UNIVERSIDADE DE
SANTIAGO DE COMPOSTELA
DEPARTAMENTO DE
ESTATÍSTICA E INVESTIGACIÓN OPERATIVA

SEQUENCING GAMES WITHOUT A COMPLETELY SPECIFIED INITIAL ORDER

Flip Klijn and Estela Sánchez

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Sequencing Games without a Completely Specified Initial Order

Flip Klijn* and Estela Sánchez†

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Abstract

In this paper we introduce several classes of cooperative games associated to 1-machine sequencing situations in which the initial order is partially or completely unspecified. Properties of the games are studied and the core is analyzed in detail.

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1 Introduction

In operations research, sequencing situations are characterized by a finite number of jobs lined up in front of one or more machines that have to be processed on the machines. A single-decision maker wants to determine a processing order of the jobs that minimizes total costs. This single-decision maker problem can be transformed into a multiple decision makers problem by taking into account agents that own (at least) one job. In such a model, a group of agents (coalition) can save costs by cooperation.

Cooperative game theory has turned out to be a useful tool for the study of cooperation in sequencing situations (cf. Curiel *et al.* (2002)). Curiel *et al.* (1989) was the first paper that studied sequencing situations by means of cooperative game theory. The sequencing situations they dealt with consist of a set of agents who each have one job to be processed on a single machine. Moreover, they assumed the existence of an initial order, i.e., an order that is established before the processing takes place. Next, they associated to each sequencing situation a cooperative TU game, called sequencing game, in which the worth of a coalition equals the maximal cost savings the coalition can obtain by reordering their positions according to admissible rearrangements. Besides studying the properties of the games, Curiel *et al.* (1989) introduced a single-valued solution, the equal gain splitting rule, which assigns to each sequencing game a certain core allocation.

In many sequencing situations, however, there is no (clear) initial order. Usually, the initial order is thought of as being based on the first come first served principle. Nevertheless, the information on the arrival of jobs may be partially unknown or not available at all. An example that illustrates our point is the short period of time in the morning in which cars arrive at a garage to be repaired. In this situation, the order in which the cars are delivered does not impose any condition on the work scheme for the day, nor can any customer claim a particular block of time. Additionally, there may be another set of jobs that initially precede the jobs. In our example, this could be the unfinished jobs from the last working day.

*CODE and Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, Spain. e-mail: fklijn@pareto.uab.es

†Corresponding author. Address: Facultad de Economía, Universidad de Vigo, Lagoas-Marcosende, 36310 Vigo (Pontevedra), Spain. e-mail: esanchez@uvigo.es

In this paper, we study sequencing situations as given in Curiel *et al.* (1989), but with the difference that now the initial order is partially or completely unspecified. In fact, we will also deal with situations in which this uncertainty on the initial order can be expressed in the form of a probability distribution on the processing orders. Thus, we will distinguish between three basic situations, each one of them being appropriate for a particular setting of (un)certainty:
a) The classical sequencing situations as studied by Curiel *et al.* (1989) in which the initial order is known.
b) The sequencing situations in which there is a probability distribution on the set of possible initial orders. These situations have, to some extent, already been studied by Hamers and Slikker (1995).
c) The sequencing situations in which the initial order is partially or completely unspecified. This is the class of sequencing situations this paper mainly deals with. With 'partially' we mean that only the order of the first jobs is known.

The paper is organized as follows. First, in Section 2 we recall some well-known concepts on cooperative cost games. In Section 3 we present sequencing situations with uncertainty on the initial order. The associated games are discussed in Section 4. Finally, in Section 5 we analyze the core of the games, and discuss some appropriate allocation rules.

2 Preliminaries on cooperative game theory

A cooperative TU cost game (or shortly, cost game) is a pair (N, c) where $N = \{1, ..., n\}$ is a finite set of agents and $c : 2^N \to I\!R$ is a map assigning to each coalition $S \in 2^N$, a real number $c(S)$ that represents the minimum costs that the agents of S can guarantee by themselves independently of the agents of $N\backslash S$, where $c(\emptyset) = 0$. The corresponding cost savings game (N, v) is defined by $v(S) = \sum_{i \in S} c(\{i\}) - c(S)$ for all $S \in 2^N$.

Let (N, c) be a cost game. We say that (N, c) is monotone if for all $S \subset T \subseteq N$, $c(S) \leq c(T)$. It is subadditive if for all $S, T \in 2^N$ such that $S \cap T = \emptyset$, it holds that $c(S \cup T) \leq c(S) + c(T)$. Finally, it is concave if for all $i \in N$ and all $S \subset T \subset N\backslash\{i\}$, it holds that $c(T \cup \{i\}) - c(T) \leq c(S \cup \{i\}) - c(S)$.[1]

Let (N, c) be a cost game. For any $x = (x_i)_{i \in N} \in I\!R^N$ and $S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$. We define the set of pre-imputations $(I^*(N, c))$, the set of non-negative pre-imputations $(I^*_+(N, c))$, and the set of imputations $(I(N, c))$ as,

$$\begin{aligned} I^*(N,c) &= \left\{x \in I\!R^N : x(N) = c(N)\right\}, \\ I^*_+(N,c) &= \left\{x \in I^*(N,c) : x_i \geq 0 \text{ for all } i \in N\right\}, \\ I(N,c) &= \left\{x \in I^*(N,c) : x_i \leq c(\{i\}) \text{ for all } i \in N\right\}. \end{aligned}$$

The core of the cost game (N, c) is defined by

$$C(N,c) = \{x \in I(N,c) : x(S) \leq c(S) \text{ for all } S \subset N\}.$$

Games with a non-empty core are called balanced games. Each concave game is balanced, but not every balanced game is concave.

For $S \subseteq N$, we denote by $\Pi(S)$ the set of orders of S, i.e., bijective functions from S to $\{1, ..., s\}$, where $s = |S|$ is the cardinality of S. A generic order of S is denoted by $\sigma_S \in \Pi(S)$. For all $i \in N$ and $\sigma_N \in \Pi(N)$, let $P(\sigma_N, i) = \{j \in N : \sigma_N(j) < \sigma_N(i)\}$ and $F(\sigma_N, i) = \{j \in N : \sigma_N(j) > \sigma_N(i)\}$ the set of predecessors and followers of i with respect to σ_N, respectively. Given $\sigma \in \Pi(N)$ and $S \in 2^N$, let $\sigma_S \in \Pi(S)$ denote the order of S induced by σ. For each $S \in 2^N$, $\sigma_S \in \Pi(S)$ and $\sigma_{N\backslash S} \in \Pi(N\backslash S)$, we define, with a slight abuse of notation, the order $\sigma^* = (\sigma_S, \sigma_{N\backslash S}) \in \Pi(N)$ by $\sigma^*(i) := \sigma_S(i)$ for all $i \in S$, and $\sigma^*(i) := \sigma_{N\backslash S}(i) + |S|$ for all $i \in N\backslash S$.

[1] $S \subseteq N$ denotes that S is a subset of N and $S \subset N$ denotes that S is a proper subset of N.

The i-th coordinate of the marginal vector $m^\sigma(c)$, $\sigma \in \Pi(N)$, is defined by

$$m_i^\sigma(c) = c(P(\sigma, i) \cup \{i\}) - c(P(\sigma, i)).$$

The Shapley value, Sh, (Shapley (1953)) of a game (N, c) is defined as the average of all marginal vectors, i.e.,

$$Sh(N, c) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(c),$$

which can be expressed alternatively as

$$Sh_i(N, c) = \sum_{S \subseteq N : i \in S} \frac{(s-1)!(n-s)!}{n!} [c(S) - c(S \backslash \{i\})] \quad \text{for all } i \in N.$$

We will denote by $ext(C(N, c))$ the set of extreme points of $C(N, c)$. If (N, c) is a concave game, then the marginal vectors $m^\sigma(c)$ are the extreme points of $C(N, c)$, i.e.,[2]

$$C(N, c) = conv\{m^\sigma(c) : \sigma \in \Pi(N)\}.$$

In that case, the Shapley value is in the bary-center of the core.

3 Sequencing situations with (un)certainty

A sequencing situation with uncertainty consists of two ingredients: a 4-tuple (N, p, α, c) and some information on the initial order.

A 4-tuple (N, p, α, c) describes a finite set $N = \{1, ..., n\}$ of agents, each one of them owning one job that has to be processed on a machine. With a slight abuse of notation we denote for $i \in N$ agent i's job by i. The processing times of the jobs are given by $p = (p_i)_{i \in N}$ with $p_i > 0$ for all $i \in N$. Each agent $i \in N$ has a cost function $c_i : [0, \infty) \to I\!R$ given by $c_i(t) = \alpha_i t$ ($t \in [0, \infty)$), where $\alpha_i > 0$. The expression $c_i(t)$ is interpreted as the cost incurred by agent i if his job is completed at time t.

Throughout this paper we will assume that the processing times of all jobs are equal to some constant. So, we can assume, without loss of generality, that $p_i = 1$ for all $i \in N$. For the sake of convenience we assume, without loss of generality, that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$. Thus, the first ingredient of a sequencing situation is captured by a pair (N, α).

Concerning the information on the initial order, the second ingredient of a sequencing situation, we distinguish among the following three classes:

a) An *initial order sequencing situation* is given by a 3-tuple (N, α, σ_N) where $\sigma_N \in \Pi(N)$ is the initial order of the jobs[3].

b) A *probabilistic initial order sequencing situation* is given by a 3-tuple (N, α, μ) where μ is a probability measure on $\Pi(N)$, i.e., $\mu(\sigma) \geq 0$ for all $\sigma \in \Pi(N)$ and $\sum_{\sigma \in \Pi(N)} \mu(\sigma) = 1$.

c) A $T-$*order sequencing situation* is given by a 3-tuple (N, α, σ_T) where T is some (possibly empty) coalition (i.e., $T \in 2^N$), and $\sigma_T \in \Pi(T)$ is the initial order of the jobs of T that are in the head of the queue.

In an initial order sequencing situation there is a fixed and known initial order before the processing of the machine starts; in a probabilistic initial order sequencing situation, the information on the initial order is reduced to some probability distribution over the set of orders; finally, $T-$*order* sequencing situations form a class that comprise initial order sequencing situations (take $T = N$). We will pay special attention

[2] Given a set $A \subseteq I\!R^N$, we denote by $conv(A)$ its convex hull.

[3] For $i \in N$, agent i's job is initially at position $\sigma_N(i)$.

to the situations in which there is no knowledge on the position of any job, that is, $T = \emptyset$ in the class c). The subclass of $\emptyset - order$ sequencing situations will be called *uncertainty sequencing situations*.

Smith (1956) solved the problem of finding the optimal order of the jobs, by ordering them in non-decreasing order of their urgency indices. In our case, with processing times equal to one, the urgency indices boil down to the cost coefficients $(\alpha_i)_{i\in N}$.

A subsequent problem is how to allocate the minimal total costs among the agents. This problem can be tackled by cooperative game theory. In the classical framework of a), a (cooperative) sequencing game is defined, by assigning to each coalition the maximal cost savings the coalition can obtain by reordering their positions according to admissible rearrangements. Clearly, in this framework the initial order of the agents plays an important role as it serves as an assignment of the initial rights of the agents. Let us now describe the sequencing games as introduced by Curiel *et al.* (1989).

To facilitate our discussion, let $c(S,\sigma)$ be the aggregate costs of coalition S in the order σ, i.e.,

$$c(S,\sigma) = \sum_{i\in S} \alpha_i(|P(\sigma,i)| + 1).$$

The (maximal) cost savings of a coalition S depend on the set of admissible rearrangements of this coalition. Given an initial order $\sigma_N \in \Pi(N)$, we call a bijection $\sigma \in \Pi(N)$ admissible for S if the players of S do not jump over players outside S. Formally, $P(\sigma,i) = P(\sigma_N,i)$ for all $i \in N\backslash S$. The set of all admissible rearrangements for a coalition S is denoted by $\Sigma_S(\sigma_N) \subseteq \Pi(N)$.

Given an initial order sequencing situation (N,α,σ_N), the corresponding (classic) sequencing game (N,v_{σ_N}) is defined in such a way that the worth of a coalition S is equal to the maximal cost savings the coalition can achieve by means of admissible rearrangements. Formally, we have

$$v_{\sigma_N}(S) = \max_{\sigma\in\Sigma_S(\sigma_N)} \{c\,(S,\sigma_N) - c\,(S,\sigma)\}. \tag{1}$$

The order $\sigma \in \Sigma_S(\sigma_N) \subseteq \Pi(N)$ that maximizes expression (1) is called an optimal order for coalition S. Curiel *et al.* (1989) proved that classic sequencing games are convex, and hence have a non-empty core.[4] Moreover, they introduced the equal gain splitting rule (EGS-rule), which assigns to each sequencing game a core allocation. Formally, for each initial order sequencing situation (N,α,σ_N),

$$EGS(N,\alpha,\sigma_N) = \left(\frac{1}{2}\sum_{j\in F(\sigma_N,i)} g_{ij} + \frac{1}{2}\sum_{k\in P(\sigma_N,i)} g_{ki}\right)_{i\in N}$$

where $g_{ij} = \max\{\alpha_j - \alpha_i, 0\}$ represents the gain attainable for player i and j in case player i is directly in front of player j.

For the sequencing situations described in class b), Hamers and Slikker (1995) propose a probabilistic egalitarian gain splitting rule (PEGS), which is a rule that generalizes the EGS-rule. Formally, for a probabilistic initial order sequencing situation (N,α,μ),

$$PEGS(N,\alpha,\mu) = \sum_{\sigma_N\in\Pi(N)} \mu(\sigma_N) EGS(N,\alpha,\sigma_N).$$

4 Sequencing cost games

In this section we present several cost games associated to different sequencing situations.

Associated to the initial order sequencing situations (type a)):

[4] The game (N,v_{σ_N}) is convex if $(N,-v_{\sigma_N})$ is concave. The core of (N,v_{σ_N}) is defined by reversing the inequalities in the definition of the core of a cost game.

- *The classic sequencing cost game* (cf. Curiel *et al.* (1989)):

  Given an initial order sequencing situation (N, α, σ_N), the associated cost game, denoted by (N, c_{σ_N}), is defined by

$$c_{\sigma_N}(S) := \sum_{i \in S} c(\{i\}, \hat{\sigma}^S) = c(S, \hat{\sigma}^S) = c_{\hat{\sigma}^S}(S) \quad \text{for all } S \subseteq N,$$

  where $\hat{\sigma}^S \in \Sigma_S(\sigma_N) \subseteq \Pi(N)$ is an optimal rearrangement for S.

Associated to probabilistic initial order sequencing situations (type b)):

- *The probabilistic cost game:*

  Given a probabilistic initial order sequencing situation (N, α, μ), we define the game (N, c_μ) by

$$c_\mu(S) := \sum_{\sigma_N \in \Pi(N)} \mu(\sigma_N) c_{\sigma_N}(S) \quad \text{for all } S \subseteq N.$$

  This game measures, for every coalition S, the expected minimal costs if the probability distribution over $\Pi(N)$ is given by μ.

Associated to uncertainty sequencing situations (type c) with $T = \emptyset$) we define the next two classes of games:

- *The pessimistic cost game*:

  Given an uncertainty sequencing situation $(N, \alpha, \sigma_\emptyset)$, we define the game (N, c_{pes}) by

$$c_{pes}(S) := \max_{\sigma_N \in \Pi(N)} \{c_{\sigma_N}(S)\} = c_{\sigma^S_{pes}}(S) \quad \text{for all } S \subseteq N,$$

  where $\sigma^S_{pes} \in \Pi(N)$ is one of the worst initial orders that coalition S could be faced with. Thus, in the game (N, c_{pes}), the value of a coalition corresponds to the most pessimistic scenario.

- *The tail cost game*:

  Given an uncertainty sequencing situation $(N, \alpha, \sigma_\emptyset)$, we define the game (N, c_{tail}) by

$$c_{tail}(S) := \min_{\sigma_S \in \Pi(S)} \{c_{(\sigma_{N \setminus S}, \sigma_S)}(S)\} = \sum_{k \in S} \alpha_k(\hat{\sigma}_S(k) + n - s) \quad \text{for all } S \subseteq N,$$

  where $\hat{\sigma}_S \in \Pi(S)$. In the tail game, the coalition S allows that all the jobs of $N \setminus S$ go first meanwhile they reorder their positions at the end of the queue in an optimal way. The spirit of this game is that members of $N \setminus S$ are not worse off if they go first. Note that the order of the members of $N \setminus S$ is irrelevant for the value of coalition S in (N, c_{tail}). For an illustration see Figure 1.

Figure 1: Coalition S in the tail game

Now we present sequencing games associated to T-order *sequencing situations* (type c), not necessarily $T = \emptyset$). In this type of situations a set of agents T is known to be the head of the initial order, meanwhile there is no knowledge on the order of the agents of $N\backslash T$.

Given $T \subseteq N$ and $\sigma_T \in \Pi(T)$, and a sequencing situation (N, α, σ_T), the associated T*-order cost game*, denoted by (N, c_{σ_T}), is defined by,

$$c_{\sigma_T}(S) := \begin{cases} c_{(\sigma_T, \sigma_{N\backslash T})}(S \cap T) + c_{tail}(S \cap (N\backslash T)) & \text{if } N\backslash T \nsubseteq S; \\ c_{(\sigma_T, \sigma_{N\backslash T})}(S) & \text{if } N\backslash T \subseteq S, \end{cases}$$

where $\sigma_{N\backslash T} \in \Pi(N\backslash T)$. Note that for $T = \emptyset$ the T-order cost game coincides with the tail cost game.

Remark 1. In the definition of the T-order game the game c_{tail} can be replaced by the game c_{pes}if the set of possible orders is restricted in an appropriate way. Note, however, that the computation of the game c_{tail} is direct, meanwhile the computation of the game c_{pes} we need to find the worst order for each coalition, which in general is a cumbersome task. We have not been able to find an algorithmic procedure to calculate the game c_{pes}. Nevertheless, the following observations may be helpful for *ad hoc* calculations.

Let $S \subseteq N$ be a coalition $\{i_1, \ldots, i_s\}$ such that $\alpha_{i_1} \geq ... \geq \alpha_{i_s}$. There is a partition $\{S_1, ..., S_k\}$ of S with $k \geq 1$, such that $c_{pes}(S) = c_{\sigma^S_{pes}}(S) = \sum_{j=1}^{k} c(S_j, \sigma^S_{pes})$ for some order $\sigma^S_{pes} \in \Pi(N)$, and additionally:
i) there is no gap at the end of the queue, that is, there is $i \in S$ such that $\sigma^S_{pes}(i) = n$,
ii) all the gaps that break cooperation between the groups $S_1, ..., S_k$ are of unit size,
iii) the urgency indices of the group processed first according to σ^S_{pes}, S_1 say, are lower or equal to the urgency index of any of the other agents, the urgency indices of the group processed second according to σ^S_{pes}, S_2 say, are lower or equal to the urgency index of any of the agents in $S_3, ..., S_k$, etc.
Obviously, when $k = 1$, the value of S in the pessimistic game coincides with its value in the tail game.

We would like to point out the main differences between the pessimistic game and the tail game, and provide some relations with the literature. In a classic sequencing game, given an order $\sigma_N \in \Pi(N)$, the cost $c_{\sigma_N}(S)$ is computed using the set $\Sigma_S(\sigma_N)$ of admissible permutations of S, that are those permutations in which members of a coalition S are not allowed to jump over non-members. In the pessimistic game, this assumption is in some sense also present, but we have take into account all $\sigma_N \in \Pi(N)$. If we admit that agents of S jump over non-members as long as it does not cause a delay in the starting time of any job not in S, then the worst for a coalition is being at the end of the queue. This is the main idea of the tail game. In the literature different classes of S-admissible arrangements have been considered (cf. Curiel *et al.* (1993)).

Throughout our analysis the pair (N, α) is fixed. The next propositions provide some first elementary relations between the games introduced above.

Proposition 1

a) $c_{pes}(N) = c_{tail}(N) = c_\mu(N)$ for all probability measures μ on $\Pi(N)$.
b) For any $S \subseteq N$ and any probability measure μ on $\Pi(N)$, $c_\mu(S) \leq c_{pes}(S)$.
c) For all $S \subseteq N$, $c_{tail}(S) \leq c_{pes}(S)$.
d) For all $i \in N$, $c_\mu(\{i\}) = \sum_{\sigma_N \in \Pi(N)} \mu(\sigma_N) c_{\sigma_N}(\{i\}) \leq c_{pes}(\{i\}) = c_{tail}(\{i\}) = n\alpha_i$ for all probability measures μ on $\Pi(N)$.

Proof. a) Follows immediately from the fact that $c_{\sigma_N}(N) = c_{\sigma'_N}(N)$ for all $\sigma_N, \sigma'_N \in \Pi(N)$.

b) Taking into account that for all $\sigma_N \in \Pi(N)$ and for all $S \subset N$, $c_{\sigma_N}(S) \le c_{pes}(S)$, one obtains directly

$$\begin{aligned} c_\mu(S) &= \sum_{\sigma_N \in \Pi(N)} \mu(\sigma_N) c_{\sigma_N}(S) \\ &\le \sum_{\sigma_N \in \Pi(N)} \mu(\sigma_N) c_{pes}(S) \\ &= c_{pes}(S). \end{aligned}$$

c) and d) are obvious. ■

We are now going to study the games c_{tail} and c_{pes} in more detail. As we mentioned earlier, it is much easier to compute c_{tail} than c_{pes}. Therefore one may wonder if there are any conditions on α that guarantee that both games coincide. The next lemma will be helpful in this respect. It also provides expressions for the marginal contributions in some specific cases.

Lemma 2

a) For $S \subseteq N$ and $i \in S$, it holds that $c_{tail}(S) - c_{tail}(S \backslash \{i\}) = (n-s+1)\alpha_i + \sum\limits_{k \in S: \alpha_k > \alpha_i} (\alpha_i - \alpha_k)$.

b) For $\sigma_N \in \Pi(N)$ and $i \in N$, $m_i^{\sigma_N}(c_{tail}) = (|F(\sigma_N, i)| + 1)\alpha_i + \sum\limits_{k \in P(\sigma_N, i): \alpha_k > \alpha_i} (\alpha_i - \alpha_k)$.

c) Let $T = \{i_1, ..., i_t\} \subset N$, where $\alpha_{i_1} \ge \alpha_{i_2} \ge ... \ge \alpha_{i_t}$. Then,

c1) $c_{tail}(T) - c_{tail}(T \backslash \{i_t\}) = n\alpha_{i_t} - \sum\limits_{l=1}^{t-1} \alpha_{i_l}$.

c2) If $c_{pes}(S) = c_{tail}(S)$ for all $S \subset T$, and $c_{pes}(T) \neq c_{tail}(T)$, then $c_{pes}(T) - c_{pes}(T \backslash \{i_t\}) = (n-t)\alpha_{i_t}$.

Proof. a) One easily obtains that for all $k \in S \backslash \{i\}$,

$$\hat{\sigma}_{S \backslash \{i\}}(k) = \begin{cases} \hat{\sigma}_S(k) & \text{if } \alpha_k < \alpha_i; \\ \hat{\sigma}_S(k) - 1 & \text{if } \alpha_k > \alpha_i, \end{cases}$$

where $\hat{\sigma}_S \in \Pi(S)$ is the optimal order of S. Then the result follows readily.
b) Take $S = P(\sigma_N, i) \cup \{i\}$, note that $n - s = F(\sigma_N, i)$, and apply a).
c) The proof of c1) is straightforward. We will prove c2).

From the hypothesis, it follows that for finding the value $c_{pes}(T)$ we can restrict ourselves to at most $k \le 2$ disconnected groups (as specified in Remark 1). Then, some easy calculations show that $c_{pes}(T) = c_{\sigma^*}(T)$ where

$$\sigma^*(i_l) = \begin{cases} n - (t-1) + l & \text{for all } l = 1, \ldots, t-1; \\ n - t & \text{for } l = t. \end{cases}$$

Hence,

$$\begin{aligned} c_{pes}(T) - c_{pes}(T \backslash \{i_t\}) &= c_{\sigma^*}(T) - c_{tail}(T \backslash \{i_t\}) \\ &= c_{\sigma^*}(T) - c_{\sigma^*}(T \backslash \{i_t\}) \\ &= (n-t)\alpha_{i_t}. \end{aligned}$$

■

We would like to point out that the marginal contribution of a player to a coalition S (Lemma 2.a)) using the tail cost game, is the sum of two quantities that have natural interpretations. The first term, $(n-s+1)\alpha_i$, measures what the player's costs are for being at a position $\ge n-s+1$. The second term, $\sum\limits_{k \in S: \alpha_k > \alpha_i} (\alpha_i - \alpha_k)$, represents the additional costs to put the player in its position of the optimal order. Note also that marginal contributions need not be positive (i.e., the game (N, c_{tail}) is not monotone).

Corollary 3

The Shapley value of the game (N, c_{tail}) has an expression in terms of the alphas:

$$Sh_i(N, c_{tail}) = \sum_{S \subset N : i \in S} \frac{(s-1)!(n-s-1)!}{n!} \left((n-s+1)\alpha_i + \sum_{k \in S : \alpha_k > \alpha_i} (\alpha_i - \alpha_k) \right) \quad \text{for all } i \in N.$$

Proposition 4

Let $S = \{i_1, ..., i_s\} \subset N$ be such that $\alpha_{i_1} \geq \alpha_{i_2} \geq ... \geq \alpha_{i_s}$. Suppose that $c_{tail}(T) = c_{pes}(T)$ for all $T \subset S$. Then, $c_{tail}(S) = c_{pes}(S)$ if and only if $\sum\limits_{l=1}^{s-1} \alpha_{i_l} \leq s\alpha_{i_s}$.

Proof.

Suppose $\sum\limits_{l=1}^{s-1} \alpha_{i_l} > s\alpha_{i_s}$. We prove that $c_{tail}(S) \neq c_{pes}(S)$. Let $\sigma^* \in \Pi(N)$ be an order such that $\sigma^*(i_s) = n - s$ and $\sigma^*(i_l) = n - s + 1 + l$ for all $l = 1, ..., s-1$. Then,

$$\begin{aligned}
c_{pes}(S) &\geq c_{\sigma^*}(S) \\
&= (n-s)\alpha_{i_s} + \sum_{l=1}^{s-1} (n-s+1+l)\alpha_{i_l} \\
&= (n-s)\sum_{l=1}^{s} \alpha_{i_l} + \sum_{r=1}^{s-1}\sum_{l=r}^{s-1} \alpha_{i_l} + \sum_{l=1}^{s-1} \alpha_{i_l} \\
&> (n-s)\sum_{l=1}^{s} \alpha_{i_l} + \sum_{r=1}^{s}\sum_{l=r}^{s} \alpha_{i_l} \\
&= \sum_{l=1}^{s} (n-s+l)\alpha_{i_l} \\
&= c_{tail}(S).
\end{aligned}$$

where the strict inequality holds by the hypothesis.

Now suppose that $c_{pes}(S) \neq c_{tail}(S)$, but $c_{pes}(T) = c_{tail}(T)$ for all $T \subset S$, $T \neq S$. By Lemma 2. a) and c2),

$$\begin{aligned}
c_{tail}(S \backslash \{i_s\}) &= c_{tail}(S) - n\alpha_{i_s} + \sum_{l=1}^{s-1} \alpha_{i_l} \\
&< c_{pes}(S) - n\alpha_{i_s} + \sum_{l=1}^{s-1} \alpha_{i_l} \\
&= c_{pes}(S \backslash \{i_s\}) + (n-s)\alpha_{i_s} - n\alpha_{i_s} + \sum_{l=1}^{s-1} \alpha_{i_l} \\
&= c_{pes}(S \backslash \{i_s\}) - s\alpha_{i_s} + \sum_{l=1}^{s-1} \alpha_{i_l}.
\end{aligned}$$

Since $c_{tail}(S \backslash \{i_s\}) = c_{pes}(S \backslash \{i_s\})$, it follows that

$$\sum_{l=1}^{s-1} \alpha_{i_l} > s\alpha_{i_s},$$

which completes the proof. ■

Corollary 5

$c_{tail} = c_{pes}$ if and only if for all $S = \{i_1, ..., i_s\} \subset N$, $1 < s < n$, $\sum_{l=1}^{s-1} \alpha_{i_l} \leq s\alpha_{i_s}$ where $\alpha_{i_1} \geq ... \geq \alpha_{i_s}$.

Proposition 6

If $\sum_{l=1}^{k} \alpha_l \leq (k+1)\alpha_{k+1}$ for all $k = 1, ..., n-1$, then $c_{tail} = c_{pes}$.

Proof.
Suppose that for some $S = \{i_1, ..., i_s\} \subset N$ where $\alpha_{i_1} \geq \alpha_{i_2} \geq ... \geq \alpha_{is}$.we have that $c_{tail}(S) \neq c_{pes}(S)$ but $c_{tail}(T) = c_{pes}(T)$ for all $T \subset S$. By Proposition 4, $\sum_{l=1}^{s-1} \alpha_{i_l} > s\alpha_{i_s}$. Take $T = \{1, 2, ..., i_s\} \backslash S$. As for all $r \in T$ we have $\alpha_r \geq \alpha_{i_s}$, it follows that

$$\begin{aligned}
\sum_{r=1}^{i_s-1} \alpha_r &= \sum_{l=1}^{s-1} \alpha_{i_l} + \sum_{r \in T} \alpha_r \\
&> s\alpha_{i_s} + t\alpha_{i_s} \\
&= (s+t)\alpha_{i_s} \\
&= i_s\alpha_{i_s}
\end{aligned}$$

which is in contradiction with one of the inequalities. ■

Corollary 7

If $\alpha_i = \alpha_j$ for all $i, j \in N$, then $c_{tail} = c_{pes}$.

Remark 2. The sufficient condition in Proposition 6 is not a necessary condition, as the following example shows. Let (N, α) be given by $N = \{1, 2, 3\}$ and $\alpha = (70, 60, 43)$. It can easily be checked that for all $S \subseteq N$, $c_{pes}(S) = c_{tail}(S)$ (by Corollary 5), but $\sum_{l=1}^{2} \alpha_l = 130 > 129 = 3\alpha_3$.

Some properties of the games are studied in the next proposition.

Proposition 8

a) (N, c_{pes}) is subadditive.
b) (N, c_{tail}) is concave.
c) (N, c_{σ_T}) is concave for all $T \subseteq N$.

Proof.
a) Take $T_1, T_2 \subseteq N$ with $T_1 \cap T_2 = \emptyset$. Note that for all $S \subset N$ there is an order $\sigma_{pes}^S \in \Pi(N)$ with $c_{pes}(S) = \sum_{i \in S} \alpha_i \sigma_{pes}^S(i)$. Hence,

$$\begin{aligned}
c_{pes}(T_1 \cup T_2) &= \sum_{i \in T_1 \cup T_2} \alpha_i \sigma_{pes}^{T_1 \cup T_2}(i) \\
&\leq \sum_{i \in T_1} \alpha_i \sigma_{pes}^{T_1}(i) + \sum_{i \in T_2} \alpha_i \sigma_{pes}^{T_2}(i) \\
&= c_{pes}(T_1) + c_{pes}(T_2).
\end{aligned}$$

b) Take $S \subset T \subset N\backslash\{i\}$. From Lemma 2. a) it follows that

$$c_{tail}(S) - c_{tail}(S\backslash\{i\}) = (n - s + 1)\alpha_i + \sum_{k \in S:\alpha_k > \alpha_i} (\alpha_i - \alpha_k)$$

and

$$c_{tail}(T) - c_{tail}(T\backslash\{i\}) = (n - t + 1)\alpha_i + \sum_{k \in T:\alpha_k > \alpha_i} (\alpha_i - \alpha_k).$$

As $s \leq t$ and $\{k \in T : \alpha_k > \alpha_i\} \supseteq \{k \in S : \alpha_k > \alpha_i\}$ we find $c_{tail}(S) - c_{tail}(S\backslash\{i\}) \geq c_{tail}(T) - c_{tail}(T\backslash\{i\})$. Hence, the game (N, c_{tail}) is concave.
c) We have to show that for all $i \in N$, $T \subseteq N\backslash\{i\}$, $R \subseteq T$ it holds that

$$c_{\sigma_S}(T \cup \{i\}) - c_{\sigma_S}(T) \leq c_{\sigma_S}(R \cup \{i\}) - c_{\sigma_S}(R). \tag{2}$$

Let $T \cap (N\backslash S) = \{i_1, i_2, \ldots , i_q\}$, where $\{i_1, i_2, \ldots , i_p\} = T \cap (N\backslash S) \cap (N\backslash R)$. Note that possibly $p = 0$ or $q = 0$.

To prove (2) we distinguish among the following cases.
I)a) $N\backslash S \not\subseteq T \cup \{i\}$ and $i \in S$.
I)b) $N\backslash S \not\subseteq T \cup \{i\}$ and $i \notin S$.
II)a) $N\backslash S \subseteq T \cup \{i\}$ and $i \in S$.
II)b) $N\backslash S \subseteq T \cup \{i\}$ and $i \notin S$.

Case I)a): (2) follows from the concavity of $(S, c_{\sigma_S|S})$ (cf. Curiel *et al.* (1989)).

Case I)b): (2) follows from the concavity of the sum k of two concave games k_1 and k_2, defined on $2^{N\backslash S}$:

$$k(U) := k_1(U) + k_2(U),$$

where $k_1(U) := \sum_{i \in U} \alpha_i \sum_{j \in S} p_j = |S| \sum_{i \in U} \alpha_i$ and k_2 is the tail game on $(N\backslash S, (p_j)_{j \in N\backslash S}, (\alpha_j)_{j \in N\backslash S})$.

Case II)a): Let $\tau := (\sigma_S, (i_1, \ldots , i_q)) \in \Pi(N)$. Then, $c_{\sigma_S}(V) = c_\tau(V)$ for $V = T \cup \{i\}, T, R \cup \{i\}, R$. Hence, (2) follows from the concavity of (N, c_τ) (cf. Curiel *et al.* (1989)).

Case II)b): In case $N\backslash S \subseteq R \cup \{i\}$, let $\tau := (\sigma_S, i, (i_1, \ldots , i_q)) \in \Pi(N)$ and apply the argument of the previous case.
In case $N\backslash S \not\subseteq R \cup \{i\}$, let $\hat{R} := R\backslash S$ and $\tau := (\sigma_S, i, (i_{p+1}, \ldots , i_q), (i_1, \ldots , i_p)) \in \Pi(N)$. Now note that $c_{\sigma_S}(R \cup \{i\}) - c_{\sigma_S}(R) = c_\tau(\hat{R} \cup \{i\}) - c_\tau(\hat{R})$ and $c_{\sigma_S}(T \cup \{i\}) - c_{\sigma_S}(T) = c_\tau(T \cup \{i\}) - c_\tau(T)$. Then (2) follows from the concavity of (N, c_τ) (cf. Curiel *et al.* (1989)). This completes the proof. ■

Remark 3. The concavity of the pessimistic game is an open problem. Note that subadditivity is satisfied (Proposition 8. a)).

5 The core

This section is devoted to a more detailed study of the core of the games discussed in the previous section.

Proposition 9

The games (N, c_μ), (N, c_{tail}), (N, c_{pes}), and (N, c_{σ_T}), where $\sigma_T \in \Pi(T)$, $T \subseteq N$, are balanced.

Proof.
Since the games (N, c_{tail}) and (N, c_{σ_T}) are concave (Proposition 8. b) and c), respectively), they are balanced. By Proposition 1, $\emptyset \neq C(N, c_{tail}) \subseteq C(N, c_{pes})$, and hence the game (N, c_{pes}) is also balanced. Notice that $(c_\mu(\{i\}))_{i\in N} - PEGS(N, \alpha, \mu) \in C(N, c_\mu)$, and hence (N, c_μ) is balanced. ■

Example 1. Let (N, α) be given by $N = \{1, 2, 3\}$ and $\alpha = (7, 3, 1)$. The next table describes the characteristic functions associated to several sequencing cost game.

S	$c_{(1,2,3)}$	$c_{(1,3,2)}$	$c_{(2,1,3)}$	$c_{(2,3,1)}$	$c_{(3,1,2)}$	$c_{(3,2,1)}$	c_{tail}	c_{pes}
$\{1\}$	7	7	14	21	14	21	21	21
$\{2\}$	6	9	3	3	9	6	9	9
$\{3\}$	3	2	3	2	1	1	3	3
$\{1,2\}$	13	16	13	24	23	23	23	24
$\{1,3\}$	10	9	17	17	9	22	17	22
$\{2,3\}$	9	9	6	5	10	5	9	10
$\{1,2,3\}$	16	16	16	16	16	16	16	16

The next figure shows the cores of the tail game and pessimistic game. The triangle represents the set of non-negative pre-imputations ($I^*_+(N, c_{tail}) = I^*_+(N, c_{pes})$).

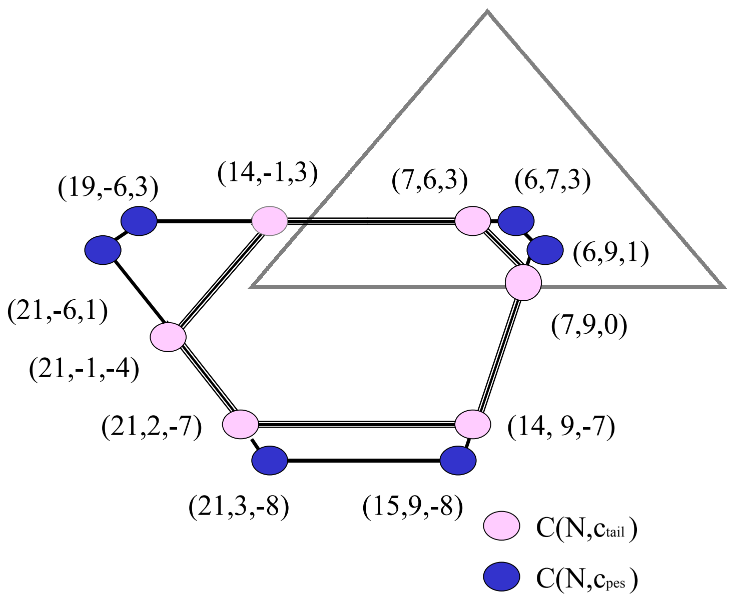

Figure 2: Cores of the tail and pessimistic game

Proposition 10

The following conditions are equivalent:

a) $C(N, c_{tail}) \subseteq I^*_+(N, c_{tail})$.
b) $m_i^\sigma(c_{tail}) \geq 0$ for all $\sigma \in \Pi(N)$ and all $i \in N$.
c) $\sum\limits_{l=1}^{k} \alpha_l \leq (k+1)\alpha_{k+1}$ for all $k = 1, ..., n-1$.

Proof.

- a) $\Leftrightarrow$ b) is obvious since the concavity of the game (N, c_{tail}) implies that

$$C(N, c_{tail}) = conv\left\{m^\sigma(c_{tail}) : \sigma \in \Pi(N)\right\}.$$

- b) $\Leftrightarrow$ c)

  Note that by Lemma 2. b), $m_i^\sigma(c_{tail}) = (|F(\sigma,i)|+1)\alpha_i + \sum_{k\in P(\sigma,i):\alpha_k>\alpha_i}(\alpha_i - \alpha_k)$ for all $i \in N$ and all $\sigma \in \Pi(N)$. Let us first prove b) $\Rightarrow$ c). Let $i \in N$. Take $\sigma \in \Pi(N)$ such that $\sigma(i) = n$. Then,

  $$m_i^\sigma(c_{tail}) = \alpha_i + \sum_{k\in P(\sigma,i):\alpha_k>\alpha_i}(\alpha_i - \alpha_k) \geq 0, \tag{3}$$

  which implies c).

  Assume now c). Let $i \in N$ and $\sigma' \in \Pi(N)$. Let $\sigma \in \Pi(N)$ be such that $\sigma(i) = n$. From the concavity of the game (N, c_{tail}) it follows that

  $$0 \leq m_i^\sigma(c_{tail}) = c_{tail}(N) - c_{tail}(N\backslash\{i\}) \leq m_i^{\sigma'}(c_{tail}).$$

  The first inequality follows from $\sigma(i) = n$ and (3). ■

One of the things that seem interesting now is the study of the core of the classic sequencing cost game and its relation with the cores of the pessimistic game and the tail game.

Proposition 11

$$ext(C(N, c_{tail})) \subseteq \bigcup_{\sigma\in\Pi(N)} ext(C(N, c_\sigma))$$

Proof.
Let $x \in ext(C(N, c_{tail}))$. We will show that there is an order $\sigma \in \Pi(N)$ such that $x \in ext(C(N, c_\sigma))$. Since (N, c_{tail}) is a concave game, there is an order $\tau \in \Pi(N)$, $\tau = (\tau^{-1}(1), ..., \tau^{-1}(n))$ such that $x = m^\tau(c_{tail})$. We will show that $x = m^\tau(c_\sigma)$, where $\sigma = (\tau^{-1}(n), ..., \tau^{-1}(1))$. Let $i \in N$ such that $i = \tau^{-1}(p)$, then

$$\begin{aligned} x_i &= m_i^\tau(c_{tail}) \\ &= c_{tail}(\{\tau^{-1}(1), ..., \tau^{-1}(p)\}) - c_{tail}(\{\tau^{-1}(1), ..., \tau^{-1}(p-1)\}) \\ &= c_\sigma(\{\tau^{-1}(1), ..., \tau^{-1}(p)\}) - c_\sigma(\{\tau^{-1}(1), ..., \tau^{-1}(p-1)\}) \\ &= m_i^\tau(c_\sigma). \end{aligned}$$

So, $x \in ext(C(N, c_\sigma))$ since (N, c_σ) is concave. ■

Proposition 12

$$a)\ conv\left\{\bigcup_{\sigma\in\Pi(N):C(N,c_\sigma)\subseteq C(N,c_{tail})} C(N, c_\sigma)\right\} \subseteq C(N, c_{tail})$$

$$b)\ C(N, c_{tail}) \subseteq conv\left\{\bigcup_{\sigma\in\Pi(N)} C(N, c_\sigma)\right\}.$$

$$c)\ conv\left\{\bigcup_{\sigma\in\Pi(N)} C(N, c_\sigma)\right\} \subseteq C(N, c_{pes})$$

Proof.
a) Follows directly from the convexity of the set $C(N, c_{tail})$.
b) A direct consequence of proposition 11.
c) By definition of c_{pes}, $c_\sigma(S) \le c_{pes}(S)$ for all $\sigma \in \Pi(N)$ and for all coalitions $S \subseteq N$ (with equality for $S = N$). Hence, $C(N, c_\sigma) \subseteq C(N, c_{pes})$ for all $\sigma \in \Pi(N)$. The result now follows from the convexity of $C(N, c_{pes})$. ■

Corollary 13

If $c_{tail} = c_{pes}$ then $C(N, c_{tail}) = conv\left\{\bigcup_{\sigma\in\Pi(N)} C(N, c_\sigma)\right\} = C(N, c_{pes})$.

Example 1 (continuation). The relations in Proposition 12 are in general not equalities.

a) Let $x = (14, 9, -7)$. It is easy to see that $x \in C(N, c_{tail}) \cap C(N, c_{(3,1,2)})$.
Nevertheless, $x \notin conv\left\{\bigcup_{\sigma\in\Pi(N):C(N,c_\sigma)\subseteq C(N,c_{tail})} C(N, c_\sigma)\right\}$ since $\tau = (3, 1, 2)$ is the only order in $\Pi(N)$ with $x \in C(N, c_\tau)$, but $C(N, c_\tau) \nsubseteq C(N, c_{tail})$. (see b) below)
b) It is easy to check that $y = (6, 9, 1) \in C(N, c_{(3,1,2)})$. Nevertheless, $y \notin C(N, c_{tail})$, since $y_2 + y_3 = 10 > 9 = c_{tail}(\{2, 3\})$.
c) Let $z = (6, 7, 3) \in C(N, c_{pes})$ and note $z \notin conv\left\{\bigcup_{\sigma\in\Pi(N)} C(N, c_\sigma)\right\}$. ◊

The next figure shows for Example 1 the cores of the classic sequencing cost games in relation with the core of the pessimistic cost game.

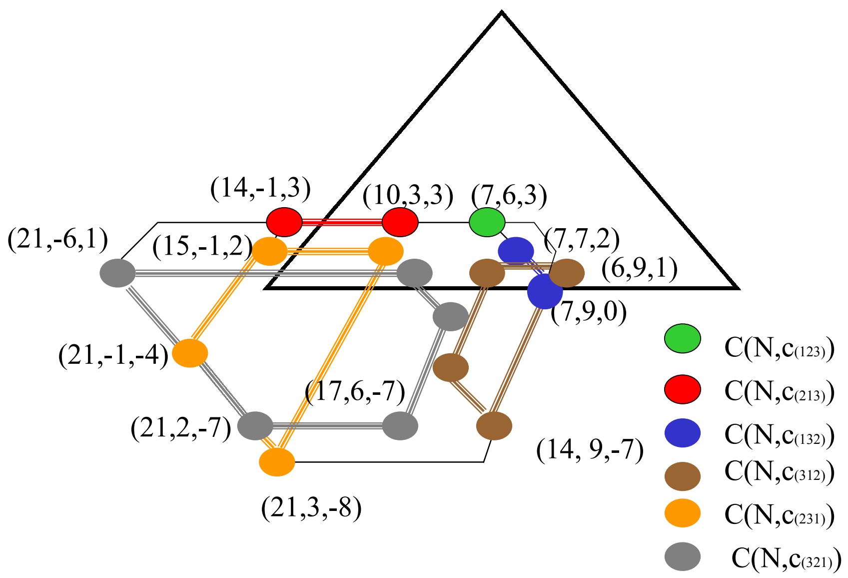

Figure 3: Cores of the classic sequencing cost games

The next proposition tells us that if an optimal order is formed, and the dynamical process of adding players is studied making use of the corresponding games, the core converges to a unique allocation.

Proposition 14

$$C(N, c_{tail}) \supseteq C(N, c_{(1)}) \supseteq C(N, c_{(1,2)}) \supseteq ... \supseteq C(N, c_{(1,2,...,n)}) = \{(\alpha_1, ..., n\alpha_n)\}.$$

Proof. Given that the value of the grand coalition is the same in all the games involved, it is sufficient to prove that for $S \subset N$ it holds that

$$c_{tail}(S) \geq c_{(1)}(S) \geq c_{(1,2)}(S) \geq ... \geq c_{(1,2,...,n)}(S).$$

Let $S \subset N$. Consider, with a slight abuse of notation, the coalition $T = (1, 2, ..., t)$, $1 \leq t < n$. It is not difficult to check that $c_T(S) = \sum_{i \in S \cap T} d_i + \sum_{i \in S \cap (N \setminus T)} b_i(S)$ where for all $i \in S$

$$\begin{aligned} d_i &= |N \setminus \{i+1, ..., n\}| \, \alpha_i = i\alpha_i, \\ b_i(S) &= |N \setminus (S \cap \{i+1, ..., n\})| \, \alpha_i. \end{aligned}$$

It is clear that $b_i(S) \geq d_i$ for all $i \in S$. Take $T' = (1, 2, ..., t+1)$, then

$$\begin{aligned} c_T(S) &= \sum_{i \in S \cap T} d_i + \sum_{i \in S \cap (N \setminus T)} b_i(S) \\ &\geq \sum_{i \in S \cap T'} d_i + \sum_{i \in S \cap (N \setminus T')} b_i(S) \\ &= c_{T'}(S). \end{aligned}$$

It remains to prove that $c_{tail}(S) \geq c_{(1)}(S)$, which is immediate since $c_{tail}(S) = \sum_{i \in S} b_i(S)$. ■

Remark 4. Let us note that if the optimal order is not formed, the previous result is not true. Let (N, α) be given by $N = \{1, 2, 3, 4, 5\}$ and $\alpha = (100, 1, 1, 1, 1)$. Take $S = (1, 2, 3)$, then $c_{tail}(S) = 300 + 4 + 5 < c_{(2)}(S) = 1 + 400 + 5$.

We focus our attention now on the class of uncertainty sequencing situations, i.e., the collection of tuples $(N, \alpha, \sigma_\emptyset)$ that we will denote simply by (N, α). Let us denote by $\mathcal{C}$ the class of all uncertainty sequencing situations. A rule on $\mathcal{C}$ is a map $\varphi : \mathcal{C} \to I\!R^n$.

The proportional rule PRO: for any uncertainty sequencing situation (N, α) we define

$$PRO(N, \alpha) = \left(\frac{\alpha_i}{\sum_{k \in N} \alpha_k} \sum_{k \in N} k\alpha_k \right)_{i \in N}.$$

The egalitarian gain splitting rule of the optimal orders ψ
Let us consider the set of all optimal orders Ω, i.e.,

$$\sigma = (\sigma^{-1}(1), ..., \sigma^{-1}(n)) \in \Omega \Leftrightarrow \alpha_{\sigma^{-1}(1)} \geq ... \geq \alpha_{\sigma^{-1}(n)}$$

For any uncertainty sequencing situation (N, α) we define

$$\psi(N, \alpha) = \left(\frac{1}{|\Omega|} \sum_{\sigma \in \Omega} \alpha_i \sigma(i) \right)_{i \in N}.$$

In the particular case $\Omega = \{\sigma_N\}$, we have $\psi_i(N,\alpha) = c_{\sigma_N}(\{i\}) - EGS_i(N,\alpha,\sigma_N) = c_{\sigma_N}(\{i\})$ for all $i \in N$.

Remark 5. If there is a constant $\delta > 0$ such that $\alpha_i = \delta$ for all $i \in N$, then $PRO_i(N,\alpha) = \psi_i(N,\alpha) = Sh_i(N, c_{tail}) = \frac{\delta}{2}(n+1)$ for all $i \in N$.

Proposition 15

Let (N, c_{tail}) be the tail game associated to (N,α). Then,

$$PRO(N,\alpha) \in C(N, c_{tail}).$$

Proof. Denote $x = PRO(N,\alpha)$. Let $S \subset N$, $S = \{i_1, ..., i_s\}$ with $\alpha_{i_1} \geq ... \geq \alpha_{i_s}$. Then,

$$\begin{aligned} x(S) &= \sum_{i\in S} \frac{\alpha_i}{\sum\limits_{k\in N}\alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n) \\ &= \frac{\alpha_{i_1}}{\sum\limits_{k\in N}\alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n) + \\ &\quad\ \frac{\alpha_{i_2}}{\sum\limits_{k\in N}\alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n) + \\ &\quad\ \vdots \\ &\quad\ \frac{\alpha_{i_s}}{\sum\limits_{k\in N}\alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n) \end{aligned} \tag{4}$$

Let $X_S = (x_{kl})$ be the $s \times n$-matrix defined by

$$x_{kl} := \frac{\alpha_{i_k}}{\sum\limits_{k\in N}\alpha_k} l\alpha_l \quad \text{for all } k = 1, ..., s \text{ and } l = 1, ..., n.$$

Note that $x(S) = \sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} x_{kl}$.

The proof proceeds now as follows. We construct an $s \times n$-matrix matrix $C_S = (c_{kl})$ such that $\sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} c_{kl} = c_{tail}(S)$, and show that $\sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} x_{kl} \leq \sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} c_{kl}$, which will complete the proof.

First we define the sets U and V of matrix coordinates as follows:

$$\begin{aligned} U &:= \{(l,m) \in \{1,...,s\} \times \{1,...,n\} : m > (n-s) + l\}, \\ V &:= \{(x,y) \in \{1,...,s\} \times \{1,...,n\} : y \leq x + (n-s-1)\}. \end{aligned}$$

Note that $U \cap V = \emptyset$. Let $f : U \to V$ be the map defined by $f(l,m) := (m-(n-s), l)$ for all $(l,m) \in U$. It can be shown easily that f is injective.

Then, the matrix C_S is defined as follows

$$c_{kl} := \begin{cases} \frac{\alpha_{i_k}}{\sum\limits_{k\in N}\alpha_k} l\alpha_l - \frac{\alpha_{i_k}\alpha_l}{\sum\limits_{k\in N}\alpha_k}(s-k-(n-l)) & \text{if } (k,l) \in U; \\ \frac{\alpha_{i_k}}{\sum\limits_{k\in N}\alpha_k} l\alpha_l + \frac{\alpha_{i_k}\alpha_l}{\sum\limits_{k\in N}\alpha_k}(k-l+(n-s)) & \text{if } (k,l) \in V; \\ \frac{\alpha_{i_k}}{\sum\limits_{k\in N}\alpha_k} l\alpha_l & \text{otherwise.} \end{cases} \tag{5}$$

It can be shown easily that $c_{kl} = \frac{\alpha_{i_k}}{\sum\limits_{k\in N}\alpha_k}\alpha_l(n-s+k)$ for all pairs $(k,l) \in \{1,...,s\} \times \{1,...,n\}$. (For $(k,l) \notin U \cup V$, take into account that $n-s+k = l$.) This implies that $\sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} c_{kl} = \sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} \frac{\alpha_{i_k}}{\sum\limits_{k\in N}\alpha_k}\alpha_l(n-s+k) = \sum\limits_{k=1}^{s} \alpha_{i_k}(n-s+k) = c_{tail}(S)$, as desired.

It remains to prove that $\sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} x_{kl} \leq \sum\limits_{k=1}^{s}\sum\limits_{l=1}^{n} c_{kl}$. Taking into account (4), (5), and the fact that f is injective, one deduces that we only need to prove that

$$\frac{\alpha_{i_k}\alpha_l}{\sum\limits_{k\in N}\alpha_k}(s-k-(n-l)) \leq \frac{\alpha_{i_{l-(n-s)}}\alpha_k}{\sum\limits_{k\in N}\alpha_k}(l-k) \quad \text{for all } (k,l) \in U.$$

This inequality follows readily since for all $(k,l) \in U$
(i) $(s-k-(n-l)) \leq l-k$ since $s \leq n$;
(ii) $\alpha_{i_k} \leq \alpha_k$ for all $k = 1,...,s$;
(iii) $\alpha_l \leq \alpha_{i_{l-(n-s)}}$ since $i_{l-(n-s)} \leq l$ as there are $s-(l-(n-s)) = n-l$ players in S after player $i_{l-(n-s)}$.
■

Remark 6. The proportional rule is in general not a core allocation of the classic sequencing cost game. To see this, take in Example 1 $\sigma = (3,2,1)$, and note that $PRO(N,\alpha) = (\frac{112}{11}, \frac{48}{11}, \frac{16}{11}) \notin C(N, c_{(321)})$, since $PRO_2(N,\alpha) + PRO_3(N,\alpha) = \frac{64}{11} > 5 = c_{(3,2,1)}(2,3)$.

Proposition 16

Let (N, c_{tail}) be the tail game associated to (N,α). Then, $\psi(N,\alpha,\sigma_\emptyset) \in C(N, c_{tail})$.

Proof. It is well known that for all $\sigma_N \in \Omega$, $(c_{\sigma_N}(\{i\}))_{i\in N} - EGS(N,\alpha,\sigma_N) \in C(N, c_{\sigma_N})$ (Curiel *et al.* (1989)). Then, by Proposition 14, $(c_{\sigma_N}(\{i\}))_{i\in N} - EGS(N,\alpha,\sigma_N) \in C(N, c_{tail})$. Because the core is a convex set, $\psi(N,\alpha,\sigma_\emptyset) \in C(N, c_{tail})$. ■

Corollary 17

Let (N, c_{pes}) be the pessimistic game associated to (N,α). Then, $PRO(N,\alpha), \psi(N,\alpha) \in C(N, c_{pes})$.

Consider the following properties which a rule on $\mathcal{C}$ may satisfy:
EFF: φ satisfies *efficiency* if for all $(N,\alpha) \in \mathcal{C}$ it holds that $\sum\limits_{i\in N} \varphi_i(N,\alpha) = c(N,\hat{\sigma})$, where $\hat{\sigma}$ is an optimal order of N.
SYM: φ satisfies *symmetry* if for all $(N,\alpha) \in \mathcal{C}$ and $i,j \in N$ with $\alpha_i = \alpha_j$ it holds that $\varphi_i(N,\alpha) = \varphi_j(N,\alpha)$.
URG: φ satisfies *urgency* if for all $(N,\alpha) \in \mathcal{C}$ and all $i,j \in N$, $i \neq j$ with $\alpha_i > \alpha_j$, it holds that $\varphi_i(N\backslash\{j\}, \alpha_{|N\backslash\{j\}}) = \varphi_i(N,\alpha)$.
PROP: φ satisfies *proportionality* if for all $(N,\alpha) \in \mathcal{C}$ and all $i,j \in N$, it holds that $\frac{\varphi_i(N,\alpha)}{\varphi_j(N,\alpha)} = \frac{\alpha_i}{\alpha_j}$.

Proposition 18

a) ψ is the unique rule that satisfies EFF, SYM and URG.
b) PRO is the unique rule that satisfies EFF and PROP.

Proof. a) It can easily be checked that ψ satisfies EFF, SYM and URG. We prove the uniqueness. Let φ be a rule satisfying EFF, SYM and URG.

Case 1) Suppose $\alpha_1 > \alpha_2 \geq \alpha_3 \geq ... \geq \alpha_n$.

By URG and EFF, we have that

$$\begin{aligned}\varphi_1(N,\alpha) &= \varphi_1(N\backslash\{2\},\alpha_{|N\backslash\{2\}}) = \varphi_1(N\backslash\{2,3\},\alpha_{|N\backslash\{2,3\}}) = \cdots = \varphi_1(\{1\},\alpha_{|\{1\}})\\ &= \alpha_1 = \psi_1(N,\alpha).\end{aligned}$$

Case 2) Suppose $\alpha_1 = \alpha_2 \geq \alpha_3 \geq ... \geq \alpha_n$.
Let $S_1 = \{j \in N : \alpha_j = \alpha_1\}$. For all $j \in S_1$, by URG and SYM we have

$$\varphi_j(N,\alpha) = \varphi_j(S_1,\alpha_{|S_1}) = \frac{\alpha_1}{2}(s_1+1) = \psi_j(S_1,\alpha_{|S_1}) = \psi_j(N,\alpha)$$

Let j_2 be the agent such that $\alpha_{j_2} = \max_{j\in N\backslash S_1} \alpha_j$ and $S_2 = \{j \in N\backslash S_1 : \alpha_j = \alpha_{j_2}\}$. Then, for all $j \in S_2$, by URG

$$\varphi_j(N,\alpha) = \varphi_j(S_1 \cup S_2, \alpha_{|S_1\cup S_2})$$

Hence, by EFF,

$$\begin{aligned}\sum_{j\in S_2}\varphi_j(N,\alpha) &= \sum_{j\in S_2}\varphi_j(S_1\cup S_2,\alpha_{|S_1\cup S_2})\\ &= \alpha_{j_2}(s_1+\frac{s_2+1}{2})s_2\end{aligned}$$

and by SYM, for all $j \in S_2$,

$$\varphi_j(N,\alpha) = \alpha_{j_2}(s_1+\frac{s_2+1}{2}) = \psi_j(S_1\cup S_2,\alpha_{|S_1\cup S_2}) = \psi_j(N,\alpha).$$

Obviously, we can repeat the same argument for the agents of $N\backslash(S_1 \cup S_2)$, which shows that $\varphi = \psi$.
b) Straightforward. ■

Remark 7. The proportional rule PRO satisfies EFF and SYM, but not URG. To see this, simply take $N = \{1,2\}$ and $\alpha = (1,2)$. Then $PRO_2(N,\alpha) = \frac{8}{3} \neq 2 = PRO_2(N\backslash\{1\},\alpha_{|\{2\}})$.

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