8,3) = \{3,5\}$, $g_{N(8,3)} < 0$, $h_{N(8,3)} = 0$. In this step, the owner of job 8 gets 65 from $h_{N(8,2)}$ and the owners of jobs in $N(8,2)$ share equally 65, i.e., 1, 2, 4, 6 and 7 get 13 each.

Hence, an optimal order after reallocating 8 is $\hat{\sigma}_8 = \tau_8^2 = (3\ 5\ 8\ 1\ 7\ 2\ 4\ 6\ 9)$ and the cost savings obtained after this reorder are $h_{N(8,1)} + h_{N(8,2)} + h_{N(8,3)} = 130$. $\diamond$

3.4 Proportionate flow shop games

In this section we study proportionate flow shop games and show that they are balanced. Moreover, if the initial order is the urgency order, then they are convex and an explicit expression of the Shapley value is provided based on the decomposition of the proportionate flow shop games into unanimity games. Besides, it is shown that the γ-rule leads to a core element.

Before stating our main results we will recall some basic notions from cooperative game theory.

A cooperative TU game in characteristic function form is an ordered pair (N, v) where N is a finite set (the set of players) and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. The *core* of a cooperative TU game (N, v) is defined by

$$Core(v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\},$$

i.e., the core is the set of efficient allocations of $v(N)$ such that there is no coalition with an incentive to split off. A game is said to be *balanced* (see Bondareva, 1963; Shapley, 1967) if the core is nonempty.

An important subclass of balanced games is the class of convex games (cf. Shapley, 1971). A game (N, v) is said to be *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \tag{3.2}$$

for all $S, T \subseteq N$.

Let (N, v) be a game and let $\pi : \{1, \ldots, |N|\} \to N$ be a bijection. The *marginal vector* $m^\pi(v)$, is defined by

$$m^\pi_{\pi(k)}(v) := v(\{\pi(1), \ldots, \pi(k)\}) - v(\{\pi(1), \ldots, \pi(k-1)\})$$

for all $k \in \{1, \ldots, |N|\}$. It is known that convexity of a game is equivalent to every marginal vector being a core element (see Shapley (1953) and Ichiishi (1981)). The Shapley value of a game (N, v) is defined as the average of its marginal vectors.

Next, we start the game theoretical study of proportionate flow shops. Let (M, N, p, α)

be a PFS problem and let $\sigma_0 \in \Pi(N)$ be an initial order on the jobs. We assume without loss of generality that $\sigma_0 = (1 \ \dots \ n)$. By associating jobs with players (or clients) the associated PFS game (N, v) is defined by

$$v(S) := \max_{\sigma \in \mathcal{A}(S)} \{c_N(\sigma_0) - c_N(\sigma)\} \tag{3.3}$$

for every $S \subseteq N$, where $\mathcal{A}(S)$ is the set of admissible rearrangements for coalition S. An order $\sigma \in \Pi(N)$ is said to be *admissible for* coalition S if $P(\sigma_0, j) = P(\sigma, j)$ for all $j \in N \setminus S$. This implies that in an admissible rearrangement the initial schedule for jobs outside S does not change, i.e., the starting time in each machine of each player outside S does not change with respect to the initial order. Moreover, agents of S are only allowed to be reordered within maximally connected components of S with regard to σ_0. Here, a *coalition* R is called *connected* (with respect to σ_0) if for all $i, j \in R$ and $k \in N$ such that $\sigma_0(i) < \sigma_0(k) < \sigma_0(j)$ it holds that $k \in R$. Given a coalition $S \subseteq N$, we denote by S/σ_0 the set of all maximally connected components of S according to σ_0. Due to the definition of admissible rearrangements, we can write the value of coalition $S \subseteq N$ as

$$v(S) = \sum_{R \in S/\sigma_0} v(R). \tag{3.4}$$

It is readily seen that PFS games are σ_0-component additive and therefore balanced (see Curiel et al., 1994).

Example 3.3. Let (M, N, p, α) be a PFS situation where $N = \{1, 2, 3\}$, $M = \{M_1, M_2, M_3\}$, $p = (3, 1, 4)$ and $\alpha = (4, 1, 7)$. Let $\sigma_0 = (1\ 2\ 3)$. The situation is represented in Figure 3.3.

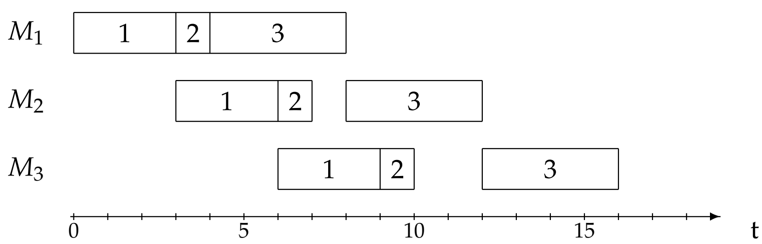

Figure 3.3: Gantt Chart of the PFS situation in Example 3.3.

The corresponding PFS game (N, v) is

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	0	3	0	1	3

We explain in detail how to calculate the value of coalition $\{1, 2\}$ below. The total cost with the initial order is $c_N(\sigma_0) = 158$. The set of admissible rearrangements for coalition $\{1, 2\}$ is

$\mathcal{A}(\{1,2\}) = \{\sigma_0, \sigma_1\}$, with $\sigma_1 = (2\ 1\ 3)$, and the total cost for the order σ_1 is $c_N(\sigma_1) = 155$. Then,

$$v(\{1,2\}) = \max_{\sigma \in \{\sigma_0, \sigma_1\}} \{c_N(\sigma_0) - c_N(\sigma)\} = \max\{0,3\} = 3. \qquad \diamond$$

Note that the initial order in Example 3.3 is not an urgency order. Moreover, the game is balanced but not convex (take $S = \{1,2\}$ and $T = \{2,3\}$).

From now on we will study PFS games with an urgency order as the initial order, i.e.,

$$\sigma_0 = \sigma_u = (1\ 2\ \ldots\ n).$$

We will give an expression for the value of a coalition based on the cost savings that each player can obtain if a similar procedure as the method in Section 3.3 is followed. Due to equation (3.4) we will restrict our study to connected coalitions. Let $S \subseteq N$ be a connected coalition, $S = \{j, j+1, \ldots, k-1, k\}$. To find the optimal order for S we will generate orders $\hat{\sigma}_j^S, \ldots, \hat{\sigma}_k^S$ in the following way: $\hat{\sigma}_j^S := \sigma_u$ and $\hat{\sigma}_i^S$ is obtained from $\hat{\sigma}_{i-1}^S$ as follows. If for every r it follows that $A_r^{\hat{\sigma}_{i-1}^S} \cap \{j, \ldots, i-1\}$ does not contain any new-max job according to $\hat{\sigma}_{i-1}^S$, then i and j belong to the same segment and j is not a new-max job. In this case, $\hat{\sigma}_i^S = \sigma_u$ by Lemma 3.3. If $A_r^{\hat{\sigma}_{i-1}^S} \cap \{j, \ldots, i-1\}$ contains a new-max job according to $\hat{\sigma}_{i-1}^S$ for some r, then we define

$$\begin{aligned}
r_i &= \min\left\{ r \;\middle|\; A_r^{\hat{\sigma}_{i-1}^S} \cap \{j, \ldots, i-1\} \text{ contains a new-max job according to } \hat{\sigma}_{i-1}^S \right\},\\
t_i &= \max\left\{ r \;\middle|\; A_r^{\hat{\sigma}_{i-1}^S} \cap \{j, \ldots, i-1\} \text{ contains a new-max job according to } \hat{\sigma}_{i-1}^S \right\},\\
s_i &= t_i - r_i + 1.
\end{aligned}$$

Analogously than in the method described in Section 3.3, we define $A^S(i,1), \ldots, A^S(i,s_i)$ as $A^S(i,1) = A_{t_i}^{\hat{\sigma}_{i-1}^S} \cap \{j, \ldots, i-1\}$, $A^S(i,r) = A_{t_i - r+1}^{\hat{\sigma}_{i-1}^S}$ for $r = 2, \ldots, s_i$. Subsequently, $\hat{\sigma}_i^S$ is obtained by placing i in position j or in between two consecutive segments or remain in its initial position. The decision will be taken in such a way that

$$c_N(\hat{\sigma}_i^S) \text{ is minimal and } \max_{k \in \bar{P}(\hat{\sigma}_i^S, i)} \{p_k\} \text{ is maximal.} \tag{3.5}$$

Note that Lemma 3.2 and Lemma 3.3 are still applicable in S. Hence, job i may be placed in j-th position only if $A_{r_i}^{\hat{\sigma}_{i-1}^S} \cap \{j, \ldots, i-1\} = A_{r_i}^{\hat{\sigma}_{i-1}^S}$, otherwise Lemma 3.2 would be violated. Hence, the value of coalition S can be written as

$$v(S) = \sum_{i=j}^{k} (c_N(\hat{\sigma}_{i-1}^S) - c_N(\hat{\sigma}_i^S))$$

with $\hat{\sigma}^S_{j-1} := \sigma_u$. We define $G^S_i := c_N(\hat{\sigma}^S_{i-1}) - c_N(\hat{\sigma}^S_i)$ for $i \in S$. Here, G^S_i denotes the cost savings obtained after reordering job $i \in S$ in S. Hence,

$$v(S) = \sum_{i \in S} G^S_i.$$

Note that as a consequence of Lemma 3.2 it follows that if $j \in P(\sigma_u, a)$, then $j \in P(\hat{\sigma}^S_i, a)$ with a a new-max job according to σ_u.

Next, we provide some lemmas that will be used in the proofs of our main results. Their proofs can be found in the appendix. The first lemma states that, in a PFS problem, the new-max jobs according to the urgency order remain new-max jobs during the proposed process of finding an optimal order for an arbitrary coalition S.

Lemma 3.4. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Let $S \subseteq N$. Then, every new-max job according to σ_u is also new-max job according to $\hat{\sigma}^S_i$ for every $i \in S$.*

Next, we provide a result on the "monotonicity" of new-max jobs and cost savings.

Lemma 3.5. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Let $S, T \subseteq N$, with $S \subseteq T \subseteq N$, be connected coalitions. Let $S = \{i_S, \ldots, j_S\}$ with $i_S < \ldots < j_S$, and let a be the new-max job according to σ_u such that $p_a = \max_{k \in P(\sigma_u, i_S)}\{p_k\}$. Then, the following assertions hold.*

(i) $\hat{\sigma}^T_i(i) = \hat{\sigma}^S_i(i)$ for every $i \in S$ with $p_i \geq p_a$.

(ii) $\hat{\sigma}^T_i(i) \leq \hat{\sigma}^S_i(i)$ for every $i \in S$ with $p_i < p_a$. Moreover, if $\hat{\sigma}^T_i(i) < \hat{\sigma}^S_i(i)$, then $\hat{\sigma}^T_i(i) < \hat{\sigma}^T_i(a)$.

(iii) Every new-max job according to $\hat{\sigma}^S_i$ is also a new-max job according to $\hat{\sigma}^T_i$.

(iv) $G^S_i \leq G^T_i$. Moreover, if $p_i \geq p_a$, then $G^S_i = G^T_i$.

The following lemma states that the cost savings achievable for a coalition by the reallocation of job i are at most the total cost savings that job i can achieve for the grand coalition during its reallocation.

Lemma 3.6. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Let $S \subseteq N$ be a connected coalition. Then,*

$$G^S_i = \sum_{r : N(i,r) \subseteq S} h_{N(i,r)}$$

for all $i \in S$.

Next, we will show that the γ-rule leads to a core element of the associated PFS game.

Theorem 3.1. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Then, the γ-rule provides a core element of the associated PFS game.*

Proof. Efficiency holds by definition. Let $S \subseteq N$ be a connected coalition, then

$$\begin{aligned}
\sum_{i\in S}\gamma_i(M,N,p,\alpha) &= (e^S)^t \sum_{i\in N}\sum_{r=1}^{s_i}\left(\frac{h_{N(i,r)}}{2}e^{\{i\}}+\frac{h_{N(i,r)}}{2|N(i,r)|}e^{N(i,r)}\right)\\
&= \frac{1}{2}\sum_{i\in S}\sum_{r=1}^{s_i}h_{N(i,r)}+\frac{1}{2}(e^S)^t\sum_{i\in N}\sum_{r=1}^{s_i}\frac{h_{N(i,r)}}{|N(i,r)|}e^{N(i,r)}\\
&\geq \frac{1}{2}\sum_{i\in S}\sum_{r=1}^{s_i}h_{N(i,r)}+\frac{1}{2}(e^S)^t\sum_{i\in S}\sum_{r=1}^{s_i}\frac{h_{N(i,r)}}{|N(i,r)|}e^{N(i,r)}\\
&\geq \frac{1}{2}\sum_{i\in S}\sum_{r=1}^{s_i}h_{N(i,r)}+\frac{1}{2}(e^S)^t\sum_{i\in S}\sum_{r:N(i,r)\subseteq S}\frac{h_{N(i,r)}}{|N(i,r)|}e^{N(i,r)}\\
&= \frac{1}{2}\sum_{i\in S}\sum_{r=1}^{s_i}h_{N(i,r)}+\frac{1}{2}\sum_{i\in S}\sum_{r:N(i,r)\subseteq S}h_{N(i,r)}\\
&\geq \sum_{i\in S}\sum_{r:N(i,r)\subseteq S}h_{N(i,r)}\\
&= \sum_{i\in S}G_i^S = v(S),
\end{aligned}$$

where $(e^S)^t$ is the transposed matrix of e^S. The first, second, and third inequalities follow because $h_{N(i,r)} \geq 0$ and the last equality is a consequence of Lemma 3.6. $\square$

The next result gives the decomposition into unanimity games of a PFS game. We denote by $\{a_1,\ldots,a_s\}$, with $a_1 < \ldots < a_s$, the set of new-max jobs according to σ_u. For $i \in N$ we denote by $r(i)$ either the index of the new-max job which precedes i if i is not a new-max job according to σ_u, or the index of i if i is a new-max job according to σ_u (i.e., $i = a_{r(i)}$). Consequently, $p_{a_{r(i)}} = \max_{k\in \bar{P}(\sigma_u,i)}\{p_k\}$.

Theorem 3.2. *Let (M,N,p,α) be a PFS problem and let σ_u be the initial order. Let (N,v) be the associated PFS game. Then,*

$$v(T)=\sum_{k\in N}\sum_{r=1}^{r(k)}\left(G_k^{\{a_r,\ldots,n\}}-G_k^{\{a_{r+1},\ldots,n\}}\right)u_{\{a_r,\ldots,k\}}(T)$$

for every $T \subseteq N$, where $G_k^{\{a_{r(k)+1},\ldots,n\}}$ is defined as 0.

Proof. Let $T \subseteq N$ be a connected coalition and set $T = \{i,\ldots,j\}$. We will distinguish between two cases.

Case 1: $T \cap \{a_1,\ldots,a_s\} = \emptyset$. Then, $\hat{\sigma}_k^T(k) = k$ for all $k \in T$ by Lemma 3.3 and therefore $G_k^T = 0$ for all $k \in T$. Hence, $v(T) = 0$. Moreover, $\{a_r,\ldots,k\} \not\subseteq T$ for every new-max job a_r and every

$k \geq a_r$. Hence, $u_{\{a_r,\dots,k\}}(T) = 0$ and

$$\sum_{k \in N} \sum_{r=1}^{r(k)} \left(G_k^{\{a_r,\dots,n\}} - G_k^{\{a_{r+1},\dots,n\}} \right) u_{\{a_r,\dots,k\}}(T) = 0 = v(T).$$

<u>Case 2:</u> $T \cap \{a_1, \dots, a_s\} = \{a_v, \dots, a_w\}$ with $a_v \leq \dots \leq a_w$. Then, $\hat{\sigma}_k^T(k) = k$ for all $k < a_v$ by Lemma 3.3 and $\hat{\sigma}_k^T(k) = \hat{\sigma}_k^{\{a_v,\dots,n\}}(k)$ for all $k \geq a_v$ by the mechanism of the algorithm. Hence, $G_k^T = 0$ for all $i \leq k < a_v$ and $G_k^T = G_k^{\{a_v,\dots,n\}}$ for all $k \geq a_v$. Therefore, $v(T) = \sum_{k=a_v}^{j} G_k^{\{a_v,\dots,n\}}$. Moreover,

$$\begin{aligned}
\sum_{k \in N} \sum_{r=1}^{r(k)} \left(G_k^{\{a_r,\dots,n\}} - G_k^{\{a_{r+1},\dots,n\}} \right) u_{\{a_r,\dots,k\}}(T) &= \sum_{k=a_v}^{j} \sum_{r=v}^{r(k)} \left(G_k^{\{a_r,\dots,n\}} - G_k^{\{a_{r+1},\dots,n\}} \right) u_{\{a_r,\dots,k\}}(T) \\
&= \sum_{k=a_v}^{j} \Bigg[\left(G_k^{\{a_v,\dots,n\}} - G_k^{\{a_{v+1},\dots,n\}} \right) \\
&\quad + \left(G_k^{\{a_{v+1},\dots,n\}} - G_k^{\{a_{v+2},\dots,n\}} \right) \\
&\quad + \dots \\
&\quad + \left(G_k^{\{a_{r(k)-1},\dots,n\}} - G_k^{\{a_{r(k)},\dots,n\}} \right) \\
&\quad + G_k^{\{a_{r(k)},\dots,n\}} \Bigg] \\
&= \sum_{k=a_v}^{j} G_k^{\{a_v,\dots,n\}} = v(T),
\end{aligned}$$

where the first equality follows because if k and r are such that

(i) $a_r < a_v \leq k$, then $\{a_r, \dots, k\} \not\subseteq \{i, \dots, j\} = T$ and $u_{\{a_r,\dots,k\}}(T) = 0$,

(ii) $k > j$ and $a_r \leq k$, then $\{a_r, \dots, k\} \not\subseteq \{i, \dots, j\} = T$ and $u_{\{a_r,\dots,k\}}(T) = 0$.

The second equality is satisfied because if k and r are such that $a_v \leq a_r \leq a_{r(k)}$, $a_r \leq k \leq j$, then $\{a_r, \dots, k\} \subseteq \{i, \dots, j\} = T$ and $u_{\{a_r,\dots,k\}}(T) = 1$.

Let $T \subseteq N$. If T is unconnected, then $v(T) = \sum_{U \in T/\sigma_0} v(U)$ and

$$\sum_{k \in N} \sum_{r=1}^{r(k)} \left(G_k^{\{a_r,\dots,n\}} - G_k^{\{a_{r+1},\dots,n\}} \right) u_{\{a_r,\dots,k\}}(T) = \sum_{U \in T/\sigma_0} \sum_{k \in N} \sum_{r=1}^{r(k)} \left(G_k^{\{a_r,\dots,n\}} - G_k^{\{a_{r+1},\dots,n\}} \right) u_{\{a_r,\dots,k\}}(U)$$

since the unanimity games are defined for connected coalitions. □

As a direct consequence of Lemma 3.5 (iv) and Theorem 3.2 we have that PFS games are convex.

Corollary 3.1. *PFS games are convex if the initial order is the urgency order.*

Proof. By Theorem 3.2 and by Lemma 3.5 (iv) we know that PFS games are decomposed in non-negative linear combination of unanimity games. Hence, PFS games are convex. □

If the initial order is an urgency order, PFS games are convex and the Shapley value belongs to the core. The next result provides a game independent expression of the Shapley value for PFS games.

Theorem 3.3. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Then, the Shapley value of the associated PFS game (N, v) is given by*

$$\Phi_i(v) = \sum_{k=i}^{n} \sum_{r=1}^{r(i)} \frac{G_k^{\{a_r,\ldots,n\}} - G_k^{\{a_{r+1},\ldots,n\}}}{|\{a_r, \ldots, k\}|}$$

for every $i \in N$.

The Shapley value of PFS games can be interpreted as follows: player $i \in N$ needs the players $a_{r(i)}, \ldots, i-1$ in order to obtain some cost savings, and the Shapley value shares the gain $G_i^{\{a_{r(i)},\ldots,i\}} = G_i^{\{a_{r(i)},\ldots,n\}}$ equally among all the players involved, i.e., $a_{r(i)}, \ldots, i$. If a new segment is added to the left of this group of jobs, i.e., if $a_{r(i)-1}, \ldots, a_{r(i)}, \ldots, i-1$ cooperate with i, extra gains, $G_i^{\{a_{r(i)-1},\ldots,i\}} - G_i^{\{a_{r(i)},\ldots,i\}} = G_i^{\{a_{r(i)-1},\ldots,n\}} - G_i^{\{a_{r(i)},\ldots,n\}} \geq 0$ can be obtained by Lemma 3.5 (iv). The Shapley value shares equally these extra gains among all the players involved, i.e., $a_{r(i)-1}, \ldots, a_{r(i)}, \ldots, i$. Step by step, additive-gains are shared equally among all who are responsible.

In our model, the computation of all marginal vectors is very hard. However, there exists a class of marginal vectors which are not so hard to compute and their interpretation is very intuitive. Moreover, averaging over these marginal vectors, we obtain an allocation rule which is in the core wherever the game is convex. We finish this section given an illustrative example where we consider this specific class of marginal vectors. This class is componed by all marginal vectors which only depend on the cost savings that are attainable by the grand coalition.

Example 3.4. Let (M, N, p, α) be a PFS situation where $N = \{1, \ldots, 7\}$, $M = \{M_1, M_2\}$, $p = (3, 2, 1, 7, 6, 5, 4)$ and $\alpha = (30, 18, 8, 49, 36, 25, 16)$. Let $\sigma_0 = (1 \ \ldots \ 7)$, i.e., the initial order is the urgency order. Hence, there are two new-max jobs according to the initial order: 1 and 4, and two initial segments: $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$.

Let (N, v) be the associated PFS game. We will study some marginal vectors that depend only on values G_i^N. For this we will consider marginal vectors whose associated order verifies:

$$1, 2, \text{ and } 3 \text{ enter first,then } 4, 5, 6 \text{ and } 7. \tag{3.6}$$

In Table 3.2 and Table 3.3 we give the marginal vectors for segment $\{1, 2, 3\}$ and segment

$\{4,5,6,7\}$, respectively, satisfying (3.6). The last but one column gives the relative orders in which these vectors are obtained. By $(i\ j)$ we denote the set of permutations of i and j. The last column gives the total number of orders satisfying (3.6) leading to the corresponding relative order.

1	2	3	relative order	number of marginal vectors
$G_1^N + G_2^N + G_3^N$	0	0	(2 3) 1	2!4!
$G_1^N + G_2^N$	0	G_3^N	2 1 3	4!
G_1^N	$G_2^N + G_3^N$	0	(1 3) 2	2!4!
G_1^N	G_2^N	G_3^N	1 2 3	4!
				3!4!

Table 3.2: Marginal vectors for segment $\{1,2,3\}$ satisfying (3.6).

4	5	6	7	relative order	number of marginal vectors
$G_4^N + G_5^N + G_6^N + G_7^N$	0	0	0	(5 6 7) 4	3!3!
$G_4^N + G_5^N + G_6^N$	0	0	G_7^N	(5 6) 4 7	3!2!
$G_4^N + G_5^N$	0	$G_6^N + G_7^N$	0	(5 7) 4 6 5 4 7 6	3!(2!+1)
$G_4^N + G_5^N$	0	G_6^N	G_7^N	5 4 6 7	$3! \cdot 1$
G_4^N	$G_5^N + G_6^N + G_7^N$	0	0	(4 6 7) 5	3!3!
G_4^N	$G_5^N + G_6^N$	0	G_7^N	(4 6) 5 7	3!3!
G_4^N	G_5^N	$G_6^N + G_7^N$	0	4 5 7 6 4 7 5 6 7 4 5 6	$3! \cdot 3$
G_4^N	G_5^N	G_6^N	G_7^N	4 5 6 7	$3! \cdot 1$
					3!4!

Table 3.3: Marginal vectors for segment $\{4,5,6,7\}$ satisfying (3.6).

If we average over these marginal vectors, player 1 will obtain:

$$\frac{3!4!3!}{3!4!3!}G_1^N + \frac{(2!+1)4!3!}{3!4!3!}G_2^N + \frac{2!4!3!}{3!4!3!}G_3^N = G_1^N + \frac{1}{2}G_2^N + \frac{1}{3}G_3^N.$$

Table 3.4 describes the outcome of the average for all players. $\diamond$

1	$G_1^N + \frac{1}{2}G_2^N + \frac{1}{3}G_3^N$
2	$\frac{1}{2}G_2^N + \frac{1}{3}G_3^N$
3	$\frac{1}{3}G_3^N$

4	$G_4^N + \frac{1}{2}G_5^N + \frac{1}{3}G_6^N + \frac{1}{4}G_7^N$
5	$\frac{1}{2}G_5^N + \frac{1}{3}G_6^N + \frac{1}{4}G_7^N$
6	$\frac{1}{3}G_6^N + \frac{1}{4}G_7^N$
7	$\frac{1}{4}G_7^N$

Table 3.4: Average of the marginal vectors described in Tables 3.2 and 3.3.

3.A Appendix

Proof of Lemma 3.4. Let a be a new-max job according to σ_u and let $i \in S$. By definition, $p_a > \max_{j \in P(\sigma_u, a)}\{p_j\}$. We have to show that a is a new-max job according to $\hat{\sigma}_i^S$, i.e., $p_a > \max_{j \in P(\hat{\sigma}_i^S, a)}\{p_j\}$.

Note that $P(\sigma_u, a) \subseteq P(\hat{\sigma}_i^S, a)$. Moreover, $p_j < p_a$ for all $j \in P(\hat{\sigma}_i^S, a) \setminus P(\sigma_u, a)$ by Lemma 3.2. Hence,

$$p_a > \max\{\max_{j \in P(\sigma_u, a)}\{p_j\}, \max_{j \in P(\hat{\sigma}_i^S, a) \setminus P(\sigma_u, a)}\{p_j\}\} = \max_{j \in P(\hat{\sigma}_i^S, a)}\{p_j\}. \qquad \square$$

For the proof of Lemma 3.5, we need the following additional lemmas. The first lemma is an immediate consequence of Lemma 3.2 and the definition of new-max job and therefore the proof will be omitted. It states that a new-max job a according to σ_u does not change its initial position in $\hat{\sigma}_a^S$, for all coalition $S \subseteq N$, $a \in S$.

Lemma 3.7. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Let $S \subseteq N$ and let $a \in S$ be a new-max job according to σ_u. Then, $\hat{\sigma}_a^S(a) = a$.*

The following result is a direct consequence of the algorithm. It says that the set of predecessors of a certain job once reordered can only increase with the consecutive application of the algorithm.

Lemma 3.8. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Let $S \subseteq N$ and $i, j \in S$ with $j < i$. Then, $P(\hat{\sigma}_j^S, j) \subseteq P(\hat{\sigma}_i^S, j)$.*

Next, we will show that if a job becomes new-max job during its reordering, then it will remain new-max job during the successive application of the algorithm.

Lemma 3.9. *Let (M, N, p, α) be a PFS problem and let σ_u be the initial order. Let $S \subseteq N$ and $i, j \in S$ with $j < i$. Then, j is new-max job according to $\hat{\sigma}_i^S$ if and only if j is new-max job according to $\hat{\sigma}_j^S$.*

Proof. If j is new-max job according to σ_u, then j is new-max job according to $\hat{\sigma}_k^S$ for all $k \in S$ by Lemma 3.4 and the result follows. Hence, we may assume that j is not new-max job according to σ_u.

We will first show the only if part. Let j be a new-max job according to $\hat{\sigma}_i^S$. Then,

$$p_j > \max_{k \in P(\hat{\sigma}_i^S, j)} \{p_k\} \geq \max_{k \in P(\hat{\sigma}_j^S, j)} \{p_k\}$$

where the first inequality follows since j is new-max job according to $\hat{\sigma}_i^S$ and the second one by Lemma 3.8. Hence, j is a new-max job according to $\hat{\sigma}_j^S$.

Next, we show the if part. Let j be a new-max job according to $\hat{\sigma}_j^S$. By Lemma 3.8 we have that $P(\hat{\sigma}_j^S, j) \subseteq P(\hat{\sigma}_i^S, j)$. Hence, we can write $P(\hat{\sigma}_i^S, j) = P(\hat{\sigma}_j^S, j) \cup P(\hat{\sigma}_i^S, j) \setminus P(\hat{\sigma}_j^S, j)$. Observe that $p_j > p_k$ for all $k \in P(\hat{\sigma}_j^S, j)$ since j is a new-max job according to $\hat{\sigma}_j^S$ by assumption. Besides, $p_j > p_k$ for all $k \in P(\hat{\sigma}_i^S, j) \setminus P(\hat{\sigma}_j^S, j)$ by Lemma 3.2. Hence, j is a new-max job according to $\hat{\sigma}_i^S$. □

For the proof of Lemma 3.5 we need some additional notation. Let $S \subseteq N$, $S = \{i_S, \ldots, j_S\}$, with $i_S < \ldots < j_S$, and $i \in S$. We define $B^S(i, 1), \ldots, B^S(i, s_i)$ as $B^S(i, r) = A^S(i, s_i - r + 1)$ for every $r \in \{1, \ldots, s_i\}$. We denote by $a(i, S)$ the new-max job according to $\hat{\sigma}_{i-1}^S$ such that i is placed at the tail of the segment defined by $a(i, S)$ after being reordered. Note that $p_{a(i,S)} = \max_{k \in P(\hat{\sigma}_i^S, i)} \{p_k\}$. Let $i \in S$ and let $a \in S$ be a new-max job according to $\hat{\sigma}_{i-1}^S$. We denote by $r(i, a)$ the index of the segment defined by a according to $\hat{\sigma}_{i-1}^S$, i.e., $r(i, a) \in \{1, \ldots, s_i\}$ such that $a \in B^S(i, r(i, a))$. Moreover, we denote by $g_{B^S(i,r)i}$ the gains obtained when, starting from $\hat{\sigma}_{i-1}^S$, we change i from the tail of segment $B^S(i, r)$ to the tail of segment $B^S(i, r-1)$. Formally, it can be written as

$$g_{B^S(i,r)i} := \sum_{j \in B^S(i,r)} (\alpha_i p_j - \alpha_j p_i) + \alpha_i (m-1)\Big(\max_{j \in B^S(i,r) \cup \{i\}} \{p_j\} - \max_{j \in B^S(i,r-1) \cup \{i\}} \{p_j\} \Big), \tag{3.7}$$

with $r \in \{1, \ldots, s_i\}$ and $B^S(i, 0) := \begin{cases} \emptyset & \text{if } i_S = 1 \\ A_{r(i,a^*)}^{\hat{\sigma}_{i-1}^S} & \text{if } i_S > 1 \end{cases}$, where a^* is the new-max job according to σ_u such that $p_{a^*} = \max_{k \in P(\sigma_u, i_S)} \{p_k\}$.

Recall that $\{a_1, \ldots, a_s\}$, with $a_1 < \ldots < a_s$, is the set of new-max jobs according to σ_u. Let $S \subseteq N$ be a connected coalition, $S = \{i_S, \ldots, j_S\}$, satisfying:

(i) $S \cap \{a_1, \ldots, a_s\} = \{a_u, \ldots, a_v\}$, with $a_u \leq \ldots \leq a_v$ and $a_u \neq a_1$;

(ii) there exists $l_1 \in S$ verifying the following three conditions

$$p_{l_1} < p_{a_{u-1}} \tag{3.8}$$

$$a_{u-1} \neq a(l_1, S) \tag{3.9}$$

$$a_{u-1} = a(l, S) \text{ for every } l \in S \text{ with } l < l_1 \text{ and } p_l < p_{a_{u-1}}. \tag{3.10}$$

Consider the following partition of S

$$\{\{i_S, \ldots, l_1 - 1\}, \{l_1\}, \{l_1 + 1, \ldots, l_2 - 1\}, \{l_2\}, \ldots, \{l_m\}, \{l_m + 1, \ldots, j_S\}\} \tag{3.11}$$

where $p_j \geq p_{a(l_k,S)}$ for every $j \in \{l_k + 1, \ldots, l_{k+1} - 1\}$ and every $k \in \{1, \ldots, m\}$ (with $l_{m+1} - 1 := j_S$), and l_k satisfying $p_{l_k} < p_{a(l_{k-1},S)}$, with $l_0 := a_{u-1}$.

Lemma 3.10. *The two following assertions hold*

(i) for every $k, \tilde{k} \in \{1, \ldots, m\}$ with $\tilde{k} > k$ we have

$$g_{B^S(l_{\tilde{k}},r)l_{\tilde{k}}} \leq \frac{p_{l_{\tilde{k}}}}{p_{l_k}} g_{B^S(l_k,r)l_k}$$

for every $r \in \{r(l_{\tilde{k}}, a(l_{k-1}, S)) + 1, \ldots, r(l_{\tilde{k}}, a(l_k, S))\}$ with $r(l_{\tilde{k}}, a(l_0, S)) := 0$;

(ii) $\hat{\sigma}^S_{l_{\tilde{k}}}(l_{\tilde{k}}) > \hat{\sigma}^S_{l_{\tilde{k}}}(l_{\tilde{k}-1})$ for every $\tilde{k} \in \{2, \ldots, m\}$.

Proof. First, we prove the result for $\tilde{k} = 2$. Since $p_j \geq p_{a(l_1,S)}$ for every $j \in \{l_1 + 1, \ldots, l_2 - 1\}$ we have $\hat{\sigma}^S_j(j) > \hat{\sigma}^S_j(l_1)$ by Lemma 3.2 and Lemma 3.3. Therefore,

$$\hat{\sigma}^S_{l_2-1}(j) > \hat{\sigma}^S_{l_2-1}(l_1) \text{ for every } j \in \{l_1 + 1, \ldots, l_2 - 1\}.$$

Hence, the set of new-max jobs preceding $a(l_1, S)$ according to $\hat{\sigma}^S_{l_1-1}$ and $\hat{\sigma}^S_{l_2-1}$ coincide. Consequently,

$$r(l_2, a(l_1, S)) = r(l_1, a(l_1, S))$$

and we will denote $r(a(l_1, S)) = r(l_1, a(l_1, S))$. Moreover,

$$B^S(l_2, r) = B^S(l_1, r) \text{ for every } r \in \{1, \ldots, r(a(l_1, S)) - 1\} \tag{3.12}$$

and

$$B^S(l_2, r(a(l_1, S))) = B^S(l_1, a(l_1, S)) \cup \{l_1\}. \tag{3.13}$$

We first show (i). For $r \in \{1, \dots, r(a(l_1, S)) - 1\}$

$$
\begin{aligned}
g_{B^S(l_2,r)l_2} &= \sum_{j\in B^S(l_2,r)} (\alpha_{l_2} p_j - \alpha_j p_{l_2}) + \alpha_{l_2}(m-1)(\max_{j\in B^S(l_2,r)\cup\{l_2\}}\{p_j\} - \max_{j\in B^S(l_2,r-1)\cup\{l_2\}}\{p_j\}) \\
&= \sum_{j\in B^S(l_1,r)} (\alpha_{l_2} p_j - \alpha_j p_{l_2}) + \alpha_{l_2}(m-1)(\max_{j\in B^S(l_2,r)\cup\{l_2\}}\{p_j\} - \max_{j\in B^S(l_2,r-1)\cup\{l_2\}}\{p_j\}) \\
&\leq \sum_{j\in B^S(l_1,r)} (\alpha_{l_2} p_j - \alpha_j p_{l_2}) + \alpha_{l_2}(m-1)(\max_{j\in B^S(l_2,r)}\{p_j\} - \max_{j\in B^S(l_2,r-1)}\{p_j\}) \\
&= \sum_{j\in B^S(l_1,r)} (\alpha_{l_2} p_j - \alpha_j p_{l_2}) + \alpha_{l_2}(m-1)(\max_{j\in B^S(l_1,r)\cup\{l_1\}}\{p_j\} - \max_{j\in B^S(l_1,r-1)\cup\{l_1\}}\{p_j\}) \quad (3.14) \\
&\leq \sum_{j\in B^S(l_1,r)} (\frac{p_{l_2}}{p_{l_1}}\alpha_{l_1} p_j - \alpha_j p_{l_2}) + \frac{p_{l_2}}{p_{l_1}}\alpha_{l_1}(m-1)(\max_{j\in B^S(l_1,r)\cup\{l_1\}}\{p_j\} - \max_{j\in B^S(l_1,r-1)\cup\{l_1\}}\{p_j\}) \\
&= \frac{p_{l_2}}{p_{l_1}}(\sum_{j\in B^S(l_1,r)} (\alpha_{l_1} p_j - \alpha_j p_{l_1}) + \alpha_{l_1}(m-1)(\max_{j\in B^S(l_1,r)\cup\{l_1\}}\{p_j\} - \max_{j\in B^S(l_1,r-1)\cup\{l_1\}}\{p_j\})) \\
&= \frac{p_{l_2}}{p_{l_1}} g_{B^S(l_1,r)l_1}
\end{aligned}
$$

where the second equality follows by equation (3.12). For the first inequality note that

$$\max_{j\in B^S(l_2,r-1)}\{p_j\} < \max_{j\in B^S(l_2,r)}\{p_j\}.$$

Hence, if $p_{l_2} \geq \max_{j\in B^S(l_2,r)}\{p_j\} > \max_{j\in B^S(l_2,r-1)}\{p_j\}$, then

$$\max_{j\in B^S(l_2,r)\cup\{l_2\}}\{p_j\} - \max_{j\in B^S(l_2,r-1)\cup\{l_2\}}\{p_j\} = 0 < \max_{j\in B^S(l_2,r)}\{p_j\} - \max_{j\in B^S(l_2,r-1)}\{p_j\};$$

if $\max_{j\in B^S(l_2,r)}\{p_j\} > p_{l_2} \geq \max_{j\in B^S(l_2,r-1)}\{p_j\}$, then

$$\max_{j\in B^S(l_2,r)\cup\{l_2\}}\{p_j\} - \max_{j\in B^S(l_2,r-1)\cup\{l_2\}}\{p_j\} \leq \max_{j\in B^S(l_2,r)}\{p_j\} - \max_{j\in B^S(l_2,r-1)}\{p_j\};$$

finally, if $\max_{j\in B^S(l_2,r)}\{p_j\} > \max_{j\in B^S(l_2,r-1)}\{p_j\} \geq p_{l_2}$, then

$$\max_{j\in B^S(l_2,r)\cup\{l_2\}}\{p_j\} - \max_{j\in B^S(l_2,r-1)\cup\{l_2\}}\{p_j\} = \max_{j\in B^S(l_2,r)}\{p_j\} - \max_{j\in B^S(l_2,r-1)}\{p_j\}.$$

The third equality is a consequence of equation (3.12) together with the fact that l_1 does not become new-max job since $p_{l_1} < p_{a_{u-1}} < p_{a(l_1,S)}$ by definition of l_1. The second inequality follows since $l_1 < l_2$, then: $u_{l_1} = \frac{\alpha_{l_1}}{p_{l_1}} \geq \frac{\alpha_{l_2}}{p_{l_2}} = u_{l_2}$ and therefore $\frac{p_{l_2}}{p_{l_1}}\alpha_{l_1} \geq \alpha_{l_2}$.

Analogously, one can see that $g_{B^S(l_2,r(a(l_1,S)))l_2} \leq \frac{p_{l_2}}{p_{l_1}} g_{B^S(l_1,r(a(l_1,S)))l_1}$. The only difference is that the second equality becomes an inequality by equation (3.13) and the fact that $\alpha_{l_2}p_{l_1} - \alpha_{l_1}p_{l_2} \leq 0$ since $u_{l_1} \geq u_{l_2}$.

Next, we will show (ii). Note that by definition of $r(a(l_1, S))$ and assumption (3.5) we have

$$\sum_{r=\bar{r}}^{r(a(l_1,S))} g_{B^S(l_1,r)l_1} \leq 0 \text{ for every } \bar{r} \in \{1, \ldots, r(a(l_1, S))\}. \tag{3.15}$$

Then,

$$\sum_{r=\bar{r}}^{r(a(l_1,S))} g_{B^S(l_2,r)l_2} \leq \frac{p_{l_2}}{p_{l_1}} \sum_{r=\bar{r}}^{r(a(l_1,S))} g_{B^S(l_1,r)l_1} \leq 0 \text{ for every } \bar{r} \in \{1, \ldots, r(a(l_1, S))\},$$

where the first inequality holds by (i) and the second one by equation (3.15). Therefore, $\hat{\sigma}^S_{l_2}(l_2) > \hat{\sigma}^S_{l_2}(l_1)$ by assumption (3.5) and Lemma 3.3.

Now, let $\tilde{k} > 2$ and suppose that the result is true for $l_1, \ldots, l_{\tilde{k}-1}$. Then,

$$\hat{\sigma}^S_{l_{\tilde{k}}-1}(l_1) < \hat{\sigma}^S_{l_{\tilde{k}}-1}(l_2) < \ldots < \hat{\sigma}^S_{l_{\tilde{k}}-1}(l_{\tilde{k}-1}). \tag{3.16}$$

Since $p_j \geq p_{a(l_k,S)}$ for every $k \in \{1, \ldots, \tilde{k}-1\}$ and every $j \in \{l_k + 1, \ldots, l_{k+1} - 1\}$, we have that $\hat{\sigma}^S_j(j) > \hat{\sigma}^S_j(l_k)$ by Lemma 3.2 and Lemma 3.3. Therefore, for every $k \in \{1, \ldots, \tilde{k}-1\}$ it follows

$$\hat{\sigma}^S_{l_{\tilde{k}}-1}(j) > \hat{\sigma}^S_{l_{\tilde{k}}-1}(l_k) \text{ for every } j \in \{l_k + 1, \ldots, l_{k+1} - 1\}. \tag{3.17}$$

Hence, for every $k \in \{2, \ldots, \tilde{k}-1\}$ we have that the set of new-max jobs in between $a(l_{k-1}, S)$ and $a(l_k, S)$ according to $\hat{\sigma}^S_{l_k}$ and $\hat{\sigma}^S_{l_{\tilde{k}}}$ coincide. Therefore, for $k \in \{1, \ldots, \tilde{k}-1\}$ we have

$$r(l_{\tilde{k}}, a(l_k, S)) = r(l_k, a(l_k, S)) \tag{3.18}$$

then, we can denote $r(a(l_k, S)) = r(l_k, a(l_k, S))$. Moreover, for every $k \in \{1, \ldots, \tilde{k}-1\}$ we have

$$B^S(l_{\tilde{k}}, r) = B^S(l_k, r) \tag{3.19}$$

for every $r \in \{r(a(l_{k-1}, S)) + 1, \ldots, r(a(l_k, S)) - 1\}$ and,

$$B^S(l_k, r(a(l_k, S))) \subseteq B^S(l_{\tilde{k}}, r(a(l_k, S)))$$

with

$$B^S(l_{\tilde{k}}, r(a(l_k, S))) \setminus B^S(l_k, r(a(l_k, S))) \subseteq \{l_k, \ldots, l_{\tilde{k}-1}\}. \tag{3.20}$$

In order to show (i) take $k \in \{1, \ldots, \tilde{k}-1\}$ and $r \in \{r(a(l_{k-1}, S)) + 1, \ldots, r(a(l_k, S))\}$, then one can see that

$$g_{B^S(l_{\tilde{k}},r)l_{\tilde{k}}} \leq \frac{p_{l_{\tilde{k}}}}{p_{l_k}} g_{B^S(l_k,r)l_k}$$

by using the same kind of arguments as in equation (3.14).

Next, we will show (ii). Note that by definition of $r(a(l_k,S))$, $k \in \{1,\ldots,\tilde{k}-1\}$, and assumption (3.5) we have

$$\sum_{r=\bar{r}}^{r(a(l_k,S))} g_{B^S(l_k,r)l_k} \leq 0 \tag{3.21}$$

for every $\bar{r} \in \{r(a(l_{k-1},S))+1,\ldots,r(l_k,a(l_k,S))\}$. Then, for every $k^* \in \{1,\ldots,\tilde{k}-1\}$ and every $\bar{r} \in \{r(a(l_{k^*-1},S))+1,\ldots,r(a(l_{k^*},S))\}$ we have

$$\begin{aligned}\sum_{r=\bar{r}}^{r(a(l_{\tilde{k}-1},S))} g_{B^S(l_{\tilde{k}},r)l_{\tilde{k}}} &= \sum_{k=k^*+1}^{\tilde{k}-1} \sum_{r=r(a(l_{k-1},S))+1}^{r(a(l_k,S))} g_{B^S(l_{\tilde{k}},r)l_{\tilde{k}}} + \sum_{r=\bar{r}}^{r(a(l_{k^*},S))} g_{B^S(l_{\tilde{k}},r)l_{\tilde{k}}} \\ &\leq \sum_{k=k^*+1}^{\tilde{k}-1} \frac{p_{l_{\tilde{k}}}}{p_{l_k}} \sum_{r=r(a(l_{k-1},S))+1}^{r(a(l_k,S))} g_{B^S(l_k,r)l_k} + \frac{p_{l_{\tilde{k}}}}{p_{l_{k^*}}} \sum_{r=\bar{r}}^{r(a(l_{k^*},S))} g_{B^S(l_{k^*},r)l_{k^*}} \\ &\leq 0\end{aligned}$$

where the first inequality holds by (i) and the second one by equation (3.21). Therefore, $\hat{\sigma}^S_{l_{\tilde{k}}}(l_{\tilde{k}}) > \hat{\sigma}^S_{l_{\tilde{k}}}(l_{\tilde{k}-1})$ by assumption (3.5) and Lemma 3.3. □

Proof of Lemma 3.5. Recall that $\sigma_0 = \sigma_u = (1\ \ldots\ n)$ and $\{a_1,\ldots,a_s\}$ is the set of new-max jobs according to σ_u with $a_1 < \ldots < a_s$. We distinguish three cases.

Case 1: $S \cap \{a_1,\ldots,a_s\} = \emptyset$. Then, $\hat{\sigma}^S_i = \sigma_u$ for every $i \in S$ and assertions (i) and (ii) are direct consequence of the definition of $\hat{\sigma}^T_i$, assertion (iii) follows by Lemma 3.4, and assertion (iv) follows since $G^T_i \geq 0 = G^S_i$ by definition of G^T_i.

Case 2: $S \cap \{a_1,\ldots,a_s\} = \{a_u,\ldots,a_v\}$ and $T \cap \{a_1,\ldots,a_s\} = \{a_u,\ldots,a_w\}$ with $a_u \leq \ldots \leq a_v \leq \ldots \leq a_w$. Then, we have $\hat{\sigma}^S_i = \hat{\sigma}^T_i = \sigma_u$ for every $i \in S$ with $i < a_u$ and $\hat{\sigma}^S_i = \hat{\sigma}^T_i$ for every $i \in S$ with $i \geq a_u$. Hence, assertions (i), (ii), (iii), and (iv) are immediate.

Case 3: $S \cap \{a_1,\ldots,a_s\} = \{a_u,\ldots,a_v\}$ and $T \cap \{a_1,\ldots,a_s\} = \{a_{\tilde{u}},\ldots,a_{\tilde{v}}\}$ with $a_{\tilde{u}} < a_u \leq a_v \leq a_{\tilde{v}}$. Let $S = \{i_S,\ldots,j_S\}$ and partition S according to (3.11). Let $i \in S$ and let a be a new-max job according to $\hat{\sigma}^S_{i-1}$ ($\hat{\sigma}^T_{i-1}$). During the remaining of this proof we will denote by $r^S(i,a)$ ($r^T(i,a)$) the index of the segment defined by a according to $\hat{\sigma}^S_{i-1}$ ($\hat{\sigma}^T_{i-1}$). Moreover, by $s_i(S)$ we denote the number of segments before reordering player i in S.

Note that for every $i \in \{i_S,\ldots,a_u\}$ we have $\hat{\sigma}^S_i = \sigma_u$ and therefore assertions (i), (ii), (iii), and (iv) follow using the same kind of reasoning as in Case 1.

Subsequently, assume that the result holds for $\{i_S,\ldots,l_k-1\}$ for some $k \in \{1,\ldots,m\}$. Then, we have

$$B^T(l_k, r^T(l_k,a_{u-1})+r) = B^S(l_k,r) \tag{3.22}$$

for every $r \in \{1,\ldots,s_{l_k}(S)\} \setminus \{r^S(l_k,a(l_1,S)),\ldots,r^S(l_k,a(l_{k-1},S))\}$.

Besides, $B^T(l_k, r^T(l_k, a_{u-1}) + r) \subseteq B^S(l_k, r)$ for every $r \in \{r^S(l_k, a(l_1, S)), \ldots, r^S(l_k, a(l_{k-1}))$ with

$$B^S(l_k, r) \setminus B^T(l_k, r^T(l_k, a_{u-1}) + r) \subseteq \{l_1, \ldots, l_{k-1}\}. \tag{3.23}$$

Note that it may be the case that $a(l_k, S) = a(l_{k+1}, S)$ for some $k \in \{1, \ldots, k-2\}$. We define recursively k^*_w, with $w \in \{1, \ldots, t\}$, as

$$k^*_w = \min\{\bar{k} \in \{k^*_{w-1} + 1, \ldots, k-1\} \mid a(l_{\bar{k}}, S) \neq a(l_{\bar{k}-1}, S)\} \tag{3.24}$$

where $k^*_0 = 0$. Note that $\hat{\sigma}^S_{l_k-1}(l_{k^*_{w-1}}) < \hat{\sigma}^S_{l_k-1}(l_{k^*_w})$ for every $w \in \{2, \ldots, t\}$ by Lemma 3.10 (ii).

Then, by equation (3.22) it follows

$$g_{B^T(l_k, r^T(l_k, a_{u-1}) + r) l_k} = g_{B^S(l_k, r) l_k} \tag{3.25}$$

for every $r \in \{1, \ldots, s_{l_k}(S)\} \setminus \{r^S(l_k, a(l_1, S)), \ldots, r^S(l_k, a(l_{k-1}, S))\}$.

Besides, one can see analogously as in equation (3.14) that

$$g_{B^T(l_k, r^T(l_k, a_{u-1}) + r^S(l_k, a(l_{k^*_w}, S))) l_k} \leq \frac{p_{l_k}}{p_{l_{k^*_w}}} g_{B^S(l_{k^*_w}, r^S(l_{k^*_w}, a(l_{k^*_w}, S))) l_{k^*_w}} \tag{3.26}$$

for every $w \in \{1, \ldots, t\}$.

By definition of $r^S(l_{k^*_w}, S)$, $w \in \{1, \ldots, t\}$, and assumption (3.5) we know that

$$\sum_{r=\bar{r}}^{r^S(l_{k^*_w}, a(l_{k^*_w}, S))} g_{B^S(l_{k^*_w}, r) l_{k^*_w}} \leq 0 \tag{3.27}$$

for every $\bar{r} \in \{r^S(l_{k^*_w}, a(l_{k^*_{w-1}}, S)) + 1, \ldots, r^S(l_{k^*_w}, a(l_{k^*_w}, S))\}$. Then,

$$\begin{aligned} \sum_{r=\bar{r}}^{r^S(l_k, a(l_{k^*_t}, S))} g_{B^T(l_k, r^T(l_k, a_{u-1}) + r) l_k} &= \sum_{w=\bar{w}+1}^{t} \sum_{r=r^S(l_k, a(l_{k^*_{w-1}}, S))}^{r^S(l_k, a(l_{k^*_w}, S))} g_{B^T(l_k, r^T(l_k, a_{u-1}) + r) l_k} + \sum_{r=\bar{r}}^{r^S(l_k, a(l_{k^*_{\bar{w}}}, S))} g_{B^T(l_k, r^T(l_k, a_{u-1}) + r) l_k} \\ &\leq \sum_{w=\bar{w}+1}^{t} \frac{p_{l_k}}{p_{l_{k^*_w}}} \sum_{r=r^S(l_k, a(l_{k^*_{w-1}}, S))}^{r^S(l_k, a(l_{k^*_w}, S))} g_{B^S(l_{k^*_w}, r) l_{k^*_w}} + \frac{p_{l_k}}{p_{l_{k^*_{\bar{w}}}}} \sum_{r=\bar{r}}^{r^S(l_k, a(l_{k^*_{\bar{w}}}, S))} g_{B^S(l_{k^*_{\bar{w}}}, r) l_{k^*_{\bar{w}}}} \\ &\leq 0 \end{aligned} \tag{3.28}$$

for every $\bar{w} \in \{1, \ldots, t\}$ and every $\bar{r} \in \{r^S(l_k, a(l_{k^*_{\bar{w}-1}}, S)) + 1, \ldots, r^S(l_k, a(l_{k^*_{\bar{w}}}, S))\}$. Here, the first inequality holds by equation (3.26) and the second one by equation (3.27).

First, suppose that $p_{l_k} \geq p_{a_u}$. Then, $\hat{\sigma}^T_{l_k}(l_k) > \hat{\sigma}^T_{l_k}(a_u)$ by Lemma 3.2. Hence, $\hat{\sigma}^T_{l_k}(l_k) = \hat{\sigma}^S_{l_k}(l_k)$ by equations (3.25) and (3.28), and by assumption (3.5).

Second, suppose that $p_{l_k} < p_{a_u}$. Then, it can be the case that for some $\hat{r} \in \{1, \ldots, r^T(l_k, a_{u-1})\}$

we have $\sum_{r=\hat{r}}^{r^T(l_k,a(l_k,S))} g_{B^T(l_k,r)l_k} > 0$. In such a case $\hat{\sigma}_{l_k}^T(l_k) < \hat{\sigma}_{l_k}^S(l_k)$ and $\hat{\sigma}_{l_k}^T(l_k) < \hat{\sigma}_{l_k}^T(a_u)$, otherwise $\hat{\sigma}_{l_k}^T(l_k) = \hat{\sigma}_{l_k}^S(l_k)$ by equation (3.25) and assumption (3.5).

Hence (i) and (ii) are satisfied. Assertion (iii) is an immediate consequence of Lemma 3.9, and the fact that l_k is not a new-max job according $\hat{\sigma}_{l_k}^S$ since $p_{l_k} < p_{a(l_{k-1},S)} < p_{a(l_k,S)}$. Assertion (iv) is a direct consequence of (ii) together with equation (3.25).

Finally, suppose that the result is true for $\{i_S,\ldots,i-1\}$ with $l_k < i < l_{k+1}$. Then, we have

$$B^T(i, r^T(l_k, a_{u-1}) + r) = B^S(i,r) \tag{3.29}$$

and

$$g_{B^T(i,r^T(l_k,a_{u-1})+r)i} = g_{A^S(i,r)i} \tag{3.30}$$

for every $r \in \{1,\ldots,r(i,a(l_k,S))-1\}$.

Moreover, $\hat{\sigma}_i^T(i) > \hat{\sigma}_i^T(a_u)$ by Lemma 3.2. Hence, $\hat{\sigma}_i^T(i) = \hat{\sigma}_i^S(i)$ by equation (3.30) and assumption (3.5). Assertion (iii) follows by induction together with Lemma 3.9, and (i). Assertion (iv) is a direct consequence of (i) together with equation (3.30). □

Proof of Lemma 3.6. Recall that $\sigma_0 = \sigma_u = (1\ \ldots\ n)$ and $\{a_1,\ldots,a_s\}$ is the set of new-max jobs according to σ_u with $a_1 < \ldots < a_s$. We will distinguish three cases.

Case 1: $S \cap \{a_1,\ldots,a_s\} = \emptyset$. Then, $\hat{\sigma}_i^S = \sigma_u$ for every $i \in S$ and

$$G_i^S = 0 = \sum_{r:N(i,r)\subseteq S} h_{N(i,r)}$$

by definition of $h_{N(i,r)}$.

Case 2: $a_1 \in S$. Then, we have $\hat{\sigma}_i^S = \hat{\sigma}_i^N$ for every $i \in S$ and

$$G_i^S = G_i^N = \sum_{r:N(i,r)\subseteq S} h_{N(i,r)}.$$

Case 3: $a_1 \notin S$, $S \cap \{a_1,\ldots,a_s\} = \{a_u,\ldots,a_v\}$ with $a_u \leq \ldots \leq a_v$. Let $S = \{i_S,\ldots,j_S\}$ and consider the partition (3.11). Let $k \in \{1,\ldots,m\}$ and let $i \in \{l_k+1,\ldots,l_{k+1}-1\}$. By Lemma 3.5 (i) and (iv) we know that $\hat{\sigma}_i^N(i) = \hat{\sigma}_i^S(i)$ and $G_i^N = G_i^S$. Hence,

$$G_i^S = G_i^N = \sum_{r:N(i,r)\subseteq S} h_{N(i,r)}$$

where the second equality follows by $\hat{\sigma}_i^N(i) = \hat{\sigma}_i^S(i)$ and the fact that $h_{N(i,r)} = 0$ for every $r \geq r(i,a(i,S))$.

Next, consider l_k with $k \in \{1,\ldots,m\}$. If $\hat{\sigma}_{l_k}^N(l_k) = \hat{\sigma}_{l_k}^S(l_k)$ we are in the previous situation.

Assume that $\hat{\sigma}^N_{l_k}(l_k) < \hat{\sigma}^S_{l_k}(l_k)$. Then,

$$G_i^S = \sum_{r=1}^{r(i,a(S,i))-1} g_{A^S(i,r)i} = \sum_{r=1}^{r(i,a(S,i))-1} g_{A^N(i,r)i} = \sum_{r=1}^{r(i,a(S,i))-1} h_{N(i,r)} = \sum_{r:N(i,r)\subseteq S} h_{N(i,r)}$$

where the first equality follows by definition of G_i^S and $r(i, a(S, i))$, the second one by equation (3.25) with $T = N$ and the fact that $B^S(i,r) = A^S(i, s_i - r + 1)$, the third equality is a direct consequence of the definition of $h_{N(i,r)}$ and the last one follows by equation (3.28) with $B^S(i,r) = A^S(i, s_i - r + 1)$ and the definition of $h_{N(i,r)}$. □

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Part II

Cautious Behavior

Introduction to cautious behavior

Now, we start with the Part II of the dissertation. In this part we focus on the non cooperative issue of the game theory. It is structured in two independent but related chapters.

The following two chapters deal with the *cautious behavior* of an agent facing to a decision problem. In this context, maximin behavior is the most common representation of cautiousness. Choosing between alternatives according to the maximin behavior consists of taking those alternatives which guarantee the best possible result if the worst-case scenario occurs. There also exist some refinements of this behavior like the *prudent* behavior introduced by Moulin (1981), or the *protective* behavior introduced by Barberà and Dutta (1982). There exist several classes of decision making situations where both notions coincide.

Chapter 4 is devoted to the study of the foundations of maximin behavior. Milnor (1954) started this issue given a characterization of the order induced by the maximin criterion for finite decision problems. Latter on, Vilkas (1963) and Tijs (1981) characterize the maximin behavior in the context of zero-sum games. In this chapter we go further. We characterize the solution that assigns to each decision problem its set of maximin actions, in the setting where the unique restriction on the problems is that the decision maker's utility function is bounded. This chapter is based on Mosquera et al. (2005).

In Barberà and Jackson (1988), besides of maximin criterion, is also characterized the protective criterion. This notion, jointly with the prudent notion, were recently studied in the context of finite games in strategic form by Fiestras-Janeiro et al. (1998). The aim of Chapter 5 is to characterize the solution that assigns to each decision problem in an standard class its set of protective actions. This chapter is based on Mosquera et al. (2006).

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Chapter 4

Characterizing cautious choice

Contents

4.1 Introduction

Choosing between alternatives according to the maximin criterion essentially involves associating with each alternative the worst possible consequence and then choosing the alternative(s) for which this worst-case scenario offers the best possible result. Different ways of modeling these actions, consequences (or states), and preferences/utilities over them yield an abundance of applications of this decision principle and its sibling, minimax behavior, in the social sciences:

- GAME THEORY: The minimax theorem of von Neumann (cf. von Neumann, 1928) is one of the corner stones of game theory. It establishes maximin behavior as an equilibrating device that assigns to every mixed extension of a finite two person zero-sum (or purely antagonistic) game a well-defined value.

- EXPERIMENTAL ECONOMICS: Sarin and Vahid (1999, 2001) show that maximin behavior is the outcome of a natural and simple dynamic process of strategy adjustment and provides a good prediction of human behavior in several experimental settings.

- STATISTICAL DECISION THEORY: Next to the Bayesian paradigm, the maximin approach is standard in statistical decision theory (cf. Blackwell and Girshick, 1954; Ferguson, 1967).

- SOCIAL CHOICE AND WELFARE: Rawlsian welfare aims for the maximization of the utility of the least "happy" member of a society; see Moulin (1988) for a textbook treatment.

- OPERATIONS RESEARCH: Problems like the optimal location of warehouses often involve the minimization of suitable distance functions. Among these distance functions, the Chebychev/supremum norm is a common one, transforming the problem in one of the minimax type (cf. Love et al., 1988).

- CONSTRAINED OPTIMIZATION: The Lagrangean dual of a constrained minimization problem is of the maximin type (cf. Bazaraa et al., 1993, Ch. 6).

Given the ubiquity of the maximin principle, it is hardly surprising that also its fundaments have been the subject of study. These studies tend to focus on one of two aspects: *(a)* characterizing the *order* induced by the maximin criterion, like in the classical study Milnor (1954) and in Barberà and Jackson (1988), or *(b)* characterizing the maximin *value* associated with zero-sum games, like Vilkas (1963) and Tijs (1981), or, more recently, Norde and Voorneveld (2004)[1] and Carpente et al. (2005).

To our knowledge, the current chapter is the first to characterize a third aspect, namely the solution that assigns to each decision problem its *set* of maximin actions. The purpose of our next section is to formally define decision problems, list the properties used in our characterization,

[1]These authors take payoffs/utilities in the game as given. Hart et al. (1994) goes one step further by first deriving utilities from a number of properties on players' preferences and then making the step to evaluations using the value function.

and state the characterization theorem. The proof of our characterization is contained in the final section.

4.2 A characterization of the set of maximin actions

A *decision problem* is a tuple (A, Ω, u), where A is a nonempty set of actions, Ω is a nonempty set of states, and $u : A \times \Omega \to \mathbb{R}$ is a bounded function which represents the decision-maker's payoff/utility function. The set of all decision problems is denoted by $\mathcal{D}$. A *solution* on $\mathcal{D}$ is a correspondence φ that assigns to every $(A, \Omega, u) \in \mathcal{D}$ a set $\varphi(A, \Omega, u) \subseteq A$ of actions. Our aim is to characterize the solution M that assigns to every decision problem $(A, \Omega, u) \in \mathcal{D}$ its set of *maximin actions*

$$M(A, \Omega, u) = \left\{ a \in A \;\middle|\; \inf_{\omega \in \Omega} u(a, \omega) = \sup_{a' \in A} \inf_{\omega \in \Omega} u(a', \omega) \right\}.$$

Since only the order of the payoffs matters, order-preserving transformations do not affect the solution and the assumption that our payoffs are bounded entails no loss of generality:

$$M(A, \Omega, u) = M(A, \Omega, \arctan u).$$

In our general setting, some properties of simpler, finite problems no longer hold: (all) maximin actions can be strictly dominated (Example 4.1) and the set of maximin actions may be empty (Example 4.2). Recall that an action $a \in A$ in a decision problem $D = (A, \Omega, u) \in \mathcal{D}$ is *strictly dominated* if there is an action $a' \in A$ with $u(a', \omega) > u(a, \omega)$ for all $\omega \in \Omega$.

Example 4.1. Consider a decision problem (A, Ω, u) with $A = \Omega = \mathbb{Z}$ and $u(a, \omega) = \arctan(a - \omega)$ for all $(a, \omega) \in \mathbb{Z} \times \mathbb{Z}$. Then $\inf_{\omega \in \mathbb{Z}} u(a, \omega) = -\pi/2$ for all $a \in \mathbb{Z}$: every $a \in \mathbb{Z}$ is maximin, yet also strictly dominated, for instance by $a + 1$. ◇

Example 4.2. Consider a decision problem (A, Ω, u) with $A = \Omega = \mathbb{N}$ and $u(a, \omega) = a/(a + 1)$ for all $(a, \omega) \in A \times \Omega$. Then $\inf_{\omega \in \Omega} u(a, \omega) = a/(a + 1)$, a function which does not achieve a maximum: $M(A, \Omega, u) = \emptyset$. ◇

We introduce some properties for a solution φ on $\mathcal{D}$. They are standard and are mostly taken from earlier publications, particularly Milnor (1954); Barberà and Jackson (1988). Anonymity requires that the solution does not depend on the way actions and states are labeled.

> **Anonymity (ANO).** Let $(A, \Omega, u), (A', \Omega', u') \in \mathcal{D}$. If there are bijections $f : A \to A'$ and $g : \Omega \to \Omega'$ such that $u(a, \omega) = u'(f(a), g(\omega))$ for all $(a, \omega) \in A \times \Omega$, then $\varphi(A', \Omega', u') = f(\varphi(A, \Omega, u))$.

Independence of irrelevant actions states that if the action set of a decision problem is reduced, but some elements in the solution of the large problem remain feasible, then the solution of the small problem consists of the feasible elements in the solution of the original problem.

Independence of irrelevant actions (IIA). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A \subset A'$ and $u'_{|A\times\Omega} = u$. If $\varphi(A', \Omega, u') \cap A \neq \emptyset$, then $\varphi(A', \Omega, u') \cap A = \varphi(A, \Omega, u)$.

Inheritance of nonemptiness states that adding finitely many actions to a decision problem with a nonempty solution yields a new decision problem whose solution is also nonempty.

Inheritance of nonemptiness (INH-NEM). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A \subset A'$ and $u'_{|A\times\Omega} = u$. If $\varphi(A, \Omega, u) \neq \emptyset$ and $A' \setminus A$ is a finite set, then $\varphi(A', \Omega, u') \neq \emptyset$.

In a decision problem $(A, \Omega, u) \in \mathcal{D}$, action $a' \in A$ *weakly dominates* action $a \in A$ if $u(a', \omega) \geq u(a, \omega)$ for all $\omega \in \Omega$, with strict inequality for some $\omega \in \Omega$. The weak-domination property states that if an action weakly dominates an action in the solution of the problem, then also the weakly dominating action belongs to the solution.

Weak domination (WDOM). Let $(A, \Omega, u) \in \mathcal{D}$ and $a^*, a' \in A$. If $a^* \in \varphi(A, \Omega, u)$ and a' weakly dominates a^*, then $a' \in \varphi(A, \Omega, u)$.

The next property requires that duplicating states does not affect the solution.

Duplication of states (DOS). Let $(A, \Omega, u), (A, \Omega', u') \in \mathcal{D}$ with $\Omega \subset \Omega'$. If there is a surjection $g : \Omega' \to \Omega$ such that $u'(a, \omega') = u(a, g(\omega'))$ for all $(a, \omega') \in A \times \Omega'$, then $\varphi(A, \Omega', u') = \varphi(A, \Omega, u)$.

Continuity states that if an action is always contained in the solution of a sequence of decision problems in $\mathcal{D}$ with fixed action and state spaces and pointwise convergent utility functions, then this action is also contained in the solution of the limiting problem.

Continuity (CONT). Let $(A, \Omega, u) \in \mathcal{D}$ and let $\{(A, \Omega, u_k)\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ such that $\lim_{k\to\infty} u_k(a, \omega) = u(a, \omega)$ for all $(a, \omega) \in A \times \Omega$. If there is an $a^* \in A$ with $a^* \in \varphi(A, \Omega, u_k)$ for all $k \in \mathbb{N}$, then $a^* \in \varphi(A, \Omega, u)$.

Restricted nonemptiness states that, for a given decision problem, if there exists some maximin action, then there also exists some element of the solution. This is not a new property in the literature, it is used in both cooperative games (cf. Voorneveld and van den Nouweland, 1998) and noncooperative games (cf. Norde et al., 1996; Voorneveld et al., 1999; Dufwenberg et al., 2001). In our context, it is related with the possibility of nonemptiness of the set of maximin actions.

Restricted Nonemptiness (r-NEM). Let $(A, \Omega, u) \in \mathcal{D}$. If $M(A, \Omega, u)$ is nonempty, then $\varphi(A, \Omega, u)$ is also nonempty.

Convexity states that if two actions belong to the solution of a decision problem and an action is added whose payoff is the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the above actions' payoffs, then the new action belongs to the solution of the new problem. This is a standard risk neutrality property already present in Milnor (1954): if two actions belong to the problem's solution, the decision-maker does not mind tossing a coin to decide between them.

Convexity (CONV). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A' = A \cup \{a'\}$ for some $a' \notin A$ and $u'_{|A \times \Omega} = u$. If there are $a^*, \tilde{a} \in \varphi(A, \Omega, u)$ such that

$$u'(a', \omega) = \frac{1}{2}u(a^*, \omega) + \frac{1}{2}u(\tilde{a}, \omega)$$

for all $\omega \in \Omega$, then $a' \in \varphi(A', \Omega, u')$.

Finally, if there is only one state, then the solution chooses the actions that maximize the payoff.

One state rationality (OSR). Take $(A, \Omega, u) \in \mathcal{D}$ with $|\Omega| = 1$; then, writing $\Omega = \{\omega\}$: $\varphi(A, \Omega, u) = \arg\max_{a \in A} u(a, \omega)$.

The former properties characterize the solution M on $\mathcal{D}$ which assigns to each decision problem its set of maximin actions:

Theorem 4.1. *The maximin solution M is the unique solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, WDOM, DOS, CONT, r-NEM, CONV, and OSR.*

Its proof is given in the next section.

4.3 Proof of the characterization theorem

The purpose of this section is to prove our characterization theorem. The proof is based on a series of lemmas.

The properties ANO and IIA of a solution guarantee that if an action has the same payoff function as an element of the solution of the problem — up to relabeling of the states — then also the former action is part of the solution. We only use a simple version:

Lemma 4.1. *Let φ be a solution on $\mathcal{D}$ satisfying ANO and IIA, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ is such that, for some $\omega_1, \omega_2 \in \Omega$,*

(i) $u(a', \omega_1) = u(a^*, \omega_2)$ *and* $u(a', \omega_2) = u(a^*, \omega_1)$,

(ii) $u(a', \omega) = u(a^*, \omega)$ *for all* $\omega \in \Omega \setminus \{\omega_1, \omega_2\}$,

then $a' \in \varphi(D)$.

Proof. Assume that $u(a^*, \omega_1) \neq u(a^*, \omega_2)$ (otherwise ANO concludes the result). The utility functions for actions a^* and a' are represented in the table below, where $\square$ and $\times$ represent two different values:

States / Actions	$\cdots$	ω_1	$\cdots$	ω_2	$\cdots$
a^*	$\boxed{\cdots}$		$\boxed{\cdots}$	$\times$	$\boxed{\cdots}$
	$\parallel$		$\parallel$		$\parallel$
a'	$\boxed{\cdots}$	$\times$	$\boxed{\cdots}$		$\boxed{\cdots}$

Consider decision problems

$$D_1 = (\{a^*, a'\}, \Omega, u_{|\{a^*,a'\}\times\Omega}),\ D_2 = (\{a^*, a'\}, \Omega, v),$$

where the utility for a^* and a' is interchanged, i.e.

$$\begin{array}{rcccl} v(a^*, \omega_1) & = & v(a', \omega_2) & := & u(a^*, \omega_2), \\ v(a^*, \omega_2) & = & v(a', \omega_1) & := & u(a^*, \omega_1), \end{array}$$

and $v(b, \omega) := u(b, \omega)$ for all other $(b, \omega) \in \{a^*, a'\} \times (\Omega \setminus \{\omega_1, \omega_2\})$. By *(i)* and *(ii)*, D_2 is isomorphic to D_1, either via switching the labels of a^* and a' or via switching the labels of ω_1 and ω_2.

Note that D can be obtained from D_1 by adding actions and, moreover, $a^* \in \varphi(D) \cap \{a^*, a'\}$. Therefore, by IIA:

$$\varphi(D_1) = \varphi(D) \cap \{a^*, a'\}, \tag{4.1}$$

so that $a^* \in \varphi(D_1)$. It is shown that also $a' \in \varphi(D_1)$. Consider the bijection $f : \{a^*, a'\} \to \{a^*, a'\}$ with $f(a^*) = a', f(a') = a^*$ and let $g : \Omega \to \Omega$ be the identity function. Since, for all $(a, \omega) \in \{a^*, a'\} \times \Omega$, we have $u(a, \omega) = v(f(a), g(\omega))$, ANO implies that $\varphi(D_2) = f(\varphi(D_1))$, so $a' = f(a^*) \in \varphi(D_2)$. Next, consider the bijection $\bar{g} : \Omega \to \Omega$ with $\bar{g}(\omega_1) = \omega_2$, $\bar{g}(\omega_2) = \omega_1$, keeping other states unchanged, and let $\bar{f} : \{a^*, a'\} \to \{a^*, a'\}$ be the identity function. Since $v(a, \omega) = u(\bar{f}(a), \bar{g}(\omega))$ for all $(a, \omega) \in \{a^*, a'\} \times \Omega$, ANO implies that $\varphi(D_1) = \bar{f}(\varphi(D_2)) = \varphi(D_2)$. Remember that $a' \in \varphi(D_2)$, so $a' \in \varphi(D_1)$. This shows that $\{a^*, a'\} = \varphi(D_1)$.

Finally, by (4.1), $a' \in \varphi(D)$. □

With the INH-NEM property and Lemma 4.1 one can establish the following consequence. If we add an action to a decision problem with the same utility as an action in the solution of the original problem, except in two states where the utilities are interchanged, then both actions belong to the solution of the new problem:

Lemma 4.2. *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, and INH-NEM, and let $D = (A, \Omega, u) \in \mathcal{D}$. Take $D' = (A', \Omega, u') \in \mathcal{D}$ satisfying that $A' = A \cup \{a'\}$ for some $a' \notin A$ and $u'_{|A\times\Omega} = u$. Suppose that there exist $a^* \in \varphi(A, \Omega, u)$ and $\omega_1, \omega_2 \in \Omega$ such that*

(i) $u'(a', \omega_1) = u'(a^*, \omega_2)$ *and* $u'(a', \omega_2) = u'(a^*, \omega_1)$*,*

(ii) $u'(a', \omega) = u'(a^*, \omega)$ *for all* $\omega \in \Omega \setminus \{\omega_1, \omega_2\}$*.*

Then $\{a^*, a'\} \subseteq \varphi(D')$*.*

Proof. Note that D' is well-defined. Suppose that $a' \notin \varphi(D')$. Since φ satisfies INH-NEM, $A' \setminus A = \{a'\}$ is a finite set, and $\varphi(D) \neq \emptyset$: $\varphi(D') \neq \emptyset$. So $\varphi(D') \cap A \neq \emptyset$ and IIA implies that $\varphi(D') \cap A = \varphi(D)$. Therefore $a^* \in \varphi(D')$. By Lemma 4.1, also $a' \in \varphi(D')$, a contradiction.

Hence, $a' \in \varphi(D')$ and using Lemma 4.1 again it follows that $a^* \in \varphi(D')$. So $\{a^*, a'\} \subseteq \varphi(D')$. □

Consider the following modification of weak dominance. In a decision problem $(A, \Omega, u) \in \mathcal{D}$, action $a' \in A$ *quasi-dominates* action $a \in A$ if there exist $\omega_1, \omega_2 \in \Omega$ such that:

(i) $u(a', \omega) \geq u(a, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1\}$, and

(ii) $u(a', \omega_2) \geq u(a, \omega_1) > u(a', \omega_1) \geq u(a, \omega_2)$.

Intuitively, a' quasi-dominates a if it is at least as good as a in all states except some ω_1, and the loss from choosing a' in state ω_1 is compensated for by a utility gain in another state ω_2.

The next Lemma shows that a solution satisfying ANO, IIA, INH-NEM, and WDOM, satisfies the following property: if an action quasi-dominates an action in the solution, then the former action also belongs to the solution.

Lemma 4.3. *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, and WDOM, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ quasi-dominates a^*, then $a' \in \varphi(D)$.*

Proof. Let $\omega_1, \omega_2 \in \Omega$ be as in the definition of quasi-dominance. Define the decision problem $D' = (A \cup \{\alpha\}, \Omega, u') \in \mathcal{D}$ with $\alpha \notin A$, $u'_{|A \times \Omega} = u$, $u'(\alpha, \omega) = u(a^*, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1, \omega_2\}$, $u'(\alpha, \omega_1) = u(a^*, \omega_2)$, and $u'(\alpha, \omega_2) = u(a^*, \omega_1)$. By Lemma 4.2: $\{a^*, \alpha\} \subseteq \varphi(D')$. Now a' weakly dominates α unless $u'(a', \omega) = u'(\alpha, \omega)$ for all $\omega \in \Omega$ (in which case $a' \in \varphi(D')$ by ANO). So, by WDOM, $a' \in \varphi(D')$.

Hence, $\{a^*, \alpha, a'\} \subseteq \varphi(D')$. Now $\varphi(D) = \varphi(D') \cap A$ by IIA, so $a' \in \varphi(D)$. □

If a solution satisfies ANO, IIA, INH-NEM, WDOM, DOS, and CONT, then whether or not an action belongs to the solution of a decision problem depends exclusively on the infimum and supremum of its payoffs.

Lemma 4.4. *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, WDOM, DOS, and CONT, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ is such that*

$$\inf_{\omega \in \Omega} u(a', \omega) = \inf_{\omega \in \Omega} u(a^*, \omega) = m \text{ and } \sup_{\omega \in \Omega} u(a', \omega) = \sup_{\omega \in \Omega} u(a^*, \omega) = M,$$

then $a' \in \varphi(D)$.

Proof. If $m = M$, then a^* and a' yield the same, constant payoff, regardless of ω, so ANO and $a^* \in \varphi(D)$ imply that $a' \in \varphi(D)$. So henceforth assume that $m < M$. This means that Ω has at least two elements. Let $\omega_1 \in \Omega$. Define for each $(\varepsilon, \delta) \in \mathbb{R}^2_+$ the decision problem

$D_{\varepsilon,\delta} = (A \cup \{\alpha, \beta\}, \Omega, u_{\varepsilon,\delta})$ with $\alpha, \beta \notin A$ as follows. For all $(\tilde{a}, \omega) \in (A \cup \{\alpha, \beta\}) \times \Omega$,

$$u_{\varepsilon,\delta}(\tilde{a}, \omega) = \begin{cases} u(a', \omega) + \delta & \text{if } \tilde{a} = a', \\ m + \varepsilon & \text{if } (\tilde{a}, \omega) = (\alpha, \omega_1), \\ m & \text{if } \tilde{a} = \beta \text{ and } \omega \neq \omega_1, \\ M & \text{if } (\tilde{a}, \omega) = (\beta, \omega_1) \text{ or } (\tilde{a} = \alpha \text{ and } \omega \neq \omega_1), \\ u(\tilde{a}, \omega) & \text{otherwise.} \end{cases}$$

The table below summarizes the definition of $D_{\varepsilon,\delta}$.

States / Actions	ω_1	$\omega \in \Omega \setminus \{\omega_1\}$
a^*	$u(a^*, \omega_1)$	$u(a^*, \omega)$
a'	$u(a', \omega_1) + \delta$	$u(a', \omega) + \delta$
α	$m + \varepsilon$	M
β	M	m
all other a	$u(a, \omega_1)$	$u(a, \omega)$

Let $D' = (A \setminus \{a'\}, \Omega, u_{|(A\setminus\{a'\})\times\Omega}) \in \mathcal{D}$. Since $a^* \in \varphi(D) \cap (A \setminus \{a'\})$, IIA implies that $\varphi(D') = \varphi(D) \cap (A \setminus \{a'\}) \neq \emptyset$. For all $(\varepsilon, \delta) \in \mathbb{R}^2_+$, $D_{\varepsilon,\delta}$ is obtained from D' by adding finitely many actions, so INH-NEM implies that $\varphi(D_{\varepsilon,\delta}) \neq \emptyset$.

Step 1: Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers with $\lim_{k\to\infty} \varepsilon_k = 0$. We show that $\alpha \in \varphi(D_{\varepsilon_k,0})$ for all $k \in \mathbb{N}$. By CONT, we then have $\alpha \in \varphi(D_{0,0})$.

Let $k \in \mathbb{N}$ and suppose, to the contrary, that $\alpha \notin \varphi(D_{\varepsilon_k,0})$. Since $\varphi(D_{\varepsilon_k,0}) \neq \emptyset$, we have two cases:

- $\beta \in \varphi(D_{\varepsilon_k,0})$. This is not possible, because α quasi-dominates β and applying Lemma 4.3 one obtains that $\alpha \in \varphi(D_{\varepsilon_k,0})$.

- $\beta \notin \varphi(D_{\varepsilon_k,0})$. Since $\varphi(D_{\varepsilon_k,0}) \neq \emptyset$ and $\alpha, \beta \notin \varphi(D_{\varepsilon_k,0})$ there is an $a \in \varphi(D_{\varepsilon_k,0}) \cap A$. By IIA: $\varphi(D_{\varepsilon_k,0}) \cap A = \varphi(D)$, so $a^* \in \varphi(D_{\varepsilon_k,0})$.

  – If $u(a^*, \omega_1) \leq m + \varepsilon_k$, then α weakly dominates a^*: $u(a^*, \omega) \leq u(\alpha, \omega)$ for all $\omega \in \Omega$, and there is an $\omega_0 \in \Omega$ such that $u(a^*, \omega_0) < u(\alpha, \omega_0)$, because otherwise $u(a^*, \omega) = u(\alpha, \omega)$ for all $\omega \in \Omega$, so that

  $$m = \inf_{\omega\in\Omega} u(a^*, \omega) = \inf_{\omega\in\Omega} u(\alpha, \omega) = \min\{m + \varepsilon_k, M\} > m,$$

  a contradiction. Using WDOM, it follows that $\alpha \in \varphi(D_{\varepsilon_k,0})$.

  – If $u(a^*, \omega_1) > m + \varepsilon_k$, then α quasi-dominates a^*: $u(\alpha, \omega) \geq u(a^*, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1\}$ and by definition of $m = \inf_{\omega\in\Omega} u(a^*, \omega)$, there is an $\omega_2 \in \Omega$, different from ω_1 (since $u(a^*, \omega_1) > m + \varepsilon_k$) with $u(a^*, \omega_2) \leq m + \varepsilon_k$. This implies that $M = u(\alpha, \omega_2) \geq u(a^*, \omega_1) > m + \varepsilon_k \geq u(a^*, \omega_2)$. By Lemma 4.3, $\alpha \in \varphi(D_{\varepsilon_k,0})$.

In both subcases, we established that $\alpha \in \varphi(D_{\varepsilon_k,0})$, in contradiction with our assumption. Conclude that $\alpha \in \varphi(D_{\varepsilon_k,0})$.

Step 2: We show that $\beta \in \varphi(D_{0,0})$.

Let $\omega_2 \in \Omega$, $\omega_2 \neq \omega_1$, and consider the decision problems

$$D_1 = \left(\{\alpha, \beta\}, \{\omega_1, \omega_2\}, u_{0,0|\{\alpha,\beta\}\times\{\omega_1,\omega_2\}}\right) \text{ and } D_2 = \left(\{\alpha, \beta\}, \Omega, u_{0,0|\{\alpha,\beta\}\times\Omega}\right).$$

D_2 can be obtained from $D_{0,0}$ by deleting actions. By step 1, $\varphi(D_{0,0}) \cap \{\alpha, \beta\} \neq \emptyset$. So IIA implies that

$$\varphi(D_{0,0}) \cap \{\alpha, \beta\} = \varphi(D_2). \tag{4.2}$$

Therefore, $\alpha \in \varphi(D_2)$. By DOS, $\varphi(D_1) = \varphi(D_2)$, so $\alpha \in \varphi(D_1)$. Now Lemma 4.1 implies that $\beta \in \varphi(D_1)$. Since $\varphi(D_1) = \varphi(D_2)$, equation (4.2) gives that $\beta \in \varphi(D_{0,0})$.

Step 3: Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers with $\lim_{k\to\infty} \delta_k = 0$. We show that $a' \in \varphi(D_{0,\delta_k})$ for all $k \in \mathbb{N}$. By CONT, we then have $a' \in \varphi(D_{0,0})$.

Consider the decision problem

$$D_3 = \left(A_3, \Omega, u_{0,0|A_3\times\Omega}\right)$$

where $A_3 = (A \cup \{\alpha, \beta\}) \setminus \{a'\}$ for some $\alpha, \beta \notin A$. By steps 1 and 2, $\varphi(D_{0,0}) \cap A_3 \neq \emptyset$, so IIA implies that $\varphi(D_{0,0}) \cap A_3 = \varphi(D_3)$. Hence, from step 2, $\beta \in \varphi(D_3)$.

Let $\delta_k > 0$ and suppose that $a' \notin \varphi(D_{0,\delta_k})$. Since $\varphi(D_{0,\delta_k}) \neq \emptyset$ one obtains that $\varphi(D_{0,\delta_k}) \cap A_3 \neq \emptyset$ and then IIA implies that $\beta \in \varphi(D_{0,\delta_k})$. So, reasoning as in step 1: if $u(a', \omega_1) + \delta_k \geq M$, then a' weakly dominates β and, by WDOM, $a' \in \varphi(D_{\delta_k,0})$; otherwise, a' quasi-dominates β and by Lemma 4.3: $a' \in \varphi(D_{\delta_k,0})$. In both cases we reach a contradiction. Conclude that $a' \in \varphi(D_{\delta_k,0})$.

Step 4: Finally, we show that $a' \in \varphi(D)$.

By step 3 $a' \in \varphi(D_{0,0}) \cap A$. Hence, IIA implies $\varphi(D_{0,0}) \cap A = \varphi(D)$, and so $a' \in \varphi(D)$. □

These results will help us prove Theorem 4.1:

Proof of Thm. 4.1. It is easy to verify that the solution M satisfies all the properties.

Let φ be a solution on $\mathcal{D}$ satisfying all the properties and let $D = (A, \Omega, u) \in \mathcal{D}$. If $\varphi(D) = \emptyset$, then by r-NEM: $M(D) = \emptyset$. So, assume that $\varphi(D) \neq \emptyset$.

Under the assumption that whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs, it is true that $\varphi(D) = M(D)$. Namely, consider the decision problem $\widehat{D} = (A, \widehat{\Omega}, \widehat{u})$ where $|\widehat{\Omega}| = 1$ and $\widehat{u}(a, \widehat{\omega}) = \inf_{\omega\in\Omega} u(a, \omega)$ for all $(a, \widehat{\omega}) \in A \times \widehat{\Omega}$. We show that

$$\varphi(D) = \varphi(\widehat{D}). \tag{4.3}$$

Consider the decision problem $\widetilde{D} = (\widetilde{A}, \Omega, \widetilde{u}) \in \mathcal{D}$ obtained from D by adding to the action space a replica $r(a)$ of every action $a \in A$, i.e., $\widetilde{A} = \{a, r(a)\}_{a\in A}$ and with payoffs $\widetilde{u}_{|A\times\Omega} = u$ and $\widetilde{u}(r(a), \omega) = \inf_{\omega\in\Omega} u(a, \omega)$ for all $a \in A$ and $\omega \in \Omega$.

By the assumption: $a \in \varphi(D)$ if and only if $\{a, r(a)\} \subseteq \varphi(\widetilde{D})$. Since $\varphi(D) \neq \emptyset$, deletion of all non-replica actions and IIA imply that

$$a \in \varphi(D) \Leftrightarrow r(a) \in \varphi((\{r(a)\}_{a\in A}, \Omega, \widetilde{u}_{\{r(a)\}_{a\in A}\times\Omega})). \tag{4.4}$$

ANO and DOS imply that

$$r(a) \in \varphi((\{r(a)\}_{a\in A}, \Omega, \widetilde{u}_{\{r(a)\}_{a\in A}\times\Omega})) \Leftrightarrow a \in \varphi(\widehat{D}). \tag{4.5}$$

The equality (4.3) now follows from (4.4) and (4.5).

Write $\widehat{\Omega} = \{\widehat{\omega}\}$. By OSR we know that $\varphi(\widehat{D}) = M(\widehat{D}) = \arg\max_{a\in A} \widehat{u}(a, \widehat{\omega})$. Finally, since M satisfies all the properties we also have that $M(\widehat{D}) = M(D)$. Therefore $\varphi(D) = M(D)$.

Now it remains to prove that whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs.

Let $a^* \in \varphi(D)$ and let $m = \inf_{\omega\in\Omega} u(a^*, \omega)$ and $M = \sup_{\omega\in\Omega} u(a^*, \omega)$. If $m = M$, then $u(a^*, \omega) = m$ for all $\omega \in \Omega$. Let $a \in A$ be such that $\inf_{\omega\in\Omega} u(a, \omega) = m$. If $\sup_{\omega\in\Omega} u(a, \omega) = m$, then $u(a, \omega) = u(a^*, \omega)$ for all $\omega \in \Omega$ and, by ANO, $a \in \varphi(D)$; otherwise, a weakly dominates a^*, so, by WDOM, $a \in \varphi(D)$. Therefore, if $m = M$, then whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs.

So henceforth assume that $m < M$. This implies in particular that Ω contains at least two elements. Choose $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$.

Take $D' = (A, \Omega', u') \in \mathcal{D}$ where $\Omega' = \{\omega_1, \omega_2, \omega_3\}$ with $\omega_3 \notin \Omega$ and, for all $a \in A$:

$$u'(a, \omega') = \begin{cases} \sup\limits_{\omega\in\Omega} u(a, \omega) & \text{if } \omega' = \omega_1 \\ \inf\limits_{\omega\in\Omega} u(a, \omega) & \text{otherwise} \end{cases}$$

The table below summarizes the definition of D'.

Actions \ States	ω_1	ω_2	ω_3
$\vdots$		$\vdots$	
a^*	M	m	m
a	$\sup\limits_{\omega\in\Omega} u(a,\omega)$	$\inf\limits_{\omega\in\Omega} u(a,\omega)$	$\inf\limits_{\omega\in\Omega} u(a,\omega)$
$\vdots$		$\vdots$	

Similar to the proof of (4.3), using Lemma 4.4 instead of the assumption, it follows that $\varphi(D) = \varphi(D')$.

Define the sequence of decision problems $\{D_k\}_{k\in\mathbb{N}} = \{(A \cup \{\alpha, \beta, \gamma\}, \Omega', u_k)\}_{k\in\mathbb{N}}$ where

$\alpha, \beta, \gamma \notin A$, $u_{k|A\times\Omega'} = u'$ and, for all $(a,\omega) \in \{\alpha,\beta,\gamma\} \times \Omega'$,

$$u_k(a,\omega) = \begin{cases} m + \dfrac{1}{2^{k-1}}(M-m) & \text{if } (a,\omega) \in \{(\alpha,\omega_1),(\beta,\omega_2)\} \\ m + \dfrac{1}{2^k}(M-m) & \text{if } (a,\omega) \in \{(\gamma,\omega_1),(\gamma,\omega_2)\} \\ m & \text{otherwise.} \end{cases}$$

The table below summarizes the definition of D_k.

Actions \ States	ω_1	ω_2	ω_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^*	M	m	m
$\vdots$	$\vdots$	$\vdots$	$\vdots$
α	$m + \frac{1}{2^{k-1}}(M-m)$	m	m
β	m	$m + \frac{1}{2^{k-1}}(M-m)$	m
γ	$m + \frac{1}{2^k}(M-m)$	$m + \frac{1}{2^k}(M-m)$	m
$\vdots$	$\vdots$	$\vdots$	$\vdots$

For all $k \in \mathbb{N}$, D_k can be obtained from D' by adding three actions. So, $\varphi(D') \neq \emptyset$ and INH-NEM imply that $\varphi(D_k) \neq \emptyset$. We show by induction that $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$.

Step 1: $\gamma \in \varphi(D_1)$.

D_1 can be obtained from D' by adding actions α, β, and γ in two steps:

First, add α and β to obtain the decision problem $D'_1 = (A \cup \{\alpha,\beta\}, \Omega', u'_1)$ with $u'_1 = u_{1|(A\cup\{\alpha,\beta\})\times\Omega'}$. Lemma 4.4 implies that $\alpha \in \varphi(D'_1)$ if and only if $\beta \in \varphi(D'_1)$. Suppose that $\alpha, \beta \notin \varphi(D'_1)$. INH-NEM and $\varphi(D') \neq \emptyset$ imply that $\varphi(D'_1) \neq \emptyset$, so there is an $a \in \varphi(D'_1) \cap A$. Then, by IIA, $\varphi(D'_1) \cap A = \varphi(D')$. Hence, $a^* \in \varphi(D'_1)$. Lemma 4.4 then implies that $\alpha, \beta \in \varphi(D'_1)$, which is a contradiction. Thus $\alpha, \beta \in \varphi(D'_1)$.

Second, add action γ, whose utility is the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the utility of the actions α and β, and by CONV: $\gamma \in \varphi(D_1)$.

Step 2: Let $k \in \mathbb{N}$ and assume that $\gamma \in \varphi(D_n)$ for all $n \in \mathbb{N}, n \leq k$. We show that $\gamma \in \varphi(D_{k+1})$.

The decision problem D_{k+1} can be obtained from D_k in two steps:

First, delete actions α and β from D_k to obtain a new decision problem. By IIA and the assumption that $\gamma \in \varphi(D_k)$, its solution contains γ. Next, introduce actions α and β again, but now with their utility functions equal to those in the problem D_{k+1}. Since α and β have the same infimum and supremum, α belongs to the solution if and only if β belongs to the solution of this new problem. Suppose that α and β do not belong to the solution. By INH-NEM and IIA, γ belongs to the solution. But then Lemma 4.4 implies that α and β should belong to the solution, which is a contradiction. Thus α and β belong to the solution.

Second, delete γ from this new problem to obtain the decision problem $D'_{k+1} = (A \cup \{\alpha, \beta\}, \Omega', u'_{k+1})$ with $u'_{k+1} = u_{k+1|(A\cup\{\alpha,\beta\})\times\Omega'}$. By IIA $\alpha, \beta \in \varphi(D'_{k+1})$. Next, introduce action γ again, but now with utility function equal to the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the payoffs of actions α and β in D'_{k+1}, so the decision problem D_{k+1} is obtained. By CONV it follows that $\gamma \in \varphi(D_{k+1})$.

Conclude, by induction, that $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$.

Let $D_\infty = (A \cup \{\alpha, \beta, \gamma\}, \Omega', u_\infty)$ be the limiting decision problem of the sequence $\{D_k\}_{k\in\mathbb{N}}$. Notice that $u_{\infty|A\times\Omega'} = u'$ and $u_\infty(\alpha, \omega) = u_\infty(\beta, \omega) = u_\infty(\gamma, \omega) = m$ for all $\omega \in \Omega'$. Since $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$, CONT implies that $\gamma \in \varphi(D_\infty)$.

Take $a \in A$ such that $\inf_{\omega\in\Omega'} u'(a, \omega) = m$. If $\sup_{\omega\in\Omega'} u'(a, \omega) = m$, then $u_\infty(a, \omega) = u'(a, \omega) = m = u_\infty(\gamma, \omega)$ for all $\omega \in \Omega$, so that $a \in \varphi(D_\infty)$ by ANO. Otherwise, a weakly dominates γ and, by WDOM, $a \in \varphi(D_\infty)$. Hence $a \in \varphi(D_\infty) \cap A$, and using IIA it follows that $\varphi(D_\infty) \cap A = \varphi(D') = \varphi(D)$.

Hence, $a \in \varphi(D)$ for all $a \in A$ with $\inf_{\omega\in\Omega} u(a, \omega) = \inf_{\omega\in\Omega} u(a^*, \omega) = m$.

$\square$

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Chapter 5

A note on axiomatizations of protectiveness

Contents

5.1 Introduction

The maximin behavior is a well-known notion that has been studied in many fields of knowledge. Choosing alternatives according to this behavior involves associating to each alternative the worst possible consequence and then choosing alternative(s) for which this worst-case scenario offers the best possible result. In this line, maximin behavior can be seen as a *cautious* behavior. This decision principle was applied in several fields of knowledge as game theory, experimental economics, statistical decision theory, ... See Mosquera et al. (2005) for a more detailed revision.

There also are several studies about the foundations of this principle. Such studies focus on: (a) characterizing the order induced by the maximin criterion, like in Milnor (1954) and in Barberà and Jackson (1988), or (b) characterizing the maximin value associated with zero-sum games, like in Norde and Voorneveld (2004) and Carpente et al. (2005), or (c) characterizing the solution that assigns to each decision problem its set of maximin actions, like in Mosquera et al. (2005).

However, in many situations, the maximin behavior is not helpful in order to select not so many alternatives, then we have to seek for other posible behaviors. We like this *"cautious"* behavior and we want to keep it in a new possible behavior. Several refinements of maximin behavior have been studied in the literature. Barberà and Dutta (1982) introduce the notion of *protective* behavior in the context of social choice functions for social choice situations with a finite number of alternatives. Moulin (1981) define another refinement of maximin notion, the *prudent* behavior, closely related with protective behavior (in fact, they coincide in many classes).

Protective behavior means that the agent maximizes his worst possible result with respect to all consequences and, in case of tie, he also searches for the minimality in terms of inclusion of the sets of consequences which provides this worst result. This behavior was studied in other different contexts from Barberà and Dutta's context. For instance, Barberà and Jackson (1988) provides an axiomatization of protective behavior as an order on the set of all finite-dimensional vectors of real numbers. Also maximin and prudent behaviors are axiomatically characterized in that paper. In the setting of finite games in strategic form, Dresher (1981) proposes to select optimal strategies for each player based on a lexicographic application of the maximin criterion. The idea underlying is that players consider the possibility that their opponents may make a mistake choosing its strategy, but they do not know which mistake will be made. Then, each of the players should follow a conservative plan of action. On the other hand, Fiestras-Janeiro et al. (1998) study protective and prudent behavior in games in strategic form. They define the notions of prudent and protective equilibria, they provide the equality of the notions in this context, and they prove that, for matrix games, the set of protective equilibria equals the set of proper equilibria (see Myerson, 1978). That paper was extended by Quant et al. (2004) to the case of multicriteria games. The two-person competitive environments are defined as bimatrix games with the features of matrix games. Borm et al. (2005) show relations among protective behavior and proper and perfect equilibria (see Selten, 1975) for competitive environments.

Given the importance that protective behavior has obtained in the last years, we want to study a bit deeper its foundations. Our goal is to characterize the solution that assigns to each decision problem its set of protective actions, taking into account that we like the *"cautiousness"*. The chapter is organized as follows. Section 5.2 provides the basic definitions of the model, the existence of protective actions and the relation with prudent behavior. Axiomatizations of the set of protective actions is given in Section 5.3, taking into account that we seek for *"cautious"* solutions. Finally, Section 5.4 concludes with the analysis of the logical independence of properties of the different axiomatizations.

5.2 The model

A *decision problem* is a tuple (A, Ω, u), where A is a nonempty, convex, compact set of actions, Ω is a nonempty, finite set of states, and $u : A \times \Omega \to \mathbb{R}$ is a continuous, concave in A function which represents the decision-maker's payoff/utility function. The set of all decision problems in these conditions is denoted by $\mathcal{D}$.

Let $(A, \Omega, u) \in \mathcal{D}$. Let $a \in A$. Recursively, we define $\alpha^r(a) \in \mathbb{R}$ and $\Omega^r(a) \subseteq \Omega$ by

(i) for $r = 1$,

$$\alpha^1(a) = \min_{\omega \in \Omega} u(a, \omega)$$

$$\Omega^1(a) = \left\{ \omega \in \Omega \;\middle|\; u(a, \omega) = \alpha^1(a) \right\},$$

(ii) for $r > 1$

$$\alpha^r(a) = \min_{\omega \in \Omega \setminus \bigcup_{k=1}^{r-1} \Omega^k(a)} u(a, \omega)$$

$$\Omega^r(a) = \left\{ \omega \in \Omega \setminus \bigcup_{k=1}^{r-1} \Omega^k(a) \;\middle|\; u(a, \omega) = \alpha^r(a) \right\}.$$

Let $a, a' \in A$, *a protectively dominates* a', $a \succ_{pro} a'$, if there exists an $\ell \in \mathbb{N}$, such that

(i) $\alpha^r(a) = \alpha^r(a')$ and $\Omega^r(a) = \Omega^r(a')$ for each $r \in \mathbb{N}$ such that $r < \ell$, and

(ii) $\alpha^\ell(a) > \alpha^\ell(a')$ or both $\alpha^\ell(a) = \alpha^\ell(a')$ and $\Omega^\ell(a) \subset \Omega^\ell(a')$.

We say that $a \in A$ is *protective* if it is not protectively dominated.

A *solution* on $\mathcal{D}$ is a correspondence φ that assigns to every decision problem $(A, \Omega, u) \in \mathcal{D}$ a subset $\varphi(A, \Omega, u) \subseteq A$ of actions. Our aim is to characterize the solution PRO that assigns to each decision problem $(A, \Omega, u) \in \mathcal{D}$ its set of *protective actions*,

$$\text{PRO}(A, \Omega, u) = \{a \in A \mid a \text{ is protective}\}.$$

Lemma 5.1. *Let $D = (A, \Omega, u) \in \mathcal{D}$. Let $\tilde{a} \in A$ be a protective action and $\bar{a} \in A$. Then, either $\tilde{a}$ and $\bar{a}$ are payoff equivalent or $\tilde{a} \succ_{pro} \bar{a}$.*

Proof. Assume that $\tilde{a}$ and $\bar{a}$ are not payoff equivalent and $\tilde{a} \not\succ_{pro} \bar{a}$. Since $\tilde{a} \in \mathrm{PRO}(D)$, $\bar{a} \not\succ_{pro} \tilde{a}$, i.e., there exists $\ell \in \mathbb{N}$ such that:

(i) $\alpha^r(\bar{a}) = \alpha^r(\tilde{a})$ and $\Omega^r(\bar{a}) = \Omega^r(\tilde{a})$ for each $r \in \mathbb{N}$, $r < \ell$,

(ii) $\alpha^\ell(\bar{a}) = \alpha^\ell(\tilde{a})$, $\Omega^\ell(\bar{a}) \setminus \Omega^\ell(\tilde{a}) \neq \emptyset$ and $\Omega^\ell(\tilde{a}) \setminus \Omega^\ell(\bar{a}) \neq \emptyset$.

Let $0 < \lambda < 1$ and let $\hat{a} = \lambda\tilde{a} + (1-\lambda)\bar{a} \in A$. We will prove that $\hat{a} \succ_{pro} \tilde{a}$.

First of all, we will prove that $\alpha^r(\hat{a}) = \alpha^r(\tilde{a})$ and $\Omega^r(\hat{a}) = \Omega^r(\tilde{a})$ for all $r < \ell$ by induction on r. Let $r = 1$. By concavity of u in A and by (i) it follows that $\alpha^1(\hat{a}) \geq \alpha^1(\tilde{a})$. If $\alpha^1(\hat{a}) > \alpha^1(\tilde{a})$, then $\hat{a} \succ_{pro} \tilde{a}$ which is a contradiction with $\tilde{a} \in \mathrm{PRO}(D)$. Then, $\alpha^1(\hat{a}) = \alpha^1(\tilde{a})$. If there exists some $\hat{\omega} \in \Omega^1(\hat{a}) \setminus \Omega^1(\tilde{a})$, then:

$$\alpha^1(\hat{a}) = u(\hat{a}, \hat{\omega}) \geq \lambda u(\tilde{a}, \hat{\omega}) + (1-\lambda)u(\bar{a}, \hat{\omega}) > \alpha^1(\tilde{a}).$$

This is a contradiction with $\tilde{a} \in \mathrm{PRO}(D)$, so that $\Omega^1(\hat{a}) \subseteq \Omega^1(\tilde{a})$ and $\alpha^1(\hat{a}) = \alpha^1(\tilde{a})$. In case that $\Omega^1(\hat{a}) \subset \Omega^1(\tilde{a})$, then $\hat{a} \succ_{pro} \tilde{a}$ which is again a contradiction with $\tilde{a} \in \mathrm{PRO}(D)$. Hence $\Omega^1(\hat{a}) = \Omega^1(\tilde{a})$. Following the same reasoning for each $r < \ell$ it is obtained that $\alpha^r(\hat{a}) = \alpha^r(\tilde{a})$ and $\Omega^r(\hat{a}) = \Omega^r(\tilde{a})$.

Now, we will prove that $\alpha^\ell(\hat{a}) > \alpha^\ell(\tilde{a})$ or both $\alpha^\ell(\hat{a}) = \alpha^\ell(\tilde{a})$ and $\Omega^\ell(\hat{a}) \subset \Omega^\ell(\tilde{a})$. Discern two cases:

(a) $\Omega^\ell(\tilde{a}) \cap \Omega^\ell(\bar{a}) = \emptyset$.
Let $\omega \in \Omega \setminus \cup_{r=1}^{\ell-1}\Omega^r(\tilde{a}) = \Omega \setminus \cup_{r=1}^{\ell-1}\Omega^r(\bar{a}) = \Omega \setminus \cup_{r=1}^{\ell-1}\Omega^r(\hat{a})$. If $\omega \in \Omega^\ell(\bar{a})$, then $\omega \notin \Omega^\ell(\tilde{a})$ and $u(\tilde{a}, \omega) > \alpha^\ell(\tilde{a}) = \alpha^\ell(\bar{a})$. Then, using concavity of u in A,

$$u(\hat{a}, \omega) \geq \lambda u(\tilde{a}, \omega) + (1-\lambda)u(\bar{a}, \omega) > \alpha^\ell(\bar{a}) = \alpha^\ell(\tilde{a}).$$

Similarly for the case when $\omega \in \Omega^\ell(\tilde{a})$.

If $\omega \notin \Omega^\ell(\tilde{a}) \cup \Omega^\ell(\bar{a})$, then $u(\tilde{a}, \omega) > \alpha^\ell(\tilde{a})$ and $u(\bar{a}, \omega) > \alpha^\ell(\tilde{a})$. Then, using concavity of u in A, $u(\hat{a}, \omega) > \alpha^\ell(\tilde{a})$.

We can conclude that $u(\hat{a}, \omega) > \alpha^\ell(\tilde{a})$ for all $\omega \in \Omega \setminus \cup_{k=1}^{\ell-1}\Omega^k(\hat{a})$. Therefore, $\alpha^\ell(\hat{a}) > \alpha^\ell(\tilde{a})$.

(b) $\Omega^\ell(\tilde{a}) \cap \Omega^\ell(\bar{a}) \neq \emptyset$.
As a consequence of (i), concavity of u in A and (ii), it follows that $\alpha^\ell(\hat{a}) \geq \alpha^\ell(\tilde{a})$. If $\alpha^\ell(\hat{a}) > \alpha^\ell(\tilde{a})$, then $\hat{a} \succ_{pro} \tilde{a}$ which is a contradiction. Let then $\alpha^\ell(\hat{a}) = \alpha^\ell(\tilde{a})$. For each $\omega \in \Omega \setminus \left(\cup_{k=1}^{\ell-1}\Omega^k(\hat{a}) \cup \left(\Omega^\ell(\tilde{a}) \cap \Omega^\ell(\bar{a})\right)\right)$, either $u(\tilde{a}, \omega) > \alpha^\ell(\tilde{a})$ or $u(\bar{a}, \omega) > \alpha^\ell(\bar{a})$, then $u(\hat{a}, \omega) > \alpha^\ell(\hat{a})$. Therefore,

$$\Omega^\ell(\hat{a}) \subseteq \Omega^\ell(\tilde{a}) \cap \Omega^\ell(\bar{a}) \subset \Omega^\ell(\tilde{a})$$

where the strict set inclusion follows from (ii).

In both cases we obtain that $\hat{a} \succ_{pro} \tilde{a}$, which is a contradiction with $\tilde{a} \in \mathrm{PRO}(D)$. □

As a consequence of Lemma 5.1, the set of protective actions equals the set of prudent actions in the class $\mathcal{D}$. We recall the definition of prudent actions. Let $a, a' \in A$, *a prudently dominates* a', $a \succ_{pru} a'$, if there exists an $\ell \in \mathbb{N}$, such that

(i) $\alpha^r(a) = \alpha^r(a')$ and $|\Omega^r(a)| = |\Omega^r(a')|$ for all $r \in \mathbb{N}$ such that $r < \ell$, and

(ii) $\alpha^\ell(a) > \alpha^\ell(a')$ or both $\alpha^\ell(a) = \alpha^\ell(a')$ and $|\Omega^\ell(a)| < |\Omega^\ell(a')|$.

We say that $a \in A$ is *prudent* if it is not prudently dominated.

Note that the difference between prudent and protective concepts is that, even though both concepts compare payoff levels, prudent only compares the cardinality of the sets of states of nature where those payoff levels are achieved instead of the inclusion relation used by the protective concept.

Theorem 5.1. *In a decision problem in $\mathcal{D}$ an action is protective if and only if it is prudent.*

Proof. By definition, prudence implies protectiveness. Then, we only have to prove that if an action is protective, then it is prudent. Let $D \in \mathcal{D}$ and let $\tilde{a} \in A$ be a protective action such that it is not prudent. Then, there exists $\bar{a} \in A$ such that $\bar{a} \succ_{pru} \tilde{a}$. Then, $\tilde{a}$ and $\bar{a}$ are not payoff equivalent and so, by Lemma 5.1, $\tilde{a} \succ_{pro} \bar{a}$. Consequently, by definition, $\tilde{a} \succ_{pru} \bar{a}$. However, this is a contradiction. □

Next, we show that, for each decision problem in $\mathcal{D}$, there always exist prudent an protective actions.

Theorem 5.2. *Each decision problem in $\mathcal{D}$ has at least one prudent action.*

Proof. Let $D \in \mathcal{D}$. Define the sets

$$\begin{aligned} M^1 &= \left\{ \bar{a} \in A \;\middle|\; \alpha^1(\bar{a}) = \max_{a \in A} \alpha^1(a) \right\}, \\ P^1 &= \left\{ \bar{a} \in M^1 \;\middle|\; |\Omega^1(\bar{a})| = \min_{a \in M^1} |\Omega^1(a)| \right\}, \end{aligned}$$

and for each $r > 1$

$$\begin{aligned} M^r &= \left\{ \bar{a} \in P^{r-1} \;\middle|\; \alpha^r(\bar{a}) = \max_{a \in P^{r-1}} \alpha^r(a) \right\}, \\ P^r &= \left\{ \bar{a} \in M^r \;\middle|\; |\Omega^r(\bar{a})| = \min_{a \in M^r} |\Omega^r(a)| \right\}. \end{aligned}$$

Note that M^1 is the set of cautious actions of D, then, by conditions on the model, $M^1 \neq \emptyset$ and $P^1 \neq \emptyset$. Moreover, for each $\ell \in \mathbb{N}$ and each pair $a, a' \in P^\ell$, $\alpha^r(a) = \alpha^r(a')$ and $|\Omega^r(a)| = |\Omega^r(a')|$

for each $1 \leq r \leq \ell$. Since Ω is a finite set, there exists $\ell_0 \in \mathbb{N}$ such that, for each $a \in P^{\ell_0}$, $\bigcup_{r=1}^{\ell_0} \Omega^r(a) = \Omega$ and $|\Omega^{\ell_0}(a)| > 0$. Define $\Omega^1 = \Omega$ and $\Omega^{r+1} = \Omega^r \setminus \bigcap_{a \in P^r} \Omega^r(a)$ for each $1 \leq r < \ell_0$. Define the following decision problems in $\mathcal{D}$:

$$\begin{aligned} D^1 &= D, \\ D^r &= (M^{r-1}, \Omega^r, u_{|M^{r-1} \times \Omega^r}) \text{ for each } 1 < r \leq \ell_0. \end{aligned}$$

For each $1 \leq r \leq \ell_0$, it is easy to check that M^r is the set of cautious actions of D^r, so that $M^r \neq \emptyset$ and $P^r \neq \emptyset$. Moreover, $M^r = M^{\ell_0}$ and $P^r = P^{\ell_0}$ for each $r > \ell_0$. By definition, P^{ℓ_0} is precisely the set of prudent actions of D. □

5.3 Axiomatizations of protectiveness

In this section we provide some characterizations of the set of protective actions using some standard properties. First of all, we will define the properties we use throughout the chapter. Let φ be a solution on $\mathcal{D}$.

Nonemptiness (NEM). $\varphi(D)$ is nonempty for each $D \in \mathcal{D}$.

Sure-thing principle (STP). Let $D, D' \in \mathcal{D}$ be such that $A' = A$, $\Omega' = \Omega \cup \{\omega'\}$ for some $\omega' \notin \Omega$, $u'_{|A \times \Omega} = u$ and let $\kappa \in \mathbb{R}$ be such that $u(a, \omega') = \kappa$ for each $a \in A$. Then, $\varphi(D) = \varphi(D')$.

Indistinguishability (IND). Let $D \in \mathcal{D}$ be such that there exists $\kappa \in \mathbb{R}$ and $u(a, \omega) = \kappa$ for each pair $(a, \omega) \in A \times \Omega$. Then, $\varphi(D) = A$.

Anonymity (ANO). Let $D, D' \in \mathcal{D}$. If there are bijections $f : A \to A'$ and $g : \Omega \to \Omega'$ such that $u(a, \omega) = u'(f(a), g(\omega))$ for each pair $(a, \omega) \in A \times \Omega$, then $\varphi(D') = f(\varphi(D))$.

Independence of irrelevant actions (IIA). Let $D, D' \in \mathcal{D}$ be such that $A' \subset A$, $\Omega' = \Omega$ and $u' = u_{|A' \times \Omega}$. If $\varphi(D) \cap A' \neq \emptyset$, then $\varphi(D') = \varphi(D) \cap A'$.

One state rationality (OSR). Let $D \in \mathcal{D}$ with $|\Omega| = 1$. Then, $\varphi(D) \subseteq \arg\max_{a \in A} u(a, \omega)$, where $\Omega = \{\omega\}$.

NEM, STP, ANO, IIA and OSR are standard properties in the literature on decision theory. IND is a kind of anonymity, indeed IND is weaker than standard anonymity (ANO).

However, we seek for characterizing solutions that are cautious. Then, we will need some non-standard properties related with cautiousness. Let $D = (A, \Omega, u) \in \mathcal{D}$, $\omega^d \in \Omega$ is called a *disaster state* of D if $u(a, \omega^d) = \min_{\omega \in \Omega} u(a, \omega)$ for each $a \in A$. A disaster state is a state in which the decision maker always obtains the worst payoff with respect to other states, independently of his choice. Note that, if a decision problem $D \in \mathcal{D}$ is such that there exists no disaster state, then it is easy to extend the problem to a problem with a disaster state. One only has to consider the decision problem $D^d = (A, \Omega \cup \{\omega^d\}, u^d)$ with $u^d_{|A\times\Omega} = u$ and ω^d is such that $u^d(a, \omega^d) = \min_{\omega\in\Omega} u(a, \omega)$. Let φ be a solution on $\mathcal{D}$.

Cautious restriction (CAR). Let $D \in \mathcal{D}$. Let $M(D)$ be the set of cautious actions of D. Then, $\varphi(D) = \varphi(M(D), \Omega, u_{|M(D)\times\Omega})$.

Disaster immunity (DII). Let $D \in \mathcal{D}$. Then, $\varphi(D) \subseteq \arg\max_{a\in A} u^d(a, \omega^d)$ where ω^d represents a disaster state for D.

Disaster restriction (DIR). Let $D \in \mathcal{D}$. Then, $\varphi(D) \subseteq \varphi(A, \{\omega^d\}, u^d_{|A\times\{\omega^d\}})$.

These properties state that a solution should choose actions in a cautious way, but some of them are more restricted than other ones. CAR states that solutions should choose the same actions even though we restrict the set of actions to the set of cautious actions. DII states that, in a disaster case, solutions select among the best actions. DIR states that solutions should choose among actions which would be selected if only the disaster is present.

Using STP, IND and CAR we can establish the first characterization of the set of protective actions.

Theorem 5.3. *The set of protective actions is the unique solution on $\mathcal{D}$ that satisfies STP, IND and CAR.*

Proof. The set of protective actions, PRO, clearly satisfies STP, IND and CAR.

Let φ be a solution on $\mathcal{D}$. Let $D = (A, \Omega, u) \in \mathcal{D}$. Denote $\overline{D^0} = D$. Let $M(\overline{D^0})$ its set of cautious actions. Define $D^1 = (A^1, \Omega^1, u^1)$ with $A^1 = M(\overline{D^0})$, $\Omega^1 = \Omega$ and $u^1 = u_{|A^1\times\Omega^1}$. By CAR, $\varphi(D^1) = \varphi(\overline{D^0})$. If u^1 is a constant function, then $\varphi(D^1) = A^1 = M(\overline{D^0}) = \text{PRO}(D)$ by IND and the result follows. Let then u^1 be a non constant function and let $\Omega^2 = \Omega^1 \setminus \bigcap_{a\in A^1} \Omega^1(a)$. Then, $\Omega^2 \neq \emptyset$. Notice that, by convexity of A^1 and concavity of u in A, $\bigcap_{a\in A^1} \Omega^1(a) \neq \emptyset$ and then, $\Omega^2 \neq \Omega^1$. Define $\overline{D^1} = (A^1, \Omega^2, u_{|A^1\times\Omega^2})$. By STP, $\varphi(\overline{D^1}) = \varphi(D^1)$. Define also $D^2 = (A^2, \Omega^2, u^2)$ with $A^2 = M(\overline{D^1})$ and $u^2 = u_{|A^2\times\Omega^2}$ and repeat the above procedure.

Since Ω is a finite set, there exists $k \in \mathbb{N}$ such that u^k is a constant function. Then, $\varphi(D^k) = M(\overline{D^{k-1}})$. Then, we obtain the chain of equalities

$$\varphi(D) = \varphi(\overline{D^0}) = \varphi(D^1) = \varphi(\overline{D^1}) = \varphi(D^2) = \cdots = \varphi(\overline{D^{k-1}}) = \varphi(D^k) = M(\overline{D^{k-1}}).$$

Therefore, $\varphi(D) = M(\overline{D^{k-1}})$.

Since PRO satisfies STP, CAR and IND, we obtain that $\text{PRO}(D) = M(\overline{D^{k-1}})$ by the above reasoning. Hence,

$$\text{PRO}(D) = \varphi(D).$$

□

The proof of Theorem 5.3 provides an easy procedure to compute the set of protective actions. Let see how it can be apply to a numerical example.

Example 5.1. Let $(A, \Omega, u) \in \mathcal{D}$ be a decision problem such that $A = [0, 8]$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and the utility function u is given by Figure 5.1.

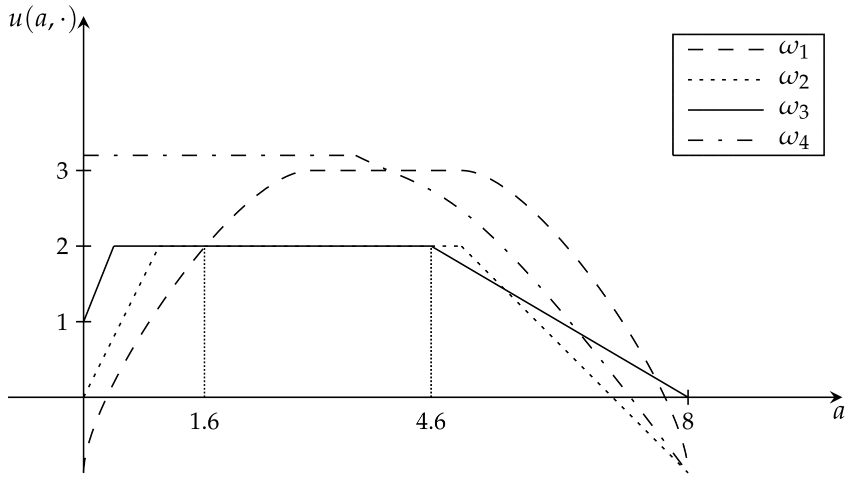

Figure 5.1: Utility function of Example 5.1.

The first step of the procedure is to seek for the set of cautious actions and for the set of states of nature where the minimax payoff is attainable for all cautious action. Let $\overline{D^0} = D$ and $\Omega^1 = \Omega$. The set of cautious actions of $\overline{D^0}$ is $M(\overline{D^0}) = A^1 = [1.6, 4.6]$ and $\bigcap_{a \in A^1} \Omega^1(a) = \{\omega_2, \omega_3\}$. Take the problem $D^1 = (A^1, \Omega^1, u^1)$ where $u^1 = u_{|A^1 \times \Omega^1}$. Since u^1 is not a constant function, we continue with the procedure.

The next step is to throw away all undesirable actions and states. Let then $\Omega^2 = \Omega^1 \setminus \bigcap_{a \in A^1} \Omega^1(a) = \{\omega_1, \omega_4\}$. We take the decision problem $\overline{D^1} = (A^1, \Omega^2, u_{|A^1 \times \Omega^2})$ whose utility function is represented in Figure 5.2.

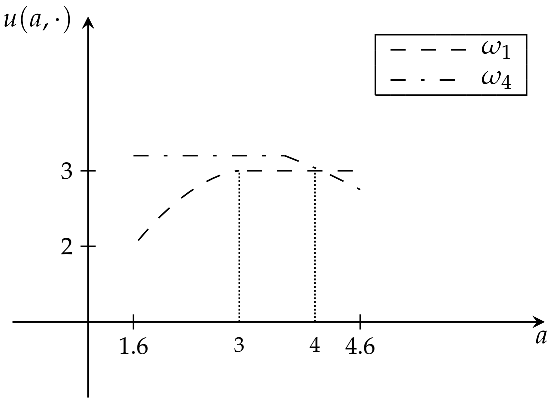

Figure 5.2: Utility function for $\overline{D^1}$ in Example 5.1.

We apply the first step of the procedure to this new decision problem. The set of cautious actions of $\overline{D^1}$ is $M(\overline{D^1}) = A^2 = [3, 4]$. We take $D^2 = (A^2, \Omega^2, u^2)$ with $u^2 = u_{|A^2 \times \Omega^2}$. Since u^2 is not a constant function, let $\Omega^3 = \{\omega_4\}$, we define the decision problem $\overline{D^2} = (A^2, \Omega^3, u_{|A^2 \times \Omega^3})$

and we continue with the procedure. The set of cautious actions of $\overline{D^2}$ is $M(\overline{D^2}) = A^3 = [3, 3.6]$ (see Figure 5.2). Let then $D^3 = (A^3, \Omega^3, u^3)$ with $u^3 = u_{|A^3 \times \Omega^3}$. Since u^3 is a constant function, then the procedure stops.

Therefore, the set of protective actions is $\text{PRO}(D) = M(\overline{D^2}) = [3, 3.6]$. $\diamond$

This procedure is known by the name of *Dresher's procedure* (Dresher, 1981) in the setting of matrix games.

From above theorem and from the relations among the properties we can state other characterization theorems. Let see the relations among properties. One can think that CAR is an ugly property. The first relation permits us to replace CAR by other three properties.

Lemma 5.2. *NEM, DII and IIA imply CAR.*

Proof. Let φ be a solution on $\mathcal{D}$ that satisfies NEM, DII and IIA. Let $D \in \mathcal{D}$. Applying DII we obtain that,

$$\varphi(D) \subseteq \arg\max_{a \in A} u^d(a, \omega^d) = M(D), \tag{5.1}$$

where ω^d is defined as a disaster state of D and the last equality is obtained by definition of set of cautious actions.

Let $D' \in \mathcal{D}$ be such that $A' = M(D)$, $\Omega' = \Omega$ and $u' = u_{|A' \times \Omega}$. If $A' = A$ then $\varphi(D') = \varphi(D)$. Otherwise, $A' \subset A$. By (5.1) and NEM, it follows that $\varphi(D) \cap A' = \varphi(D) \neq \emptyset$. Thus, by IIA, we can conclude that $\varphi(D') = \varphi(D)$. $\square$

Immediately, by Theorem 5.3 and Lemma 5.2, we obtain the second characterization of the set of protective actions.

Corollary 5.1. *The set of protective actions is the unique solution on $\mathcal{D}$ that satisfies NEM, STP, IND, DII, and IIA.*

One can think that DII is still a bit ugly. The next relation permits us to relax a little bit this property. The proof is straightforward and it is omitted.

Lemma 5.3. *DIR and OSR imply DII.*

Moreover, using NEM, STP, OSR and DIR, we can throw away IND in Corollary 5.1, as the following lemma shows.

Lemma 5.4. *NEM, STP, OSR and DIR imply IND.*

Proof. Let φ be a solution on $\mathcal{D}$ satisfying NEM, STP, OSR and DIR. Let $D = (A, \Omega, u) \in \mathcal{D}$ be such that u is a constant function. Let $\kappa_D \in \mathbb{R}$ be such that $u(a, \omega) = \kappa_D$ for each pair $(a, \omega) \in A \times \Omega$.

The proof will be done by contradiction. Suppose that $\varphi(D) \subset A$. Then, there exists $a^0 \in A$ such that $a^0 \notin \varphi(D)$. Take the following decision problem in $\mathcal{D}$. Let $D^0 = (A, \Omega \cup \{\omega^0\}, u^0) \in \mathcal{D}$ where u^0 is a concave in A, and continuous in $A \times \Omega$ function satisfying

(i) $u^0_{|A\times\Omega} = u$,

(ii) $u^0(a,\omega^0) \geq \kappa_D$ for each $a \in A$,

(iii) $u^0(a^0,\omega^0) > u(a,\omega^0)$ for each $a \in A \setminus \{a^0\}$.

Then,

$$\varphi(D^0) = \varphi(A, \{\omega^0\}, u^0_{|A\times\{\omega^0\}}) \subseteq \arg\max_{a\in A} u^0(a,\omega^0) = \{a^0\} \tag{5.2}$$

where the first equality follows from applying STP as many times as elements in Ω, the set inclusion is consequence of OSR, and the last equality follows from the fact that a^0 is the unique action where the function $u^0(\,\cdot\,,\omega^0)$ reaches the maximum value.

By (5.2) and NEM, we obtain that

$$\varphi(D^0) = \{a^0\} \tag{5.3}$$

Moreover, each $\omega \in \Omega$ is a disaster state of D^0. Then, by DIR and STP,

$$\varphi(D^0) \subseteq \varphi(D) \tag{5.4}$$

Therefore, combining (5.3) and (5.4), we have that $a^0 \in \varphi(D)$ which is a contradiction. □

Immediately, by Corollary 5.1 and lemmas 5.3 and 5.4, we obtain the third characterization of the set of protective actions.

Corollary 5.2. *The set of protective actions is the unique solution on $\mathcal{D}$ that satisfies NEM, STP, DIR, OSR and IIA.*

Finally, note that, in characterizations given by Theorem 5.3 and Corollary 5.1, IND can be replaced by ANO without any change in the proofs.

5.4 Logical independence

In this section we will show that Theorem 5.3 and corollaries 5.1 and 5.2 are adjusted, i.e., the properties used in the results are logically independents. First, we analyze Theorem 5.3. Let $D \in \mathcal{D}$, define the following solutions.

$$\begin{aligned} \varphi_1(D) &= A, \\ \varphi_2(D) &= \text{center of } \{\text{PRO}(D)\}, \end{aligned}$$

where given a convex set B, center of $\{B\}$ is defined as follows: if U_B denotes the uniform distribution over the set B and $E(\mathbb{P})$ denotes the expectation of the probability distribution $\mathbb{P}$, then

$$\text{center of } \{B\} = E(U_B).$$

It is clear that φ_1 satisfies all properties but CAR, φ_2 satisfies all properties but IND and the set of cautious actions M satisfies all properties but STP.

Now, we check the logical independence of properties in Corollary 5.1. Let $D \in \mathcal{D}$. Define the solution φ_3 by

$$\varphi_3(D) = \begin{cases} A & \text{if } u \text{ is constant in } A, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to check the following assertions. The solution φ_3 satisfies all properties but NEM. The cautious solution M satisfies all properties but STP. The solution φ_1 satisfies all properties but DII. The solution defined by

$$\varphi_4(D) = \begin{cases} A & \text{if } u \text{ is constant in } A, \\ \text{center of } \{\text{PRO}(D)\} & \text{otherwise,} \end{cases}$$

satisfies all properties but IIA. In order to check the independence of IND, we define the following solution. Let b^0 be a fixed possible action. Consider the following class of problems $\mathcal{D}^0 = \{D \in \mathcal{D} \mid b^0 \in \text{PRO}(D)\}$. The solution given by

$$\varphi_5(D) = \begin{cases} \{b^0\} & \text{if } D \in \mathcal{D}^0, \\ \text{PRO}(D) & \text{otherwise,} \end{cases}$$

satisfies all properties but IND. Namely, it is easy to check that this solution satisfies NEM, STP and DII. Let see that φ_5 satisfies IIA. Let $D, D' \in \mathcal{D}$ be such that $A' \subset A$, $\Omega' = \Omega$ and $u' = u_{|A' \times \Omega}$. We face to four cases:

- $D \in \mathcal{D}^0$ and $D' \in \mathcal{D}^0$.
  Then, $\varphi_5(D) = \varphi_5(D') = \{b^0\} = \varphi_5(D) \cap A'$.

- $D \notin \mathcal{D}^0$ and $D \notin \mathcal{D}^0$.
  Then, $\varphi_5(D) = \text{PRO}(D)$ and $\varphi_5(D') = \text{PRO}(D')$. Since PRO satisfies IIA, then $\varphi_5(D') = \varphi_5(D) \cap A'$ if $\varphi_5(D) \cap A' \neq \emptyset$.

- $D \in \mathcal{D}^0$ and $D' \notin \mathcal{D}^0$.
  $b^0 \notin \text{PRO}(D')$ because $D' \notin \mathcal{D}^0$. Then, either $b^0 \notin A'$ or there exists $a \in A' \subset A$ such that $a \succ_{pro} b^0$. However, the latter assertion is a contradiction with $b^0 \in \text{PRO}(D)$. Then, $b^0 \notin A'$ and $\varphi_5(D) \cap A' = \{b^0\} \cap A' = \emptyset$.

- $D \notin \mathcal{D}^0$ and $D' \in \mathcal{D}^0$.
  $b^0 \notin \text{PRO}(D)$ because $D \notin \mathcal{D}^0$. Let $a \in A$ be such that $a \in \text{PRO}(D)$. Then, by Lemma 5.1,

$a \succ_{pro} b^0$. However, $b^0 \in \text{PRO}(D')$ since $D' \in \mathcal{D}^0$, and then $a \notin A'$. Then, $\varphi_5(D) \cap A' = \text{PRO}(D) \cap A' = \emptyset$.

Therefore, φ_5 satisfies IIA.

Finally, we analyze the logical independence of properties in Corollary 5.2. Let $D \in \mathcal{D}$. The solution φ_3 satisfies all properties but NEM. The cautious solution M satisfies all properties but STP. The solution φ_1 satisfies all properties but OSR. The solution φ_5 satisfies all properties but DIR. In order to prove logical independence of IIA, solution φ_4 is modified in the following way. Consider the class of decision problems $\mathcal{D}^1$ where $D \in \mathcal{D}^1$ if at least one of the following assertions is fulfilled,

(i) $|\Omega| = 1$,

(ii) for each $\omega \in \Omega$, there exists $\kappa_\omega \in \mathbb{R}$ such that $u(a,\omega) = \kappa_\omega$ for each $a \in A$,

(iii) there exists $\omega^1 \in \Omega$ such that (ii) is fulfilled for each $\omega \in \Omega \setminus \{\omega^1\}$.

The solution φ_6 given by

$$\varphi_6(D) = \begin{cases} \text{PRO}(D) & \text{if } D \in \mathcal{D}^1, \\ \text{center of } \{\text{PRO}(D)\} & \text{otherwise,} \end{cases}$$

satisfies all properties but IIA. It is easy to check that φ_6 satisfies NEM and OSR. Let check that φ_6 also satisfies STP and DIR. Let $D, D' \in \mathcal{D}$ be such that $A' = A$, $\Omega' = \Omega \cup \{\omega'\}$ for some $\omega' \notin \Omega$, $u'_{|A\times\Omega} = u$ and let $\kappa \in \mathbb{R}$ be such that $u(a,\omega') = \kappa$ for each $a \in A$. It easy to check that $D \in \mathcal{D}^1$ if and only if $D' \in \mathcal{D}^1$. Since PRO satisfies STP, we have that $\varphi_6(D) = \varphi_6(D')$, so that φ_6 satisfies STP. Let $D \in \mathcal{D}$ and denote $D_d = (A, \{\omega^d\}, u^d_{|A\times\{\omega^d\}})$. Then, $D_d \in \mathcal{D}^1$ and, as a consequence, $\varphi_6(D_d) = \text{PRO}(D_d) = M(D)$. On the other hand, $\varphi_6(D) \subseteq \text{PRO}(D) \subseteq M(D)$. Then, $\varphi_6(D) \subseteq \varphi_6(D_d)$. Therefore, φ_6 satisfies DIR.

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Resumen en Castellano

Resumen en castellano

La presente tesis se enmarca dentro del campo de la *teoría de juegos*. La teoría de juegos es la teoría matemática de las *situaciones conflictivas*. Una situación conflictiva es una situación interactiva donde varios agentes tienen que tomar decisiones, el resultado final depende de las decisiones de todos los agentes y cada agente tiene sus propias preferencias sobre el conjunto de posibles resultados.

Las primeras aportaciones a la teoría de juegos datan de principios del siglo XX con los trabajos Zermelo (1913), Borel (1921) y von Neumann (1928). Sin embargo, se puede considerar que la teoría de juegos nace como disciplina científica en el año 1944 a partir de la publicación del libro *"Theory of Games and Economic Behavior"* de John von Neumann y Oskar Morgenstern. Posteriormente, en el año 1950, John Nash definió el concepto de equilibrio en juegos en forma estratégica. Este concepto tuvo una gran repercusión en el campo de la economía, muestra de ello es que Nash recibió el premio Nobel de economía en el año 1994 junto con J. Harsanyi y R. Selten. Desde los años 50 hasta ahora, la teoría de juegos ha ido evolucionando a grandes pasos debido a que una gran variedad de situaciones reales pueden ser modeladas utilizando la teoría de juegos. Cabe notar que el premio Nobel de economía del año 2005 también ha sido concedido a dos investigadores que trabajan en teoría de juegos: Robert Aumann y Thomas Schelling.

La teoría de juegos se puede dividir en dos grandes áreas, la teoría de *juegos no cooperativos* y la teoría de *juegos cooperativos*. La diferencia entre las dos está en los objetivos y en las posibilidades de los agentes involucrados en el modelo. En el modelo no cooperativo los agentes no pueden tomar acuerdos vinculantes y la teoría de juegos estudia cómo debe actuar cada uno de los jugadores para maximizar sus propios beneficios. En el modelo cooperativo los agentes sí pueden tomar acuerdos vinculantes, además pueden formar coaliciones y el objetivo es repartir el beneficio o el coste resultante. Por otro lado, la teoría de juegos no cooperativos está ampliamente relacionada con la teoría de la decisión, es más, todo problema de decisión puede ser estudiado como un juego no cooperativo con una estructura especial.

Esta tesis se estructura en dos partes: una dedicada a la teoría de juegos cooperativos y la otra dedicada al estudio de soluciones cautelosas en problemas de decisión.

Juegos Cooperativos

La Parte I de esta tesis esta dedicada al estudio del comportamiento *cooperativo* en situaciones de *investigación operativa*. Esta parte está organizada en tres capítulos independientes, cada uno de los cuales estudia el problema del reparto de costes/beneficios que surgen cuando, en un modelo de investigación operativa multi-agente, se incorpora la posibilidad de que exista cooperación entre los agentes. A esta clase de problemas que utilizan la teoría de juegos en situaciones de investigación operativa multi-agente se la conoce como la clase de los *juegos de investigación operativa*. Una buena revisión de los trabajos existentes en este campo puede encontrarse en Borm et al. (2001).

En el Capítulo 1 nos centramos en los *modelos de inventario*, en concreto trabajamos con las situaciones de centralización de inventarios. La centralización de inventarios es conocida por reducir costes en modelos de optimización de costes de inventario en ambientes multi-agente. Un problema importante que surge de esta centralización es cómo repartir los ahorros generados entre los distintos agentes y una herramienta para dar respuesta a este problema es la teoría de juegos.

El problema del reparto de costes en situaciones de inventario ha sido tratado en varios trabajos durante los últimos años. Por ejemplo, Hartman y Dror (1996) proponen una regla de reparto que verifica una serie de propiedades deseables para cualquier regla en el contexto de modelos de inventario multi-agente estocásticos y de revisión continua. Hartman et al. (2000) y Müller et al. (2002) estudian el *núcleo* de una clase de juegos cooperativos que surgen de modelos de inventario multi-agente estocásticos de un solo periodo. Meca et al. (2003) y Meca et al. (2004) definen los llamados *juegos de inventario*, los cuáles modelizan problemas de asignación de costes en modelos de inventario multi-agente determinísticos y de revisión continua. Ellos proponen una regla de reparto de costes, a la que llaman *SOC-rule* (Share Ordering Cost rule), la caracterizan y prueban que siempre proporciona repartos que pertenecen al núcleo del juego.

En este capítulo trabajamos con el modelo estudiado en Meca et al. (2003). A diferencia de este trabajo, nosotros buscamos reglas que sean *inmunes ante posibles manipulaciones* de los agentes involucrados en el problema, bien a través de fusiones artificiales de varios agentes, o bien mediante escisiones artificiales de un agente en varios agentes. Aquí probamos que la única regla eficiente para juegos de inventario que es inmune ante estas manipulaciones mediante coaliciones es la SOC-rule. Aunque la manipulación mediante coaliciones no ha sido nunca estudiada en el contexto de modelos de centralización de inventarios, es una propiedad muy interesante a la cual se ha dedicado mucha atención en trabajos económicos. Ju (2003), Bergantiños y Sánchez (2002) y de Frutos (1999) son tres ejemplos recientes en los que la manipulación mediante coaliciones ha sido estudiada en el contexto de varios problemas de asignación. Este capítulo está basado en el trabajo Mosquera et al. (2006b).

En el Capítulo 2 cambiamos el modelo de investigación operativa a estudiar. Aquí estudiamos problemas que se pueden representar a través de un *árbol lineal de coste fijo*. Nuestro problema de estudio surge a partir de la pregunta de cuál debería ser el precio por el uso de las autopistas. Una gran parte de la literatura relativa al estudio de aspectos relacionados con las carreteras se ha centrado en estudiar los problemas de congestión. Sin embargo, nosotros trabajamos con un problema diferente de éste, el problema de repartir entre los agentes potenciales de las carreteras los costes generados por la construcción y el mantenimiento de éstas, de acuerdo a los principios de igualdad y eficiencia.

Existen varios trabajos que tratan con el problema de asignar tasas a las distintas clases de vehículos (coches, camiones, etc. . . .) que usan un servicio, en este caso la autopista. Por ejemplo, Villarreal-Cavazos y García-Díaz (1985) proponen cuatro métodos de asignación de tasas basados en las diferentes características que presenta cada clase de vehículos. Para definir uno de esos métodos, el llamado *método generalizado*, los autores hacen un uso un poco tímido de alguna herramienta que proporciona la teoría de juegos para abordar un problema de estas características.

Con este capítulo nosotros queremos aportar un poco más a este campo. En primer lugar, extendemos la clase de problemas a considerar, no nos limitamos al caso del estudio de las carreteras. Nótese que una carretera o autopista puede ser considerada como un cierto recurso público dotado de una cierta estructura diferencial. Nuestro objeto de estudio será el caso del reparto del coste total de un recurso público que tiene una estructura similar a la de las autopistas: puede dividirse en secciones ordenadas e indivisibles y que cada usuario potencial puede usar un subconjunto de secciones consecutivas (en función de ese orden). Por ejemplo, en el caso de una autopista lineal, sin ramificaciones, puede considerarse que está formada por los distintos tramos delimitados entre los puntos de acceso y salida de la misma. La ordenación vendría dada por la situación geográfica de los tramos y estaría fijada desde un principio, es decir, los tramos de la autopista los podemos empezar a enumerar desde cualquiera de los dos extremos de la autopista, entonces nos fijaremos como punto inicial uno de estos dos extremos. En nuestro modelo, el coste de cada sección del recurso depende en gran parte de su tamaño, y el coste total del recurso es la suma de los costes de cada sección. Debido a la simplicidad de la función de coste, este problema se presta a abordarlo utilizando herramientas de la teoría de juegos. Así, asociado a cada problema de reparto de costes en este contexto definimos un juego cooperativo; a los juegos de tal clase los llamaremos *juegos de autopista*.

Una primera aproximación a este problema sin usar ninguna herramienta sofisticada para repartir los costes en este modelo podría ser el considerar una de las dos siguientes formas naturales de realizar este reparto de costes. La primera de ellas es repartir los costes totales proporcionalmente a los costes que tiene cada agente, y la segunda es repartir el coste de cada sección de forma igualitaria entre los agentes que la usan. En este capítulo comprobamos que estas dos posibles reglas de reparto coinciden con dos soluciones muy conocidas y estudiadas en el campo de la teoría de juegos cooperativos: el *valor de compromiso* y el *valor de Shapley*, respectivamente. Por otro lado, utilizando ya directamente una herramienta de teoría de juegos, estudiamos en profundidad el *nucleolus* de los juegos de autopista como otra posible regla de reparto de costes. Aunque en general el cálculo del nucleolus suele ser muy laborioso y complicado, en este estudio proponemos un procedimiento sencillo para calcularlo. Este procedimiento se basa en encontrar las coaliciones que determinan el valor del nucleolus, a las que llamaremos *coaliciones relevantes*, calcular los excesos de estas coaliciones cuando el reparto se hace en función del nucleolus, encontrar el valor mínimo de estos excesos y dar el valor del nucleolus para determinados agentes involucrados en el problema. A simple vista este proceso no parece demasiado sencillo puesto que hay que calcular los excesos sin conocer el valor del nucleolus y no sabemos si lo tendremos que calcular en todas las coaliciones posibles. Sin embargo, en este capítulo comprobamos que no es necesario calcular los excesos de todas las coaliciones posibles y que el número de coaliciones para las que hay que calcularlo no es muy elevado. Además, debido a la estructura de estas coaliciones, se puede calcular su exceso cuando el reparto se hace en función del nucleolus sin necesidad de conocer el valor del nucleolus y de una forma sencilla.

Cabe destacar que, los conocidos *juegos de aeropuerto* definidos por Littlechild y Owen (1973) son una subclase de los juegos de autopista aquí definidos y, a su vez, los juegos de autopista son una clase especial de *juegos de contribuciones* definidos en Koster et al. (2003). Este capítulo está basado en el trabajo Mosquera y Zarzuelo (2006).

En el capítulo 3 pasamos a estudiar un modelo *de programación de tareas*. Nos centramos en el estudio de los *problemas de secuenciación en línea proporcionales*. En un problema de secuenciación en línea, un grupo de trabajos tienen que ser procesados a través de un número fijo de máquinas y el orden de las máquinas en el que los trabajos tienen que ser procesados es el mismo para todos ellos. Cada trabajo tiene asignado un coste que depende del tiempo total que tarda en ser procesado por todas las máquinas. Cuando cada trabajo tiene el mismo tiempo de procesado sobre cada máquina decimos que estamos ante un problema de secuenciación en línea proporcional. Este tipo de problemas ha ganado mucho interés en los últimos años y se han publicado muchos trabajos en este campo. Uno de ellos es Shakhlevich et al. (1998), en el que nos basaremos para este capítulo. En él se proporciona un algoritmo para obtener una programación óptima de los trabajos, de forma que el coste de procesar todos los trabajos es mínimo.

Si asociamos cada trabajo con un cliente, un problema de secuenciación en línea proporcional da lugar a un problema de decisión interactivo. En él, cada cliente tiene unos costes, de los cuales asumimos que dependen linealmente del tiempo que tarda en ser procesado su trabajo. Si suponemos que existe una ordenación inicial de los trabajos, el primer problema al que se enfrentan los clientes es encontrar una reordenación óptima de los trabajos de forma que se maximice el ahorro obtenido con respecto a la ordenación inicial. El siguiente problema es cómo repartirse estos ahorros de una forma justa. El primero de los problemas tiene fácil solución, utilizar el algoritmo propuesto por Shakhlevich et al. (1998). Para resolver el segundo problema, a cada problema de secuenciación de esta naturaleza le asociamos un juego cooperativo. A esta clase de juegos los llamaremos *juegos de secuenciación en línea proporcionales*, abreviadamente *juegos PFS*.

Uno de los primeros trabajos en los que se estudia un problema de secuenciación general desde el punto de vista de la teoría de juegos es Curiel et al. (1989). En él, se aborda el problema de secuenciación de trabajos sobre una única máquina. A este trabajo le siguen una gran variedad de artículos con distintas generalizaciones de este sencillo modelo, como por ejemplo Hamers et al. (1995), Borm et al. (2002), Estévez-Fernández et al. (2004), van den Nouweland (1993), . . . En este capítulo probamos que los juegos PFS son equilibrados y que, además, son convexos si el orden inicial es el orden de urgencia (ver Smith, 1956). También proporcionamos una fórmula explícita, independiente de los valores del juego, para el valor de Shapley, con lo cual se hace computacionalmente más sencillo su cálculo. Bajo la suposición sobre el orden inicial, también definimos una nueva regla de reparto que sigue la misma filosofía que la *regla de división con igual ganancia* (EGS) definida en Curiel et al. (1989). Esta regla está basada en el algoritmo definido en Shakhlevich et al. (1998). Tanto el valor de Shapley como esta nueva regla proporcionan repartos que están en el núcleo del juego PFS correspondiente. Este capítulo está basado en el trabajo Estévez-Fernández et al. (2006).

Problemas de decisión

La parte II de esta tesis está estructurada en dos capítulos independientes pero relacionados y está dedicada al estudio de ciertas cuestiones en problemas de decisión unipersonales. Los capítulos estudian el comportamiento *cauteloso* de un agente que se enfrenta a un problema de decisión.

El llamado comportamiento *maximín* es la reacción más representativa de *cautela* ante cualquier problema. Escoger entre distintas alternativas de acuerdo a este comportamiento es asignarle a cada una de ellas la peor consecuencia posible y escoger aquellas alternativas para las cuales esta peor consecuencia ofrezca el mejor resultado. El capítulo 4 está dedicado al estudio de los fundamentos del comportamiento maximín. Dicho comportamiento ha sido estudiado en multitud de campos de investigación de distinta índole. Dependiendo de cómo se modelen las alternativas, las consecuencias y las preferencias, nos encontraremos con aplicaciones del comportamiento maximín a diferentes campos de las ciencias sociales como pueden ser la teoría de juegos, (ver von Neumann, 1928), economía experimental (ver Sarin y Vahid, 1999, 2001), teoría de la decisión estadística, elección social y bienestar (ver Moulin, 1988), investigación operativa (ver Love et al., 1988), ...

Dada la ubicuidad de este principio, es gratamente sorprendente que también sus fundamentos hayan sido objeto de estudio. Estos estudios se centran en uno de los dos siguientes aspectos: *(a)* caracterizar el orden inducido por el comportamiento maximín (Milnor, 1954; Barberà y Jackson, 1988) o *(b)* caracterizar el valor maximín asociado a juegos en forma estratégica de suma nula (Vilkas, 1963; Tijs, 1981). Sin embargo, hasta dónde nosotros conocemos, ningún estudio se ha centrado en caracterizar la solución que a cada problema de decisión le asigna el conjunto de sus alternativas maximín. El objetivo de este capítulo es formalizar este tercer aspecto. La clase de problemas que consideramos es una clase muy general dónde la única restricción es que la función de utilidad del decisor tiene que estar acotada. En este capítulo definimos una serie de propiedades deseables para una solución en este contexto de forma que sean lo más estándar posible. Debido a la generalidad de la clase de problemas de decisión considerada, necesitamos más propiedades de lo habitual para caracterizar esta solución además de adaptar algunas de las propiedades estándar a este contexto y de definir alguna otra propiedad no tan estándar. Por ejemplo, en nuestro contexto la solución maximín puede ser vacía, cosa que no ocurre en la mayoría de los problemas de decisión que se estudian en la literatura, por eso necesitamos relajar un poco la propiedad de *no vacío* (*nonemptiness*) para una solución. Por último, caracterizamos la *solución maximín* utilizando dichas propiedades. Este capítulo está basado en el trabajo Mosquera et al. (2005).

Aunque el comportamiento maximín sea un claro exponente de la cautela ante un problema de decisión, existen numerosas situaciones en donde aún se podría ser *más* cauteloso. Por este y otros motivos se hace necesaria la definición de algunos refinamientos de este comportamiento. En la literatura existente se han definido dos posibles refinamientos. Moulin (1981) define formalmente el comportamiento *prudente* y Barberà y Dutta (1982) introducen el comportamiento *protectivo*. Ambos comportamientos están muy relacionados, de hecho los dos coinciden en muchos problemas. El objetivo del Capítulo 5 de esta tesis es estudiar los fundamentos del comportamiento protectivo.

Actuar *protectivamente* quiere decir que el agente maximiza su peor resultado con respecto a todas las consecuencias y, en caso de empate, también minimiza, en términos de inclusión de conjuntos, el conjunto de consecuencias que proporcionan el peor resultado. Actuar *prudentemente* es actuar protectivamente salvo que, en caso de empate, se minimiza el número de consecuencias que proporcionan el peor resultado. El comportamiento protectivo ha sido estudiado en contextos distintos al estudiado en Barberà y Dutta (1982). Por ejemplo, Barberà y Jackson (1988) proporcionan una caracterización del comportamiento protectivo como una ordenación sobre el conjunto de vectores reales finito-dimensionales. Por otro lado, Fiestras-Janeiro et al. (1998) estudian los comportamientos protectivos y prudentes dentro del contexto de juegos finitos en forma estratégica.

Dada la importancia que en los últimos años ha ganado el comportamiento protectivo, queremos investigar un poco más sus fundamentos. En este capítulo caracterizamos la solución que asigna a cada problema de decisión su conjunto de alternativas protectivas. El contexto en el que trabajamos es un contexto más o menos estándar en la literatura de teoría de la decisión, en dónde el conjunto de alternativas es compacto, convexo y no vacío, el conjunto de consecuencias es finito y no vacío, y la función de utilidad del decisor es convexa en el conjunto de alternativas y continua. Las propiedades que caracterizan el comportamiento protectivo son propiedades también estándar en la literatura de problemas de decisión. Aún así, necesitamos otras propiedades no tan estándar para hacer referencia al concepto de *cautela* como, por ejemplo, que la solución no cambie si restringimos el conjunto de alternativas al conjunto de sus alternativas maximín. Con esta propiedad lo que conseguimos es que las alternativas elegidas sean siempre maximín ya que, como ya hemos indicado, escoger entre estas alternativas es el comportamiento más representativo de la *cautela*. Este capítulo está basado en el trabajo Mosquera et al. (2006a).

Conclusiones

En la primera parte de esta tesis hemos estudiado tres clases de juegos de investigación operativa que surgen de situaciones de la vida cotidiana. En estas tres clases nos hemos centrado en el problema del reparto de costes/beneficios aunque de formas un poco distintas. En los juegos de inventario nos hemos centrado en caracterizar una regla de reparto ya existente, la SOC-rule, sobre una clase de juegos ya definida anteriormente, utilizando una propiedad básica que nunca nadie había utilizado antes en este contexto: *la propiedad de inmunidad ante manipulaciones*. Sin embargo, en los otros dos capítulos hemos definido dos nuevas clases de juegos de investigación operativa, los *juegos de autopista* y los *juegos PFS*. Hemos estudiado algunas características de estas clases de juegos y nos hemos centrado en el estudio no axiomático de soluciones estándar para juegos cooperativos aplicadas a estos contextos, aunque para el caso de los juegos PFS también hemos definido una nueva regla de reparto especialmente diseñada para este contexto y que se obtiene a partir de un algoritmo diseñado para el problema de optimización originario de esta clase de juegos.

La segunda parte de esta tesis la hemos dedicado a los problemas de decisión. Nos hemos centrado en estudiar el comportamiento *cauteloso* del agente frente a problemas de esta índole. Hemos escogidos los criterios *maximín* y *protectivo* como representantes de este tipo de comportamiento y hemos estudiado sus fundamentos. Con respecto al comportamiento maximín hemos propuesto una caracterización basándonos en propiedades estándar para soluciones a problemas de decisión dentro de una clase de problemas muy general. Para el caso del comportamiento protectivo hemos dado tres caracterizaciones utilizando propiedades que hacen referencia a la *cautela* en una clase de problemas de decisión más o menos estándar en la literatura existente.

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