Allocation problems with indivisibilities when preferences are single-peaked.*

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Abstract

We deal with the problem of allocating a given amount of indivisible units of an homogeneous good among a set of agents with single-peaked preferences. We define a family of uniform and equal distance families of rules, sharing the idea of the uniform and equal distance rules used in the allocation problems with divisible goods. We do it by considering a priority ordering among the agents and peaks to allocate the units. Axiomatic characterizations of these new discrete rules by using some intuitive properties are also provided.

Keywords: Allocation problem, Indivisibilities, single-peaked preferences, uniform rule, equal distance rule.

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1 Introduction

We face up in this paper the problem of allocating an amount of indivisible units of an homogeneous good among a group of agents whose preference are single-peaked. One of the examples used to illustrate this kind of problems is that in which a group of people must supply a certain amount of labor paid hourly to complete a common task. These agents' preferences over the worked hours, and then over the earned money, are assumed to be single-peaked. This means that each agent has a most preferred amount of hours (equivalently money) to work, and if it happens that he has to work more than this preferred amount he wishes to deviate as less as possible. Similarly, if it happens that he has to work less than this preferred amount he wishes to be as close as possible to his preferred amount.

Consider now the following situation. At a university department, there are a certain amount of extra hours which have to be covered by the faculty. Here the agents are the professors of the department; each of whom, given the salary per hour, has an "ideal" amount of time to work. Usually, each hour corresponds to a complete class of a particular subject, that is, it is not possible to allocate fractions of hours. In this case, thus, the task is made out of a certain number of indivisible units (hours). Similar situations appear in the allocation of shifts in hospitals or hotels, among other examples.

The above situation is a particular instance of a general set of problems called allocation problems under single-peaked preferences with indivisible goods. These problems come described by three elements. First, a set of agents. Second, an amount of indivisible units of a certain good to be distributed, called the task. Finally, a profile of the agents' preferences over the number of units involved in the task. A *rule or solution* is a function that distributes the task among the agents.

The literature related with allocation problems when preferences are single-peaked has focused so far on the continuous case (when the task is perfectly divisible). The traditional way of supporting rules to solve the problem is by applying the so called *axiomatic method.* Rules are then defended on the basis of the properties they fulfil, and in general, suitable combinations of different appealing properties are used to differentiate among rules. The most appealing properties in this case have to do with efficiency, equity, and incentive compatibility considerations. By far, the best-known rule in the continuous case is the *uniform* rule, introduced in Sprumont (1991). It proposes to treat all agents as equally as possible, subject to efficiency. Characterizations of this rule also appear in Ching (1994), Sönmez (1994), Thomson (1994), and Dagan (1996), among others. An alternative, also well-known rule in the continuous case is the *equal distance rule* , introduced in Thomson (1994), that proposes to select the allocation at which all agents are equally far from their preferred consumptions, subject to efficiency and boundary conditions, in which case those agents whose consumptions would be negative are given zero instead. Characterizations of this rule appear in Herrero & Villar (1998, 2000) and in Herrero (2002). Both the uniform and the equal distance rules are efficient and equitable, and the uniform rule is also incentive compatible, while the equal distance rule is not.

When the good comes in indivisible units, some of the aforementioned properties cannot be met. This happens, for instance, with equity properties, that should be accomodated to this case. The traditional requirement of *equal treatment of equals,* for example, can only be partially reached, and we should allow equal agents to be alloted different amounts. The only way of keeping equality as far as possible is to forbid equal agents allotments to differ in more than one unit. Fortunately enough, efficiency and strategy-proofness can be satisfied in the indivisible case.

A natural way to solve rationing problems when the good comes in indivisible units consists of applying priority methods (see Young 1994, Moulin (2000), Moulin & Stong (2002), and Herrero & Martinez (2004)). When there is a pure priority relationship on the set of agents, the easiest way of solving the problem is by asking the agents to choose which amount of the task to consume following the priority ordering, and forcing the last agent to get whatever is left. These pure priority methods are efficient and incentive compatible, but they are far from being minimally equitable. A different method consists of applying priority orderings on the cartesian product of agents and integer numbers, so called *standards of comparison* in Young (1994). When using standards of comparison to solve rationing problems, the most natural procedure consists of starting from some predetermined allocation, and then move out of it, unit by unit by using the standard. This procedure was used in Herrero & Martinez (2005) to solve claims problems.

In this paper we analyze two methods to solve allocation problems when preferences are single-peaked when the good comes in indivisible units, by using standards of comparison. The first procedure (*Up method*) allocates the task unit by unit, according to the standard, when the numbers paired with the agents are interpreted as agents' peaks. The second procedure (*Temporary Satisfaction method*) starts by giving all agents their preferred consumptions, and then move away from this provisional allocation, unit by unit by using the standard. Here, the numbers paired with the agents are interpreted either as agents' peaks or the opposite peaks, depending upon the type of problem at hand (either an excess demand or excess supply problem).

Then we explore the properties our families of discrete rules may satisfy. As it happens in Herrero and Martínez (2005), in order to approach equality we shoul consider a sub-family of standards, those called *monotonic standards,* that always give priority to larger numbers. Then it happens that *monotonic Up methods* provide allocations very similar to those prescribed by the *equal-distance rule* when the good is perfectly divisible. Similarly, the allocations prescribed by *monotonic Temporary Satisfaction methods* are as close as possible to those provided by the continuous *uniform rule.* Then we obtain that our discrete families can be characterized by sets of properties very similar to those supporting some of the characterizations of the continuous versions of the respective rules.

The rest of the paper is structured as follows: In Section 2 we set up the problem of allocating indivisible units of a good when preferences are single-peaked. In Section 3 we introduce standards of comparison and use them to construct two allotment procedures: the *Up* and the *Temporary Satisfaction* methods, that convey to construct two families of discrete rules. Section 4 analyzes the properties our families of rules may fulfil. In Section 5 we present our characterization results. Section 6, with final comments and remarks, concludes.

2 Preliminaries

This section is devoted to provide formal statements of single-peaked preferences, allocation problems and rules.

A preference relation, R_i, defined over $\mathbb{Z}_+$ is said to be **single-peaked** if there exists an integer number $p(R_i) \in \mathbb{Z}_+$, called the **peak** of R_i, such that, for each $x, x' \in \mathbb{Z}_+$,

$$\big[(x' < x < p(R)) \text{ or } (p(R_i) < x < x')\big] \Leftrightarrow xPx'.$$

Let $\mathbb{S}$ denote the class of all single-peaked preferences defined over $\mathbb{Z}_+$. Let $\mathbb{N}$ be the set of all potential **agents**. Let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. An allocation problem with single-peaked preferences, or simply a **problem**, is a triple $e = (N, T, R)$, where $N \in \mathcal{N}$ is the set of agents ($n = |N|$), $R \equiv (R_i)_{i \in N} \in \mathbb{S}^N$ is the profile of agents' preferences, and $T \in \mathbb{Z}_{++}$ is the amount to be allocated, called the **task**. Let $\mathbb{A}^N$ denote the class of all problems with agents set N, and $\mathbb{A}$ the class of all problems, that is,

$$\mathbb{A}^N = \big\{(N, T, R) \in \{N\} \times \mathbb{Z}_+ \times \mathbb{S}^N\big\}$$

and

$$\mathbb{A} = \bigcup_{N \in \mathcal{N}} \mathbb{A}^N.$$

For each problem, we face the question of finding a division of the task among the agents.

An **allocation** for $e = (N, T, R) \in \mathbb{A}$ is a list $\boldsymbol{x} \in \mathbb{Z}_+^N$ satisfying the condition of being a complete distribution of the task among the agents, $\sum_{i \in N} x_i = T$. Let $\boldsymbol{X(e)}$ be the set of all allocations for $e \in \mathbb{A}$. An allocation, $x \in X(e)$ is **efficient** if there is no other allocation in which the agents are better-off, that is, there is no allocation $y \in X(e)$ such that for each $i \in N$, $y_i R_i x_i$, and for some $j \in N$, $y_j P_j x_j$. Let $\boldsymbol{P(e)}$ be the set of all efficient allocations for $e \in \mathbb{A}$.

A **rule** is a function, $\boldsymbol{F}$, that selects, for each problem $e \in \mathbb{A}$, a unique allocation $F(e) \in P(e)$.[1] As Sprumont (1991) points out, the requirement of efficiency is equivalent to asking for each agent to consume no more than his preferred amount if $\sum_{i \in N} p(R_i) \geq T$, and no less if $\sum_{i \in N} p(R_i) \leq T$.

3 Standards of Comparison and Up and Temporary Satisfaction Methods.

A standard of comparison is a linear order (complete, antisymmetric and transitive) over the cartesian product $\mathbb{N} \times \mathbb{Z}$. such that for each agent, larger integer numbers have priority over smaller integer numbers. This product can be interpreted as the product of the set of potencial agents and the set of their potential peaks.

Standard of Comparison, $\boldsymbol{\sigma}$ (Young, 1994): Is a function $\sigma : \mathbb{N} \times \mathbb{Z} \longrightarrow \mathbb{Z}_+$ such that for each $i \in \mathbb{N}$, and each $a \in \mathbb{Z}$, $\sigma(i, a+1) < \sigma(i, a)$. Let Σ denote the class of all standards of comparison.

This class of orders have been applied by Moulin and Stong (2002), and Herrero and Martínez (2004) in the context of claims problems with indivisiblilities.[2]

Consider a problem with only one unit of the task to allocate. The standard of comparison determines the agent who receives this unit. Alternatively, if the task differs from the sum of the peaks (aggregate demand) by just one unit, then all agents, but one, are fully satisfied. In this case, the standard of comparison determines who that agent is.

Two natural methods for solving allocation problems by using the standards of comparison. can be constructed. The first option consists of an algorithm to allocating all units of the task one by one. The second one consists of accomodating all units of either excess demand or excess supply one by one, after giving (temporarily) all agents their peaks. We shall

[1] Notice that the notion of rule refers simply to the selection of an efficient allocation for each problem. This choice can be made via a direct formulation, as it is the case for the majority of rules used when the task is perfectly divisible, or else, it can be made by using an algorithm, as we do in this work.

[2] The reader is referred to the survey by Thomson (2003) for a widely exposition of claims problems when the good is perfectly divisible.

call those *up methods* and *temporary satisfaction methods* respectively.

For each list of pairs agent-integer number $\{(i, a_i)\}_{i \in N}$, the **agent with strongest number** according to the standard of comparison σ is the agent $k \in N$ such that the pair (k, a_i) has the highest priority among all the pairs (i, a_i) according to σ. That is, k is the agent with the strongest number according to σ if for each $i \in N \setminus \{k\}$, then $\sigma(k, a_k) < \sigma(i, a_i)$.

Up method associated to $\boldsymbol{\sigma}$, $\boldsymbol{U^{\sigma}}$: Let $e = (N, T, R) \in \mathbb{A}$. Start by associating to each agent her peak, and then identifying the agent with the strongest number (peak) according to σ. Then give one unit of the task, T, to this agent. Reduce his number (peak) by one unit. Now identify the agent with the new strongest number for σ, and proceed in the same way. Repeat this process until the task runs out.

Temporary satisfaction method associated to $\boldsymbol{\sigma}$, $\boldsymbol{TS^{\sigma}}$: Let $e = (N, T, R) \in \mathbb{A}$. Start by giving all agents their peaks. Now we distinguish two cases. On the one hand, if the task in not enough, $\sum_{i \in N} p(R_i) \geq T$. In this case we have to remove some units from the temporary allocation. Associate to each agent his peak, and identify the agent with the strongest number (peak) according to σ. Subtract one unit from this agent, and reduce his number accordingly. Identify again the agent with the new strongest number according to σ, and proceed in the same way until reaching the task. On the other hand, if the task is too large, i.e., if $\sum_{i \in N} p(R_i) \leq T$, we have to allocate extra units to the agents, $T' = T - \sum_{i \in N} p(R_i)$. We shall proceed in the following way. Associate to each agent the the opposite of his peak, that is, let $a_i = -p(R_i)$. Identify the agent with the strongest number (-peak) according to σ. Then assign one unit of the remaining task, T', to this agent. Reduce the number of this agent by one unit. Now identify the agent with the new strongest number for σ, and proceed in the same way. Repeat this process until the task T' runs out.

Example 1. *Assume that the standard of comparison is such that, restricted to agents in N, it happens that $\sigma(2, x) < \sigma(1, y) < \sigma(3, z)$, for all $x, y, z \in \mathbb{Z}_+$. Now, consider the allocation problem where $N = \{1, 2, 3\}$, $T = 6$ and $R = (R_1, R_2, R_3)$ such that $p(R) = (1, 3, 5)$. Note that, in this case, $\sum_{i \in N} p(R_i) > T$. For the pairs involved in the aforementioned problem, we have*

$$\sigma(2,3) < \sigma(2,2) < \sigma(2,1) < \sigma(1,1) < \sigma(3,5) < \sigma(3,4) < \sigma(3,3) < \sigma(3,2) < \sigma(3,1).$$

The next table shows the functioning of the up-method for this problem. The first column gives the kth unit of the task. The second column gives the allocation up to that unit, $x^{(k)}$. The third column gives the updated vector of numbers, $p^{(k)}$.

T	$x^{(k)}$	$p^{(k)}$
	(0,0,0)	(1,3,5)
1	(0,1,0)	(1,2,5)
2	(0,2,0)	(1,1,5)
3	(0,3,0)	(1,0,5)
4	(1,3,0)	(0,0,5)
5	(1,3,1)	(0,0,4)
6	(1,3,2)	(0,0,3)

Example 2. *The next table we show the functioning of the temporary satisfaction method for the same standard of comparison and problem as in previous example. In this case we start by fully satisfying all agents, that is, by giving to each agent his peak. This implies allocating 9 units, but we only have 6 units to allocate. Thus we need to remove 3 units. The table shows the process of removing. The first column gives the kth unit of the task. We start from 9 units and we remove unit by unit up to reach 6 units. The second column gives the allocation up to that unit,* $x^{(k)}$*. The third column gives the updated vector of numbers,* $p^{(k)}$*.*

T	$x^{(k)}$	$p^{(k)}$
9	(1,3,5)	
8	(1,2,5)	(1,3,5)
7	(1,1,5)	(1,2,5)
6	(1,0,5)	(1,1,5)

Previous examples illustrate the way both the up method and the temporary satisfaction method work. Additionally, they show that these methods could result in *pure priority rules*, depending upon the standard of comparison used. Given the standard of comparison in previous examples, the allocation obtained by means of the up-method is the allocation prescribed by a pure priority rule in which agent 2 is fully satisfied first, then agent 1 comes to the line and he is also fully satisfied, and, finally, any remaining units go to agent 3. As for the allocation obtained by the application of the temporary satifaction method, it coincides with the allocation recommended by the pure priority rule with the reverse order: Now, agent 3 is the one going first to the line up to when he is fully satisfied, next, agent 1 comes to the line, and finally, any remaining units go to agent 2 The next examples consider a different type of standard of comparison.

Example 3. *Let* $N = \{1, 2, 3\}$*, and assume that the standard of comparison is such that, restricted to agents in* N*, it happens that for all* $i, j \in N$*, and all* $x, y \in \mathbb{Z}_{++}$*, if* $x > y$*, then* $\sigma(i, x) < \sigma(j, y)$*. Furthermore,* $\sigma(1, x) < \sigma(2, x) < \sigma(3, x)$ *if* x *is odd, and* $\sigma(2, x) < \sigma(1, x) < \sigma(3, x)$ *if* x *is even. Now, let be* $T = 14$*, and* $R = (R_1, R_2, R_3)$ *such* *that* $p(R) = (1,3,5)$. *The next table shows the functioning of the up method associated to this standard of comparison*

T	$x^{(k)}$	$p^{(k)}$
...	...	...
9	(1,3,5)	(0,0,0)
10	(1,4,5)	(0,-1,0)
11	(2,4,5)	(-1,-1,0)
12	(2,4,6)	(-1,-1,-1)
13	(3,4,5)	(-2,-1,-1)
14	(3,5,5)	(-2,-2,-1)

Example 4. *This example illustrates the functioning of the temporary satisfaction method for the same problem and standard of comparison as before. In this case,* $T = 14 > 9 = p(R_1) + p(R_2) + p(R_3)$. *Then, after fully compensating all the agents* $T' = 5 = T - (p(R_1) + p(R_2) + p(R_3))$ *remains. We associate to each agent his opposite peak:* $(1,-1)$, $(2,-3)$, *and* $(3,-5)$. *The next table shows the rest of the process.*

T'	$x^{(k)}$	$p^{(k)}$
0	(1,3,5)	(-1,-3,-5)
1	(2,3,5)	(-2,-3,-5)
2	(3,3,5)	(-3,-3,-5)
3	(4,3,5)	(-4,-3,-5)
4	(4,4,5)	(-4,-4,-5)
5	(4,5,5)	(-4,-5,-5)

4 Properties

Here we look for properties our rules may fulfil. Some of the following properties have been studied in the case where the good is perfectly divisible, and their rationale and "appealingness" are preserved in the case of indivisible goods. For some other properties, we have to adapt the fairness principle at hand so that it becomes meaningful in the case of problems with indivisibilities.

The most common and appealing requirement in the continuous case is a property of impartiality. In one of its forms, the so called *equal treatment of equals*, it says that in any problem, if two agents have identical preferences, then they should be indifferent among their respective allocations. Paired with the requirement of efficiency, it simply means that agents with identical preferences should be alloted the same amount. Unfortunately, no rule can fulfill this property in the context of problems with indivisibilities. It is enough to consider a two-agents problem with identical preferences $R_1 = R_2$, and $T = 1$. **?**, **?**, and **?** consider a weaker version of this condition, that they call **balancedness**: If in a problem two agents have equal preferences, then their allocations should differ, at most, by one unit.

Balancedness: For each $e \in \mathbb{A}$ and each $\{i, j\} \subseteq N$, if $R_i = R_j$, then $|F_i(e) - F_j(e)| \leq 1$.

The following property is straightforward. It says that an agent's allocation depends only on his preferred consumption.

Peaks only: For each $e = (e, T, (R_i, R_{-i})) \in \mathbb{A}$ and each $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}_{\mathbb{Z}}$ such that $p(R'_i) = p(R_i)$, then $F_i(e) = F_i(e')$.

The next principle, *ar-truncation,* can be interpreted as an instance of a general principle of independence of irrelevant alternatives. Given $e \in \mathbb{A}$, let $ar(e) = \frac{\sum_{j \in N} p(R_j) - E}{n}$. The number $ar(e)$ is simply the average rationing of the task among the agents in N. This property states that any information on the agents' preferences below $ar(e)$ should be ignored. In consequence, all those problems whose preferences coincide in $[ar(e), +\infty[$ are indistinguishable.

Ar-truncation: For each $e = (N, T, R) \in \mathbb{A}$ and each $e' = (N, T, R') \in \mathbb{A}_{\mathbb{Z}}$, if for all $i \in N$, $R_i = R'_i$ on $[ar(e), +\infty[$, then, $F(e) = F(e')$.

The following two properties refer to the case in which there is a change in the a problem's task, without altering agents' preferences. The first one, *one-sided resource monotonicity,* considers the case in which the change in the task does not alter the type of rationing associated to the initial problem, i.e, if initially we have to ration labor, it is still labor to be rationed after the task increasing, or else, if in the initial problem we have to ration leisure, then again, we have too much labor to allocate even after the decreasing of the task. In either case, the property states that no agent should suffer.

One-sided resource monotonicity: For all $e, e' \in \mathbb{A}$ such that $e = (N, T, R)$ and $e' = (N, T', R)$,if it happens that $(i) \sum_{j \in N} p(R_j) \geq T' > T$, or else, $(ii) \sum_{j \in N} p(R_j) \leq T' < T$,then for all $i \in N$, $F_i(e') R_i F_i(e)$.

Imagine now that when estimating the value of the task, we were pessimistic, so that the real value is larger than expected. Then two possibilities are open, either to forget about the initial allocation and just solve the new problem, or keep the tentative allocation and then allocate the rest of the task among the agents, after adjusting the preferences by shifting them by the amount already obtained. The property of *agenda independence* requires that the final allocation should not depend on this timing.

Agenda independence For $e = (N, T, R) \in \mathbb{A}$ and each $T \in \mathbb{Z}_{++}$, $F(e) = F(N, T', R) + F(N, T - T', R')$, where $R'_i = \pi^{F_i(N,T',R)}(R_i)$.[3]

The principle of *strategy-proofness* states that thruthtelling should be a (weakly) dominant strategy for all agents, or, in other words, that no agent should over benefit from misrepresenting his preferences.

Strategy-proofness: For each $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$, each $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$, and each $i \in N$, $F_i(e) R_i F_i(e')$.

The next group of properties refer to changes in the set of agents. Suppose that, after solving the problem $e = (N, T, R) \in \mathbb{A}$, a proper subset of agents $S \subset N$ decides to reallocate the total amount they have received, that is, they face a new allocation problem: $(S, \sum_{i \in S} a_i, R_S)$, where $R_S = (R_i)_{i \in S}$ and a is the allocation corresponding to apply the rule to the problem e. A rule satisfies *consistency* if the new reallocation is only a restriction to the subset S of the initial allocation.

Consistency: For each $e \in \mathbb{A}$, each $S \subset N$, and each $i \in S$, $F_i(e) = F_i(S, \sum_{j \in S} F_j(e), R_S)$.

If the previous requirement is made only for subsets of agents of size two, then it is referred to as *bilateral consistency.*

Bilateral consistency: For each $e \in \mathbb{A}$, each $S \subset N$, such that $|S| = 2$,and each $i \in S$, $F_i(e) = F_i(S, \sum_{j \in S} F_j(e), R_S)$.

Finally, we consider the possibilituy of recovering the solution for the general case out of the solutions in the two-agent case. Let us consider an allocation for a problem with the following feature: For each two-agents subset, the rule chooses the restriction of that allocation for the associated reduced problem to this agent subset. Then that allocation should be the one selected by the rule for the original problem.

Let $c.con(T, R; F) \equiv \{x \in \mathbb{Z}_+^N : \sum_{i \in N} x_i = T$ and for all $S \subset N$ such that $|S| = 2, x_S = F(S, \sum_{i \in S} x_i, R_S)\}$

Converse consistency (Chun, 1999): For each $e \in \mathbb{A}_{\mathbb{Z}}$, $c.con(T, R; F) \neq \phi$, and if $x \in c.con(T, R; F)$, then $x = F(e)$.

The next two results are also valid for the case of indivisibilities..

Proposition 1. *One-sided resource monotonicity together with consistency imply converse consistency.*

Proof. Let $e \in \mathbb{A}_{\mathbb{Z}}$. By *consistency* the set $c.con(T, R; F) \neq \phi$. Let $x, y \in c.con(T, R; F)$

[3] For a given $a \in \mathbb{Z}$, $\pi^a : \mathbb{S} \longrightarrow \mathbb{S}$ is defined as follows: For each $R \in \mathbb{S}$, $x \pi^a(R) y$ iff $(x + a) R (y + a)$. Given $R \in \mathbb{S}$, we call $\pi^a(R)$ the shifting of R by a.

with $x \neq y$. We distinguish two cases.

Case 1. If $\sum_{i \in N} p(R_i) \geq T$. There exists $k \in N$ such that $x_k > y_k$. Consider each two-agent set $S = \{k, j\}$ with $j \in N$ and $j \neq j$. Since $x, y \in c.con(T, R; F)$, $x_S = F(S, x_j + x_k, R_S)$ and $y_S = F(S, y_j + y_k, R_S)$. By *one-sided resource monotonicity*, $x_j \geq y_j$. This fact, join with $x_k > y_j$, and $\sum_{i \in N} x_i = T = \sum_{i \in N} y_i$ yields a contradiction.

Case 1. If $\sum_{i \in N} p(R_i) \leq T$. There exists $k \in N$ such that $x_k < y_k$. Consider each two-agent set $S = \{k, j\}$ with $j \in N$ and $j \neq j$. Since $x, y \in c.con(T, R; F)$, $x_S = F(S, x_j + x_k, R_S)$ and $y_S = F(S, y_j + y_k, R_S)$. By *one-sided resource monotonicity*, $x_j \leq y_j$. This fact, join with $x_k < y_j$, and $\sum_{i \in N} x_i = T = \sum_{i \in N} y_i$ yields a contradiction.

□

Lemma 2 ([Elevator lemma] Thomson, 2000). *If a rule F is bilaterally consistent and coincides with a conversely consistent rule F' in the two agent case, then it coincides with F' in general.*

5 Characterizations

As we observed in Section 3, up and temporary satisfaction methods associated to a standard of comparison may end up in pure priority methods, and thus could violate *balancedness*. In order to guarantee this property, we should concentrate on a particular subfamily of standards of comparison, that we call *monotonic standards*.

Monotonic standard of comparison: For each $\{i, j\} \subseteq \mathbb{N}$, and each $x, y \in \mathbb{Z}$, if $x > y$, then $\sigma(i, x) < \sigma(j, y)$. Let Σ^M denote the subfamily of all monotonic standards of comparison.

In other words, monotonic standards of comparison always give priority to agents with larger integer numbers.

The following result is straightforward:

Proposition 3. *Let $\sigma \in \Sigma$ be an standard of comparison. Then, the associated up and temporary satisfaction methods, U^σ and TS^σ, satisfy balancedness if and only if σ is monotonic.*

We shall call **up (temporary satisfaction) monotonic methods** to the up (temporary satisfaction) methods associated to monotonic standards of comparison.

We present now our first characterization of the temporary satisfaction monotonic methods. It constitutes a parallel result to that obtained by Ching to characterize the uniform rule (1994).

Theorem 4. *A rule F satisfies peaks only, balancedness, strategy proofness, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = TS^\sigma$.*

Proof. It is easy to check that each TS^σ satisfies the two properties. Conversely, let F bea rulle satisfying all the properties.

Step 1. *Definition of the standard of comparison.* Let us define the order $\sigma \in \Sigma^M$ as follows

$$
\begin{aligned}
&a > b \Rightarrow \sigma(i,a) > \sigma(j,b)\\
&a = b \Rightarrow [\sigma(i,a) > \sigma(j,b) \Leftrightarrow F(\{i,j\},1,(R_i,R_j)) = e_i],
\end{aligned}
$$

where R_i and R_j are two single-peaked preference relations such that $p(R_i) = a = b = p(R_j)$. It is straightforward to see that such a σ is complete and antisymmetric. Let us show that σ is transitive. Suppose that there exist $\{i,j,k\} \subseteq N$ such that $\sigma(i,x) > \sigma(j,y)$, $\sigma(j,y) > \sigma(k,z)$, but $\sigma(i,x) < \sigma(k,z)$. By construction and *peaks only*, this can only happen when $x = y = z$. By definition of σ, in such a case, $F(\{i,j\},1,(R_i,R_j)) = e_i$, $F(\{j,k\},1,(R_j,R_k)) = e_j$ and $F(\{k,i\},1,(R_k,R_i)) = e_k$, where $p(R_i) = x$, $p(R_j) = y$, and $p(R_k) = z$. Consider the problem $(\{i,j,k\},2,(R_i,R_j,R_k))$. There are only three possible allocations: $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$. Suppose that $F(\{i,j,k\},2,(R_i,R_j,R_k)) = (1,1,0)$, by *consistency*, $F(\{i,k\},1,(R_i,R_k)) = e_i$, achieving in this way a contradiction with $F(\{i,k\},1,(R_i,R_j)) = e_k$. An analogous argument is applied if $F(\{i,j,k\},2,(R_i,R_j,R_k)) = (1,0,1)$, or if $F(\{i,j,k\},2,(R_i,R_j,R_k)) = (0,1,1)$. Therefore $\sigma(i,c_i) > \sigma(k,c_k)$, and then σ is transitive.

Step 2. *Let us prove now that $F = TS^\sigma$.* By Proposition **??** and Lemma **??** it is sufficient to consider the two-agent case. Then, let us consider the problem $e = (S,T,R) \in \mathbb{A}_{\mathbb{Z}}$ where $S = \{i,j\} \subseteq N$. Without loss of generality we can assume that $p_i \equiv p(R_i) \leq p(R_j) \equiv p_j$. We analyze the case in which $p_i + p_j \geq T$. The other case is completely analogous. We distinguish the following cases:

Case 1. If $R_1 = R_2$ and $T \in \dot{2}$. By *balancedness*, $F(e) = \left(\frac{T}{2},\frac{T}{2}\right) = D^\sigma(e)$.

Case 2. If $R_i = R_j$ and $E \in \dot{2}+1$. Then, by *balancedness* and the definition of the order σ, $F(e) = D^\sigma(e)$.

Case 3. If $F_i(e) \leq F_j(e) \leq p(R_i) \leq p(R_j)$. By *strategy proofness*, $F_i(e) = F_i(S,T,(R_j,R_j))= D_i^\sigma(S,T,(R_j,R_j)) = D_i^\sigma(e)$.

Case 4. If $F_j(e) \leq F_i(e) \leq p(R_i) \leq p(R_j)$. By *strategy proofness*, $F_j(e) = F_j(S,T,(R_i,R_i)) = D_j^\sigma(S,T,(R_i,R_i)) = D_j^\sigma(e)$.

Case 5. If $F_i(e) \leq p(R_i) < F_j(e) \leq p(R_j)$. By *strategy proofness*, $F_i(e) = F_i(S,T,(R_j,R_j)) = D_i^\sigma(S,T,(R_j,R_j)) = D_i^\sigma(e)$. If $F_i(e) = D_i\sigma(e) = p(R_i)$, then $F_j(e) = T - F_i(e) = T - D_i\sigma(e) = U_j\sigma(e)$. If $F_i(e) \leq p(R_i) - 1$, then $F_j(e) \leq p(R_i)$, which is a contradiction.

Then, F coincides with TS^σ in the two agents case, and therefore they do so in general.

□

We present now a characterization of the up monotonic methods.

Theorem 5. *A rule F satisfies peaks only, balancedness, agenda independence, ar-truncation, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$.*

Proof. It is easy to check that each up monotonic method satisfies the properties. Conversely, let F be a discrte rule satisfying all the properties.

Step 1. *Definition of the standard of comparison.* Let us define the order $\sigma \in \Sigma^M$ as follows

$$a > b \Rightarrow \sigma(i,a) > \sigma(j,b)$$
$$a = b \Rightarrow [\sigma(i,a) > \sigma(j,b) \Leftrightarrow F(\{i,j\},1,(R_i,R_j)) = e_i],$$

where R_i and R_j are two single-peaked preference relation such that $p(R_i) = a = b = p(R_j)$. It is straightforward to check that σ is an order.

Step 2. *Let us prove now that $F = U^\sigma$.* By Proposition **??** and Lemma **??** it is sufficient to consider the two-agent case. Then, let us consider the problem $e = (S,T,R) \in \mathbb{A}_\mathbb{Z}$ where $S = \{i,j\} \subseteq N$. Without loss of generality we can assume that $p_i \equiv p(R_i) \leq p(R_j) \equiv p_j$. Suppose first that $p_i = p_j$. By peaks only, equal treatment of equals, and the definition of the order, $F(e) = U^\sigma(e)$. Let now $p_i = p_j$We distinguish now the following cases:

Case 1. If $p_i + p_j \geq T$. Let us define $T' = p_i + p_j$. Then $F(S,T',R) = (p_i,p_j) = U^\sigma(S,T',R)$. Once we have alloted the amount T', both agents have the same preference relation: $R'_i = R'_j$. By balancedness and the definition of the order $F(S,T-T',(R'_i,R'_j)) = U^\sigma(S,T-T',(R'_i,R'_j))$. By agenda independence, $F(e) = F(S,T',R) + F(S,T-T',(R'_i,R'_j)) = U^\sigma(S,T',R) + U^\sigma(S,T-T',(R'_i,R'_j)) = U^\sigma(e)$.

Case 2. If $p_i + p_j \leq T$. If T is such that $0 \leq T \leq p_j - p_i$, then $ar(e) \leq p_1$. By *ar*-truncation, $F(e) = (0, T) = U^\sigma(e)$. If T is such that $p_j - p_i \leq T \leq p_i + p_j$, then, by agenda independence, $F(e) = F(S, p_j - p_i, R) + F(S, T - (p_j - p_i), R')$, where $R'_i = R'_j$. By balancedness and the definition of the order, $F(e) = F(S, p_j - p_i, R) + F(S, T - (p_j - p_i), R') = U^\sigma(S, p_j - p_i, R) + U^\sigma(S, T - (p_j - p_i), R') = U^\sigma(e)$.

Then, F coincides with U^σ in the two agents case, and therefore they do so in general.

□

6 Temporary satisfaction Monotonic Methods and the uniform rule and up Monotonic Methos and equal distance rule.

In the previous section we obtained characterization results for the family of up and temporary satisfaction monotonic methods. Some of those characterizations have analogous counterparts in characterization results of the continuous equal distance and uniform rules, respectively.[4] Actually, the relationship between those monotonic methods and the uniform and equal distance rules is strongest. On the one hand, any monotonic temporary satisfaction method can be interpreted as a discrete version of the uniform rule, and, similarly, any monotonic up method could be interpreted as a discrete version of the equal distance rule. In this section we further explore the relationship between the family of monotonic temporary satisfaction methods and the *u* rule, and the relationship between the monotonic up methods and the *ed* rule. We show that, for any problem, the allocations prescribed by the uniform rule can be interpreted as the ex-ante expectations of the agents under the application of temporary satisfaction monotonic methods, if all plausible allocations prescribed by such methods are equally likely, and similarly the allocations prescribed by the equal distance rule can be interpreted as the ex-ante expectations of

[4] Under the assumption that the task were completely divisible, two of the most widely studied rules are the so called uniform and equal distance rules. The idea underlying the first one is equality distribution of the task.

Uniform rule, ***u***: For each $e \in \mathbb{A}$, selects the unique vector $e(u) \in \mathbb{R}^N$ such that: If $\sum_{i \in N} p(R_i) \geq T$, then $u(e) = \min\{p(R_i), \lambda\}$ for some $\lambda \in \mathbb{R}$. And, if $\sum_{i \in N} p(R_i) \leq T$, then $u(e) = \max\{p(R_i), \lambda\}$ for some $\lambda \in \mathbb{R}$.

The idea of the second rule is also equality, but now focusing on losses above or below, depending on the case, with respect to the peaks.

Equal distance rule, ***ed***: For each $e \in \mathbb{A}$, selects the unique vector $ed(e) \in \mathbb{R}^N$ such that $ed(e) = \max\{0, p(R_i) + \lambda\}$ for some $\lambda \in \mathbb{R}$.

the agents under the application of up monotonic methods, if all plausible allocations prescribed by such methods are equally likely. Next proposition proves the result.

Proposition 6. *Let $e = (N, T, R) \in \mathbb{A}$. Let $\Sigma^M_{(N,c)}$ denote the subset of Σ^M of the different partial orders involved in the problem (N, E, c).*[5] *Then*

(a) $u(e) = \dfrac{1}{\left|\Sigma^M_{(N,c)}\right|} \displaystyle\sum_{\sigma \in \Sigma^M_{(N,c)}} D^\sigma(N, E, c).$

(b) $ed(N, E, c) = \dfrac{1}{\left|\Sigma^M_{(N,c)}\right|} \displaystyle\sum_{\sigma \in \Sigma^M_{(N,c)}} U^\sigma(N, E, c).$

Proof. Let us prove the result for the uniform rule. On one hand, we know that the continuous uniform rule satisfies *converse consistency*, since it satisfies *one-sided resource monotonicity* and *consistency*. On the other hand, it is easy to check that the temporary satisfaction methods are *consistent*. Then the average given by the right hand side in the formula is also consistent (see Thomson 2004). By using the Elevator Lemma it is enough to consider the two-agent case. But it is straightforward that in this case both the uniform rule and the average coincide. As a result, they are equal in general. We use an analogous argument for Statement b. □

7 Final Remarks

In this work we have considered allocation problems with indivisible goods when the agents' preferences are single-peaked, that is, problems in which the task, the allocations and the preferences are only defined over the set of integer numbers. Two natural procedures, *up and temporary satisfaction methods* have been proposed to solve these problems. The construction of these methods rely on using a particular standar of comparison on the cartesian product of agents and integer numbers, interpreted eother as peaks or opposite peaks. Thus, what we propose is not a pair of solutions, but else, two families of solutions, one for each method.

When we concentrate on a certain sub-family of standards, *monotonic standards,* our two families of solutions satisfy properties very much related to some well-known properties studied in the case of perfectly divisible goods, and they have a strongest relationship with the continuous uniform and equal-distance rule, respectively.

[5] In Σ^M we consider all possible orders over $\mathbb{N} \times \mathbb{Z}_{++}$. Notice that, for a given e, no all of them rank the pairs $(i, p(R_i))$ involved in that particular problem in different ways. $\Sigma^M_{(N,c)}$ denotes precisely the subset of those different orders.

Some qualifications on the properties used in this paper are in order. The procedural properties related to changes in the set of agents, *consistency, bilateral consistency* and *converse consistency,* read exactly as in the case of a perfectly divisible good, and they maintain both their interpretation ans strenght in obtaining the characterization results. It is particularly interesting that also the incentive compatibility condition, *strategy proofness,* is not only meaningful in the case of indivisible goods, but also that all the solutions in the family of temporary satisfaction methods do satisfy this property. This means that there are a large family of allocation methods for which the agents do not have incentives to misrepresent their preferences. As for *balancedness,* this property is the best we can do to approach equal treatment, and, in this respect, we may look at our procedures as to be as impartial as possible, given the indivisibilities.

In the same way as the requirement of balancedness forces to rely on a subfamily of standards of comparison, the so called monotonic standards, we may ask whether some additional properties may also significantly reduce the family of standards. Some particular sub-families come naturally to mind, and seem to be worth studying. For instance, consider a priority relation α on the set of agents, and then, construct a monotonic standard σ, out of this priority relation in the following way

$$\begin{aligned} x > y \Longrightarrow \sigma(i,x) < \sigma(j,y) \qquad & \forall i,j \in N \\ \alpha(i) < \alpha(j) \Longrightarrow \sigma(i,x) < \sigma(j,x) \qquad & \forall x \in \mathbb{Z} \end{aligned}$$

This family of standards always respect the priority relation in the set of agents, whenever the integer numbers coincide. We may call this standards *persistent monotonic standards.* It is an open problem to see whether persistent monotonic standards are associated to some appealing additional property for both up and temporary satisfaction methods. This and related questions are left for future research..