UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

DEPARTAMENTO DE ESTATÍSTICA E INVESTIGACIÓN OPERATIVA

Goodness of Fit Test for Interest Rate Models: an approach based on Empirical Process

A. Monsalve-Cobis, W. González-Manteiga, M. Febrero-Bande

Report 10-03

Reports in Statistics and Operations Research

Goodness of Fit Test for Interest Rate Models: an approach based on Empirical Process.

Abelardo Monsalve-Cobis *

Department of Statistic , University of Santiago of Compostela, Spain

Department of Mathematics, Universidad Centroccidental "Lisandro Alvarado", Venezuela.

abelardoenrique.monsalve@rai.usc.es

amonsalve@ucla.edu.ve

Wenceslao González-Manteiga

Department of Statistic , University of Santiago of Compostela, Spain,

wenceslao.gonzalez@usc.es

and

Manuel Febrero-Bande

Department of Statistic , University of Santiago of Compostela, Spain,

manuel.febrero@usc.es

June 2010

Abstract

In this work a new test for the goodness of fit of a parametric form of the drift and volatility functions of interest rate models is proposed. The test is based on a marked empirical process of the residuals. More specifically, the marked empirical process is constructed using estimators of the integrated regression function for the drift function and integrated conditional variance function for the volatility function. The distribution of the processes is approximated using bootstrap techniques. The test is applied to simulated classical financial models and is illustrated in an empirical application to the EURIBOR data set.

Keywords: Función de Regresión Integrada, Función de Varianza Condicional Integrada, Procesos de Difusión.

1 Introduction

The objective of investing is to obtain benefits, that is, to acquire goods and capital so the foreseeable yield for the cost of a unit of investment exceeds the cost of money or the interest rate. Variations in the interest rate occur frequently, affecting directly or indirectly the investments and the economy in general. The globalization of capital markets has resulted in increased volatility

*The authors acknowledge partial support by Project MTM2008-03010 of the Ministerio de Ciencia e Innovación, Spain.

of interest rates worldwide, which has aroused the interest of both financial and academic professionals. The characterization of the dynamics of interest rates allows the determination of their temporal structure, the evaluation of the prices of a wide range of financial assets, the design of investment and hedging strategies and risk assessment. From a macroeconomic perspective, it has special relevance in the determination of appropriate monetary policy and in various transmission channels, for relations between short- and long-term rates and for the formation of expectations. Based on current information, interest rates in the future are unknown: thus, a model will be used to characterize this uncertainty. Hence, some authors consider probabilistic descriptions to characterize their evolution in the future.

In finance, continuous time models have become of vital importance, particularly continuous time diffusion processes frequently used to characterize the dynamics of major economic variables, such as exchange rates, asset valuation and interest rates. In recent years, research in this context has shown notable growth and development, starting with the work of Merton (1973), who proposed an interest rate model as a stochastic process to option pricing. Later, arbitrage arguments similar to the works of Black and Scholes (1973) were used to model the temporal structure of interest rates as observed in the works of Vasicek (1977) and Brennan and Schwartz (1979). Based on these ideas, models to further perfect the initial considerations were proposed, which include Cox et al. (1980), Cox et al. (1985), Chan et al. (1992), among others.

As for the modeling of interest rates, more and more literature has appeared in recent years. However, the selection of an appropriate model for a given data set remains a topic of discussion with no defined criterion. Clearly, the correct specification of a model describing the probabilistic behavior of short-term interest rates, and information on individual risk preferences, fully determine the associated temporal structure. An incorrect specification of the associated model would lead to an inadequate analysis of the data and to serious errors in the assessment of interest rates and risk. Hence, taking into account the various models proposed in the literature, one of the interests of researchers and finance professionals focused on searching for tools to determine which model among the wide range available in the literature is appropriate to characterize the empirical regularities of interest rates. For finance professionals parametric models are often interesting, since observations can be interpreted in terms of parameters. The question, then, is if it is appropriate to use a parametric model for a given set of financial data. Some work with question. See Aït-Sahalia (1996), Gao and King (2004), Hong and Li (2005), Chen et al. (2008), who propose tests to compare the parametric specification of a diffusion process using the marginal or transition density functions of the process; Corradi and White (1999) present a normal asymptotic test for the diffusion function; Dette and von Lieres und Wilkau (2003) propose a test for the parametric form of the volatility function based on the stochastic process of the integrated volatility; Li (2007) proposes a nonparametric test for the parametric specification of the diffusion function in a diffusion process based on the quadratic error between the estimated diffusion function, nonparametrically, and the diffusion proposed as null hypothesis; Arapis and Gao (2006), Gao and Casas (2008) propose a test to determine the parametric form of the drift and volatility based on smoothing techniques; Fan and Zhang (2003), Fan et al. (2003) propose simultaneous tests for the specification of the drift and diffusion function, based on the likelihood ratio test.

The proposals mentioned above, in most cases, are usually based on nonparametric techniques, in terms of the estimation of the model or the proposed test. Indeed, this is an additional disadvantage, since the selection of the smoothing parameter can also affect the power of the test. The aim of this paper is to present a test to compare the goodness of fit for a parametric form of the drift function and the volatility function in interest rate models based on empirical processes, see Stute (1997) and Stute et al. (1998) for goodness of fit test in regression models, and Koul and Stute (1999) for goodness of fit test for time series. There are few papers testing the goodness of fit for diffusion models based on the empirical process. The amount of literature is scarce, of course with the exception of the works of Lee and Wee (2008), who propose a test based on the empirical process of the residues of the diffusion model, and Negri and Nishiyama (2009), who proposed a goodness of fit test for ergodic diffusion models.

The proposal presented in this paper suggests using a goodness of fit test of easy and efficient implementation. As mentioned earlier, because the literature on this topic is scarce, an alternative source to address the problem is using the tests proposed for regression models. The idea is to rewrite the diffusion process as if a regression model were being used. Thus, the goodness of fit test for the drift function is based on the ideas presented in Stute (1997), and in this case considered the integrated regression function of the process variations that characterize the interest rates, thereby obtaining an empirical process based on the residues of the regression model considered. Then, the test for the volatility function is constructed from the integrated conditional variance function which gives a new empirical process which will be compared with the hypothesis of a parametric volatility function. The distributions of both statistics are approximated using bootstrap techniques, as observed in the regression models presented in Stute et al. (1998). Note that for test implementation only the estimation of the parameters of the process (using consistent "*root-n*" estimators) and the application of a bootstrap procedure is required, which will be described in detail later, for the calibration of the distribution of the test statistic, which is relatively simple.

The article is structured as follows. Section 2 presents the diffusion model on which the study and particular aspects of the same will be based. In Section 3 the goodness of fit test and its construction for both the drift function and the volatility function will be presented. In Section 4 a simulation study will be implemented for the tests proposed, to verify the level and power of the tests. Finally, Section 5 will present an application to the series of EURIBOR interest rates.

2 Diffusion Models for Interest Rates

In finance a model frequently used to characterize the dynamics of interest rates is the diffusion model in continuous time or Itô process given by the stochastic differential equation,

$$dr_t = \mu(r_t)dt + \sigma(r_t)dW_t, \tag{1}$$

where W_t is a standard Brownian motion, and $\mu(r_t)$ and $\sigma(r_t)$ depend on interest rate alone. In Equation (1), $\mu(r_t)$ is denoted as *drift* function, and $\sigma(r_t)$ is called *diffusion* or volatility function satisfying the following expressions:

$$\mu(r_t) = \lim_{\Delta\to 0} \frac{1}{\Delta}\mathbf{E}\left(r_{t+\Delta} - r_t \mid r_t\right) \quad \text{and}$$

$$\sigma^2(r_t) = \lim_{\Delta\to 0} \frac{1}{\Delta}\mathbf{E}\left((r_{t+\Delta} - r_t)^2 \mid r_t\right).$$

To study model (1) and observe its empirical behavior, basically there are two approaches. In the first approach, it is assumed that the behavior of the functions is known and determined by a parameter, θ, in a space of parameters, $\Theta \subset \mathbf{R}^d$ with d being a positive integer number, estimated by parametric techniques, for example, maximum likelihood method, generalized moments method, or estimation by approximation of the likelihood function, see e.g. Aït-Sahalia (1999), among others. Model (1) is then expressed parametrically as,

$$dr_t = \mu(r_t,\theta)dt + \sigma(r_t,\theta)dW_t. \tag{2}$$

In the second approach, no specific behavior is assumed for functions $\mu(r_t)$ and $\sigma(r_t)$, and non-parametric techniques are used for its estimation, see, for example, Stanton (1997), Arapis and Gao (2006), Gao and Casas (2008), among others. The different parametric representations of (2) have generated several types of interest rate models in the last 30 years, among which note:

$$\begin{aligned}
\text{Merton} &: \quad dr_t = \alpha dt + \sigma dW_t,\\
\text{VAS} &: \quad dr_t = (\alpha + \beta r_t)dt + \sigma dW_t,\\
\text{CIR} &: \quad dr_t = (\alpha + \beta r_t)dt + \sigma\sqrt{r_t}dW_t,\\
\text{CKLS} &: \quad dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{\gamma} dW_t,
\end{aligned}$$

proposed respectively by Merton (1973); Vasicek (1977); Cox et al. (1985) and Chan et al. (1992).

An alternative version in discrete time of the model (2) is given as:

$$\begin{aligned}
r_{t_{i+1}} - r_{t_i} \;=\;& \mu(r_{t_i},\theta)\Delta + \\
& + \sigma(r_{t_i},\theta)\Delta^{1/2}\varepsilon_{t_{i+1}}, i = 1,\ldots,n,
\end{aligned} \tag{3}$$

where $\{\varepsilon_{t_i}\}$ are independent and standard normal random variables, and $t_i = i\Delta$, with $i = 1,\ldots,n$ in a time interval $[0,T]$, for fixed Δ, that is, the observations are equidistant. For example, when the unit of time is one year, the sample is selected weekly in $t_i = i/52$, that is, $\Delta = 1/52$ for $i = 1, \ldots, n$. The discrete version (3) of model (1) is useless when simulating trajectories of the diffusion model, hence, this approximation in discrete time allows the estimation of the model parameters, as will be seen in the next section.

2.1 Estimation of the Model

There is extensive literature on the estimation of drift and volatility functions of model (2), for a detailed review of the different methods, see, for example, Prakasa-Rao (1999) and Iacus (2008). In practice, all continuous time models are observed in discrete time. Therefore, for the estimation of the parameters of the continuous time model (2), the discrete model is considered (3), that is, it is assumed that the process is observed in discrete time. It is denoted as $\mathcal{F}_n = \sigma\{r_{t_i}, i \leq n\}$ the sigma-algebra generated by the n first observations with $\mathcal{F}_0$ the trivial sigma-algebra. Thus, the likelihood function of the discrete process is given by

$$L_n(\theta) = \prod_{i=1}^{n} p_\theta(\Delta, r_{t_{i+1}} | r_{t_i}) p_\theta(r_{t_0}) \tag{4}$$

where $p_\theta(\Delta, r_{t_i} | r_{t_i - 1})$ denotes the transition density or conditional density function. $L_n(\theta)$ can be obtained by using the Markovian property of the r_t process (see, for example, Arnold (1974)). Let

$$\begin{aligned} \ell_n(\theta) &= \log L_n(\theta) = \sum_{i=1}^{n} \ell_i(\theta) + \log(p_\theta(r_{t_0})) \\ &= \sum_{i=1}^{n} \log p_\theta(\Delta, r_{t_{i+1}} | r_{t_i}) + \log(p_\theta(r_{t_0})), \end{aligned} \tag{5}$$

be the logarithm of the likelihood function. Usually $p_\theta(r_{t_0})$ is unknown, or under certain assumptions it is not always easy to determine. If the number of observations is increase with time, it can be assumed that the relative weight of $p_\theta(r_{t_0})$ in the likelihood function $L_n(\theta)$ decreases, so it can be assumed that $p_\theta(r_{t_0}) = 1$.

The maximum likelihood estimator (MLE) is expressed as $\hat{\theta} = \arg\max_\theta \ell_n(\theta)$. If the parametric model generating observations $\{r_{t_i}\}_i$ is known, then the method to be applied naturally is that of maximum likelihood. However, with the exception of some cases, like for example the diffusion models of Vasicek (VAS)-Orsntein Uhlenbeck or CIR, explicit forms are not available for the transition density function, as is the case of the CKLS model, among others.

A technique commonly used to estimate model (1) is to proceed as if the observations come from the Gaussian distribution, with mean the drift of the model and the standard deviation of the volatility function, and then obtain the maximum likelihood estimator. Note that the method is efficient if Δ, the discretization step, is sufficiently small (see Fan and Zhang (2003)), since significant biases can be produced in the estimations when Δ is large. To reduce the estimation bias Aït-Sahalia (1999) suggests an approximation of the transition density function using Hermite polynomials. By applying this approximation the author obtains a maximum likelihood estimator, see Aït-Sahalia (2002), "*root-n*" consistent for a diffusion process.

In this work, it is important to obtain consistent estimations of the parameters, hence, in what follows the maximum likelihood method suggested by Aït-Sahalia (2002) will be applied, given that its implementation and the computational cost are quite reasonable, in accordance with the requirements needed for the development of the proposal that will be presented in the following section.

3 Goodness of Fit Test for Interest Rate Models

In this section two goodness of fit tests are presented. The first is used to test the null hypothesis for the parametric form of the drift function. So, a test based on the integrated regression function of the process formed by interest rate variations is proposed, as a result obtaining an empirical process determined by the residuals of the regression model in question. Note that the volatility function in this case is then determined by the model for which the drift function is being compared. The second test aims to test the null hypothesis relative to the fact that the volatility function belongs to a parametric family. In this case, a test based on the integrated volatility function is proposed, from which an empirical process determined by the residuals is obtained. Similarly, for this test the drift function is then determined by the model for which its volatility function is being compared. The distribution of the test, in each case, will be approximated using bootstrap techniques.

3.1 Test for the drift function

For regression models, an alternative procedure, to the smoothing methods, for the construction of a goodness of fit test for the regression function is that based on the *integrated regression function*, see Stute (1997). Consider (X, Y) the random vector with F being the marginal distribution function of X and the associated regression function

$$m(x) = \mathbf{E}\left(Y|X = x\right).$$

The integrated regression function is defined as:

$$I(x) = \mathbf{E}\left(Y|\mathbf{1}_{\{X\leq x\}}\right) = \int_{-\infty}^{x} m(y)dF(y). \tag{6}$$

Function (6) can be estimated empirically using

$$I_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{X_i\leq x\}} Y_i. \tag{7}$$

The idea is to compare the estimator of the integrated regression function, $I_n(x)$, with an estimator that is based on the suppositions under the null hypothesis. So, considering the null hypothesis

$$H_0 : m = m_0$$

and $I_0(x)$ an appropriate estimator of (6) under the null hypothesis, for example

$$I_0(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq x\}} m_0(X_i),$$

then the test is defined using the empirical process determined by the residuals of the regression

$$\begin{aligned} R_n(x) &= \sqrt{n}\,(I_n(x) - I_0(x)) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq x\}}(Y_i - m_0(X_i)). \end{aligned} \tag{8}$$

For a composite null hypothesis, $H_0 : m \in \{m_\theta : \theta \in \Theta\}$, the goodness of fit test is defined from the process:

$$R_n(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq x\}}(Y_i - m_{\hat{\theta}}(X_i)).$$

with $\hat{\theta}$ an appropriate estimator of the true parameter θ. In each case, to contrast the hypothesis H_0 a functional (for example, the Kolmogorov-Smirnov statistic based on R_n) is applied to the empirical process R_n.

The purpose of this paper is to propose a general method based on the empirical process determined by the residuals, for diffusion models, to test the goodness of fit of the parametric form of the drift function of interest rate model (2). In other words, the hypothesis being studied is

$$H_0 : \mu \in \{\mu(\cdot, \theta) : \theta \in \Theta\}, \tag{9}$$

where $\mu(\cdot, \theta)$ represents the drift function of model (2) with $(\theta \in \Theta \subset \mathbf{R}^p)$. To contrast this hypothesis, the discretized version of model (2) is considered, that is, the model

$$\frac{Y_{t_i}}{\Delta} = \mu(r_{t_i}, \theta) + \sigma(r_{t_i}, \theta)\Delta^{-\frac{1}{2}}\varepsilon_{t_{i+1}},\ i = 1, \ldots, n \tag{10}$$

where $Y_{t_i} = r_{t_{i+1}} - r_{t_i}$, represents the variations or differences of process $\{r_{t_i}\}$, $\{\varepsilon_{t_i}\}$ are independent and normal standard random variables and independent of process $\{r_{t_i}\}$. Rewriting the before mentioned process as if dealing with a parametric time series model,

$$\frac{Y_{t_i}}{\Delta} = \mu(r_{t_i}, \theta) + \eta_{t_i}, \tag{11}$$

where $\eta_{t_i} = \sigma(r_{t_i}, \theta)\Delta^{-\frac{1}{2}}\varepsilon_{t_{i+1}}$, ideas relative to the goodness of fit test for regression models based on the integrated regression function can be applied.

For an appropriate estimator $\hat{\theta}$ of the true θ value, the statistic of the goodness of fit test, for diffusion models, would be based on the process

$$R_n^d(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i} \leq x\}}\left(\frac{Y_{t_i}}{\Delta} - \mu(r_{t_i}, \hat{\theta})\right),\ x \in \mathbf{R}. \tag{12}$$

Consider a continuous functional $\Psi(\cdot)$ to define the statistic $D_n = \Psi(R_n^d)$. Then, the null hypothesis H_0 is rejected if $D_n > c_{1-\alpha}$ where $c_{1-\alpha}$ satisfies:

$$\mathbf{P}\{\Psi(R_n^d) > c_{1-\alpha}\} = \alpha,$$

that is, $c_{1-\alpha}$ is a critical value for an α level test. To determine the value of $c_{1-\alpha}$, the distribution of the process R_n^d must be known. An alternative is to approximate by bootstrap techniques the distribution of process R_n^d, see Stute et al. (1998). The critical value, $c_{1-\alpha}$, is approximated by $c^*_{1-\alpha}$ such that

$$\mathbf{P}^*\{\Psi(R_n^{d*}) > c^*_{1-\alpha}\} = \alpha,$$

where $\mathbf{P}^*$ denotes a probability measure generated by the bootstrap sample and

$$R_n^{d*}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}_{\{r^*_{t_i} \leq x\}} \left(\frac{Y^*_{t_i}}{\Delta} - \mu(r^*_{t_i}, \hat{\theta}^*) \right), \; x \in \mathbf{R}. \tag{13}$$

with $\hat{\theta}^*$ being an estimator calculated from the bootstrap sample, $\{\left(r^*_{t_i}, Y^*_{t_i}\right)\}$, which will be defined later. In practice, $c^*_{1-\alpha}$ is approximated by MonteCarlo

$$c^*_{1-\alpha} = D_n^{*\lceil B(1-\alpha) \rceil}$$

being the $\lceil B(1-\alpha) \rceil$-th order statistic from B bootstrap replicates

$$D_n^{*j} = \Psi(R_n^{drift*}), \quad 1 \leq j \leq B.$$

Hence, the test is defined by the statistic $D_n = \Psi(R_n^d)$ such that H_0 is rejected if $D_n > c^*_{1-\alpha}$. As $\Psi(\cdot)$ functional, the criteria of Kolmogorov-Smirnov (*KS*) and Cramér-von Mises (*CvM*) are considered; that is,

$$D_n^{KS} = \sup_x |R_n^d(x)|, \tag{14}$$

and

$$D_n^{CvM} = \int_{\mathbf{R}} \left(R_n^d(x) \right)^2 F_n(dx) \tag{15}$$

where F_n is the empirical distribution function $\{r_{t_i}\}_i$. Depending on the case, $D_n = D_n^{KS}$ or $D_n = D_n^{CvM}$ are denoted to indicate if one or the other statistic is applied. Furthermore, the empirical *p-value* is also estimated using expression

$$\frac{\sharp\{D_n^{*j} > D_n\}}{B},$$

meaning, the proportion of $D_n^*(\cdot)$ bootstrap replicas exceeding D_n.

So, the bootstrap procedure to carry out the approximation of the critical value is as follows:

1. For each $i = 1, 2, \ldots, n$ are generated

$$Y_{t_i}^* = \mu(r_{t_i}, \hat{\theta})\Delta + \sigma(r_{t_i}, \hat{\theta})\Delta^{1/2}\varepsilon_{t_i}^*, \tag{16}$$

   the bootstrap sample $\{\left(r_{t_i}^*, Y_{t_i}^*\right)\}_{i=1}^n$, where $\hat{\theta}$ is an appropriate estimator of the process, the variable $r_{t_i}^* = r_{t_i}$ remains unaltered (fixed design), and with $\{\varepsilon_{t_i}^*\}$, random independent variables with normal standard distribution, $N(0,1)$. Variables $\{\varepsilon_{t_i}^*\}$ are independent of $\{r_{t_i}\}$.

2. $\hat{\theta}^*$ is estimated, using an appropriate estimator, from the bootstrap resample $\left\{\left(r_{t_i}, Y_{t_i}^*\right)\right\}_{i=1}^n$, obtained in step 1. ,

3.
$$R_n^{d*}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbf{1}_{\{r_{t_i} \leq x\}} \left(\frac{Y_{t_i}^*}{\Delta} - \mu(r_{t_i}, \hat{\theta}^*)\right).$$

   is determined.

4. $D_n^* = \Psi(R_n^{*d}(x))$ is calculated.

5. The previous steps are repeated B times, obtaining, $j = 1, 2, \ldots, B$ replicas of D_n^{*j}.

6. Finally, $\hat{c}_{1-\alpha}^* = D_n^{*\lceil B(1-\alpha)\rceil}$ is calculated.

Remark 1. As mentioned in Section 2.1, for the procedure described before, the estimators $\hat{\theta}$ and $\hat{\theta}^*$, considered adequate, meet the criteria of the consistent *"root-n"* type. A reasonable alternative is to apply the maximum likelihood estimator proposed by Aït-Sahalia (2002). Any other estimator meeting these criteria can be considered for the estimation of the parameters. Of course, the selection of an estimator with the characteristics before mentioned is dependent on the efficiency of said estimator in terms of computational time.

3.2 Test for the volatility function

In this case, and in the same way as with the test for the drift function, the objective is to construct a goodness of fit test to compare the null hypothesis of the parametric form of the volatility function,

$$H_0: \quad \sigma \in \{\sigma(\cdot, \theta) : \theta \in \Theta\}. \tag{17}$$

To define the test, consider the *integrated conditional variance function*

$$\begin{aligned} V_o(x) &= \int_{-\infty}^{x} \sigma^2(u) dF(u) = \mathbf{E}\left(\sigma^2(r_t, \theta)\mathbf{1}_{\{r_t \leq x\}}\right) \\ &= \mathbf{E}\left[\left(\frac{Y_t}{\Delta} - \mu(r_t, \theta)\right)^2 \mathbf{1}_{\{r_t \leq x\}}\right]. \end{aligned} \tag{18}$$

where F is a stationary distribution function of the process $\{r_t\}$. An empirical estimator of $V_o(x)$ is obtained by expression:

$$V_{o_n}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i}\le x\}} \left(\frac{Y_{t_i}}{\Delta} - \mu(r_{t_i}, \theta)\right)^2. \tag{19}$$

Then, for $\hat{\theta}$ an adequate estimator of the true parameter θ, the process used to base the goodness of fit test will be defined, for a diffusion model, as

$$R_n^v(x) \quad = \quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i}\le x\}} \left(\left(\frac{Y_{t_i}}{\Delta} - m(r_{t_i}, \hat{\theta})\right)^2 - \frac{\sigma^2(r_{t_i}, \hat{\theta})}{\Delta}\right), \quad x \in \mathbf{R} \tag{20}$$

As with the drift function test, a continuous functional $\Psi(\cdot)$ is considered to define the statistic $V_n = \Psi(R_n^v)$, such that H_0 is rejected if $c_{1-\alpha}$ is a critical value that satisfies

$$\mathbf{P}\left(\Psi(R_n^v) > c_{1-\alpha}\right) = \alpha$$

To approximate the distribution of the statistic, bootstrap techniques similar to those applied in the previous section will be used. Hence, $c_{1-\alpha}$ is approximated by $c^*_{1-\alpha}$ such that

$$\mathbf{P}^*\left(\Psi(R_n^{v*}) > c^*_{1-\alpha}\right) = \alpha$$

where, $\mathbf{P}^*$ denotes the probability measure generated by the bootstrap sample and

$$R_n^{v*}(x) \quad = \quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{1}_{\{r^*_{t_i}\le x\}} \left(\left(\frac{Y^*_{t_i}}{\Delta} - m(r^*_{t_i}, \hat{\theta}^*)\right)^2 - \frac{\sigma^2(r^*_{t_i}, \hat{\theta}^*)}{\Delta}\right) \quad x \in \mathbf{R}, \tag{21}$$

with $\hat{\theta}^*$ and estimator of the parameter θ calculated from the bootstrap sample $\{(r^*_{t_i}), Y^*_{t_i}\}$ generated in the same way as in the case of the drift function, using (16). The value $c^*_{1-\alpha}$ is approximated using MonteCarlo

$$c^*_{1-\alpha} = V_n^{*\lceil B(1-\alpha)\rceil}$$

the $\lceil B(1-\alpha)\rceil$-th order statistic from B bootstrap replicates

$$V_n^{*j} = \Psi(R_n^{v*}), \quad 1 \le j \le B.$$

Therefore, the test is defined by the statistic $V_n = \Psi(R_n^v)$ with H_0 rejection region if $V_n > c^*_{1-\alpha}$. As with the test for the drift function, as $\Psi(\cdot)$ functionals are considered the criteria of Kolmogorov-Smirnov (KS) and Cramér-von Mises (CvM), respectively denoted as,

$$V_n^{KS} = \sup_x \left|R_n^v(x)\right|, \tag{22}$$

and

$$V_n^{CvM} = \int_{\mathbf{R}} \left(R_n^v(x)\right)^2 F_n(dx) \tag{23}$$

where F_n is the empirical distribution function of $\{r_{t_i}\}_i$. Similarly, the *p-value* is estimated from the expression

$$\frac{\sharp\{V_n^{*j} > V_n\}}{B}$$

meaning, the proportion of V_n^* bootstrap replicas exceeding V_n.

4 Simulation study

For the performance of the goodness of fit test, in this section we present a simulation study to determine the level and power of the test proposed. Hence, simulations are considered to evaluate the test, to compare the parametric form of the drift function and the parametric form of the volatility function. In this section, two CKLS diffusion models (mean-reverting process and constant volatility elasticity parameter) and their respective alternatives are considered. For the first model, the CKLS model is considered as an alternative hypothesis with certain perturbations generated by a non-linear function $\rho_{(\cdot)}$ in its drift, in the case of drift function test, and in its volatility, for the volatility function test. For the second model, let us suppose that the alternative hypothesis is a jump diffusion model.

Model 1.

In the first case, one CKLS model is considered as a null hypothesis. In particular we will work with the model

$$dr_t = (0.0408 - 0.5921 r_t)dt + \sqrt{1.6704} r_t^{1.4999} dW_t, \tag{24}$$

the values of the parameters of the model (24) were worked in Chan et al. (1992) based on one-month treasury bill yields of the United States, and later used in Fan et al. (2003). The power of the goodness of fit test for the drift function will be evaluated in a series of alternative models indexed by the parameter, $0 \leq \rho_d \leq 0.1$, of a non-linear function $\rho_d(r_t) = \rho_d(1 - r_t^{\rho_d})$, (see Figure 1(a))

$$dr_t = \left(0.0408 - 0.5921 r_t + \rho_d(1 - r_t^{\rho_d})\right) dt + \sqrt{1.6704} r_t^{1.4999} dW_t. \tag{25}$$

Clearly, under the null hypothesis, $\rho_d = 0$. The power of the goodness of fit test for the parametric form of volatility will be evaluated in a series of alternative models indexed by the parameter, $0 \leq \rho_v \leq 0.02$, of a non-linear function $\rho_v(r_t) = \rho_v(1 - r_t^{\rho_v})$, (see Figure 1(b))

$$dr_t = (0.0408 - 0.5921 r_t)dt + \left(\sqrt{1.6704} r_t^{1.4999} + \rho_v(1 - r_t^{\rho_v})\right) dW_t. \tag{26}$$

Again, under the null hypothesis, $\rho_v = 0$.

Series of data are generated weekly (i.e. $\Delta = 1/52$) of model (24) with the number of observations, $n = 100, 500, 1000$. So, for each value

$$\rho_d \in \{0.01, 0.05, 0.07, 0.09, 0.1\},$$

series of data are generated weekly of the model (25) with $n = 100, 500, 1000$, respectively. Finally, for

$$\rho_v \in \{0.001, 0.005, 0.009, 0.01, 0.02\},$$

series of data are generated weekly ($\Delta = 1/52$) of the model (26) with $n = 300, 500$, respectively.

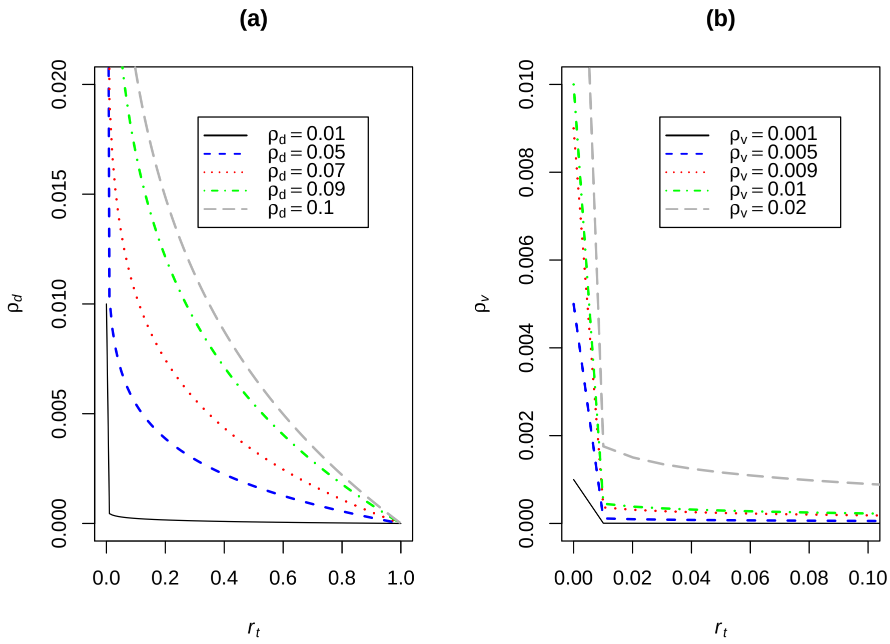

Figure 1: Behavior of the ρ functions. (a) Perturbation function of drift, ρ_d,(b) Perturbation function of volatility, ρ_v.

Based on 1000 simulations, and for $B = 1000$ bootstrap trials, the percentage of rejections is calculated using the respective statistics (14) and (15), (22) and (23) for different levels of significance, $\alpha = 0.01, 0.05, 0.10$.

As observed in table (1), for $\rho_d = \rho_v = 0$, models (25) and (26), are actually model CKLS (24), therefore, the power value under the null hypothesis should be approximately the nominal value of significance level $1\%, 5\%$, and 10%. As observed in tables (1) and (2), for $\rho_d = \rho_v = 0$, the estimated values approach the nominal level of significance. As the values of ρ_d and ρ_v are increased, in their respective models (25) and (26), the alternative hypotheses deviate from the null hypothesis, so the rejection rates are expected to increase. In fact, simulations (see Tables (1) and (2)) confirm what was stated before and show that the test presents reasonably good power. For example, in the case of the power for the goodness of fit test of the drift function, for $\rho = 0.07$, the test is able to reject the null hypothesis in approximately 70% of the time, for a significance level of 10%. In the case of the power for the goodness of fit test of the volatility function, for $\rho_v = 0.009$, the test is able to reject the null hypothesis nearly 70% of the time, for a significance level of 10%. This suggests that the test has admissible discrimination capacity for the differentiation of model (24) with respect to models (25) and (26). An important aspect in our simulation study is related with the sample size. Evidently, in the MonteCarlo simulation study the increase in sample size leads to an improvement in the estimations. For example, in the case of a test, it is the power. However, in the study presented before this finding is not totally reflected. In Table

Table 1: Rates of rejections, $\hat{\alpha}_{KS}$ and $\hat{\alpha}_{CvM}$, estimated from the model (25), for different values of ρ_d and several sample sizes, n, when the test statistic, D_n, is applied for the drift function; $\hat{p}_{KS}$ and $\hat{p}_{CvM}$ correspond to the estimated p-values. The index (*KS*) and (*CvM*) denote the criteria of Kolmogorov-Smirnov and Cramer von Misses, respectively.

				$\hat{\alpha}_{KS}$			$\hat{\alpha}_{CvM}$		
ρ_d	n	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	10%	5%	1%	10%	5%	1%
0	100	0.477	0.488	0.081	0.041	0.004	0.080	0.041	0.002
	500	0.499	0.491	0.110	0.051	0.009	0.096	0.050	0.009
	1000	0.507	0.512	0.107	0.048	0.011	0.110	0.048	0.010
0.05	100	0.271	0.258	0.327	0.199	0.039	0.373	0.250	0.077
	500	0.263	0.262	0.347	0.210	0.053	0.369	0.246	0.072
	1000	0.268	0.277	0.319	0.201	0.054	0.328	0.203	0.051
0.07	100	0.094	0.079	0.707	0.525	0.180	0.761	0.607	0.282
	500	0.093	0.076	0.731	0.567	0.219	0.787	0.643	0.303
	1000	0.106	0.092	0.684	0.500	0.189	0.735	0.576	0.258
0.1	100	0.068	0.053	0.797	0.613	0.208	0.847	0.699	0.303
	500	0.058	0.042	0.835	0.698	0.343	0.880	0.762	0.462
	1000	0.053	0.042	0.846	0.703	0.355	0.876	0.775	0.460

(1) there are cases where an increase in sample size, for example $n = 1000$ for $\rho_d = 0.07$, does not produce an increase in power, $\hat{\alpha}_{KS} = 0.684$ at a significance level of 10%, if compared with the corresponding sample size $n = 500$ which produces an estimate $\hat{\alpha}_{KS} = 0.731$. One explanation for this phenomenon is the effect produced by an increase in sample size in the observation window. The sampling considered in this work is of fixed design, that is, the observation window is $[0, n\Delta = T]$, with $\Delta = constant$, that increases when n (sample size) increases. In other words, when the sample size is increased, more observations are not being included in the same period, but rather, the observation horizon is becoming greater, which implies at first that we are not observing the same process, but rather a process with a subset of new observations. Hence, the series could be affected by this, as well as the estimations of the parameters and the power, since it is not actually an increase in sample size, especially when dealing with mean reverting processes.

Model 2.

The jump diffusion models were introduced by Das (2002) and Johanes (2004) to characterize the dynamics of interest rates. Their use is founded mainly on the fact that economic shocks, government interventions in the market, news dissemination, among others, cause big jumps in interest rates, thereby generating discontinuity in their dynamics. Consider the jump diffusion

Table 2: Rate of rejections, $\hat{\alpha}_{KS}$ and $\hat{\alpha}_{CvM}$, estimated from model (25), for different values of ρ_v and several sample sizes, n, when the test statistic , V_n is applied to the volatility function; $\hat{p}_{KS}$ and $\hat{p}_{CvM}$ correspond to the estimated p-values. The index (*KS*) and (*CvM*) denote the criteria of Kolmogorov-Smirnov and Cramer von Misses, respectively.

				$\hat{\alpha}_{KS}$			$\hat{\alpha}_{CvM}$		
ρ_v	n	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	10%	5%	1%	10%	5%	1%
0	100	0.502	0.503	0.079	0.020	0.002	0.079	0.051	0.008
	500	0.511	0.501	0.099	0.051	0.012	0.099	0.051	0.011
	1000	0.498	0.499	0.108	0.048	0.099	0.108	0.049	0.087
0.005	100	0.431	0.429	0.151	0.078	0.022	0.144	0.077	0.017
	500	0.397	0.398	0.233	0.163	0.056	0.231	0.160	0.066
	1000	0.361	0.363	0.289	0.204	0.106	0.291	0.214	0.101
0.009	100	0.318	0.319	0.271	0.171	0.048	0.279	0.174	0.053
	500	0.171	0.166	0.616	0.521	0.352	0.635	0.538	0.354
	1000	0.115	0.108	0.736	0.667	0.520	0.764	0.685	0.515
0.02	100	0.230	0.175	0.430	0.305	0.101	0.569	0.426	0.207
	500	0.030	0.018	0.911	0.868	0.757	0.946	0.914	0.832
	1000	0.003	0.001	0.992	0.983	0.956	0.998	0.996	0.977

model CKLS, based on the specification proposed by Das (2002), that is,

$$dr_t = (\alpha + \beta r_t)\, dt + \sigma r_t^{\gamma} dW_t + J(\lambda, \rho^2) d\pi_t(q_t) \tag{27}$$

So, interest rates evolve with a drift function of average return and two random terms, the volatility function (W_t standard Brownian movement) or diffusion and a Poisson process π_t with frequency of events q_t, including a random jump $J(\lambda, \rho^2)$ with distribution $N(\lambda, \rho^2)$. Let model specification (discretized) considered by Das (2002) be,

$$Y_{t_i} \quad = \quad \alpha + \beta r_{t_i-1} + \sigma r_{t_i-1}^{\gamma} \varepsilon_{t_i} + J(\lambda, \rho^2) \Delta\pi_{t_i}(q_{t_i}),\ i = 1, \ldots, n, \tag{28}$$

where $Y_{t_i} = r_{t_i} - r_{t_i-1}$ and $\{\varepsilon_{t_i}\}$ are random independent variables with distribution $N(0, 1)$ and independent of $\{r_{t_i}\}$; and $\Delta\pi_{t_i}(q_{t_i})$ is the Poisson process of discrete time increments, approximated by an independent random variable with Bernoulli distribution of parameter

$$q_{t_i} = \frac{1}{1 + exp(-a - br_{t_i-1})} \tag{29}$$

and with jump size the value of the random variable $J(\lambda, \rho^2)$ with distribution $N(\lambda, \rho^2)$. The conditioned mean for the variation in interest rate is:

$$\mathbf{E}\left(r_{t_i} - r_{t_i-1} | r_{t_i-1}\right) = \alpha + \beta r_{t_i-1} + q_{t_i}\lambda \tag{30}$$

Table 3: Rate of rejections, $\hat{\alpha}_{KS}$ and $\hat{\alpha}_{CvM}$, estimated from model (33), for different values of ρ_d and several sample sizes, n, when the test statistic D_n is applied to the drift function ; $\hat{p}_{KS}$ and $\hat{p}_{CvM}$ correspond to the p-estimated values. The index (*KS*) and (*CvM*) denote the criteria of Kolmogorov-Smirnov and Cramer von Misses, respectively.

					$\hat{\alpha}_{KS}$		$\hat{\alpha}_{CvM}$	
λ	ρ^2	n	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	10%	5%	10%	5%
0	0	300	0.470	0.478	0.080	0.042	0.070	0.041
		400	0.489	0.487	0.091	0.047	0.079	0.042
		500	0.509	0.502	0.109	0.052	0.090	0.048
0.0009	0.001	300	0.387	0.433	0.164	0.092	0.140	0.074
		400	0.384	0.434	0.206	0.110	0.148	0.078
		500	0.256	0.285	0.368	0.252	0.350	0.252
0.0079	0.001	300	0.200	0.209	0.488	0.358	0.434	0.292
		400	0.181	0.194	0.502	0.350	0.456	0.300
		500	0.145	0.156	0.578	0.404	0.516	0.376
0.019	0.001	300	0.177	0.163	0.516	0.374	0.518	0.386
		400	0.137	0.124	0.596	0.418	0.600	0.446
		500	0.108	0.092	0.676	0.508	0.624	0.504

while the variance is

$$\mathbf{E}\left[(r_{t_i} - r_{t_i-1})^2 | r_{t_i-1}\right] = \sigma^2 r_{t_i-1}^2 + q_{t_i}\left(\rho^2 + (1-q_{t_i})\lambda^2\right) \tag{31}$$

For the simulation study consider again the CKLS model as null hypothesis,

$$dr_t = (0.00739344 - 0.0876 r_t)dt + 0.7791 r_t^{1.48} dW_t, \tag{32}$$

whose parameters are taken from Aït-Sahalia (1999), based on the monthly sampling Fed funds rate, from January 1963 to December 1998. To study the power of the test, $N = 500$ are generated, for $n = 300, 400, 500$, observations of a series of alternatives indexed by $\lambda \in \{0; 0.0009; 0.0079; 0.019\}$, $\rho \in \{0, 0.001\}$, and $a = -0.5$ and $b = -0.5$, of the jumps diffusion model (27),

$$dr_t = (0.00739344 - 0.0876 r_t)\, dt + 0.7791 r_t^{1.48} dW_t + J(\lambda, \rho^2) d\pi_t(q_t) \tag{33}$$

For $B = 1000$ bootstrap replicas the following results are obtained,

It is clear that under the null hypothesis, $\lambda = \rho = 0$, the value of the power should be approximately the nominal value of the significance level 5%, and 10%. As observed in the tables (3),(4), the estimated values approach the nominal level of significance, and sample size influences the estimated values.

Table 4: Rate of rejections, $\hat{\alpha}_{KS}$ and $\hat{\alpha}_{CvM}$, estimated from model (33), for different values of ρ_d and several sample sizes, n, when the test statistic V_n is applied to the volatility function; $\hat{p}_{KS}$ and $\hat{p}_{CvM}$ correspond to the p-estimated values. The index (*KS*) and (*CvM*) denote the criteria of Kolmogorov-Smirnov and Cramer von Misses, respectively.

					$\hat{\alpha}_{KS}$		$\hat{\alpha}_{CvM}$	
λ	ρ^2	n	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	10%	5%	10%	5%
0	0	300	0.470	0.478	0.080	0.042	0.070	0.041
		400	0.489	0.487	0.091	0.047	0.079	0.042
		500	0.509	0.502	0.109	0.052	0.09	0.048
0.0009	0.001	300	0.233	0.237	0.458	0.346	0.468	0.338
		400	0.205	0.213	0.516	0.404	0.512	0.404
		500	0.009	0.008	0.978	0.966	0.984	0.970
0.0079	0.001	300	0.037	0.030	0.904	0.858	0.912	0.880
		400	0.015	0.013	0.960	0.934	0.972	0.948
		500	0.017	0.018	0.962	0.948	0.958	0.946
0.01	0.001	300	0.034	0.029	0.906	0.874	0.930	0.890
		400	0.020	0.020	0.952	0.922	0.950	0.926
		500	0.015	0.013	0.960	0.946	0.966	0.948

When the value of λ is increased, the alternative hypotheses deviate from the null hypothesis, so the rate of rejections is increased. This is observed in the simulations presented in Table (3), which confirms what was mentioned before, and shows that the test has reasonable power. Then, the influence of the parameter λ on the variance (31) of the jumps diffusion process (33) produces a set of alternative hypotheses for the volatility function, so, when the goodness of fit test is applied to the volatility function the results shown in Table (4) are obtained. As observed in the Table, there is an increase in power with respect to λ.

Remark 2. In Section (2.1) the distribution of the empirical process, $R_n^d(x)$, was approximated using bootstrap techniques. An alternative way of approximating the distribution of the empirical process, $R_n^d(x)$, is from a limit distribution function corresponding to certain zero-mean Gaussian process. In terms of goodness of fit tests for generalized mixed linear models, Pan and Lin (2005), suggest this type of approximation for methods based on cumulative sums of residuals over co-variates or predictions of the response variable. In this case, that is, in the context of diffusion models for interest rate, the approximation can be carried out as follows:

$$W_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i} \le x\}} \left(\frac{Y_{t_i}}{\Delta} - \mu(r_{t_i}, \hat{\theta}) \right), \; x \in \mathbf{R}. \tag{34}$$

Note that process $W_n(x) = R_n^d(x)$ of Section (2.1). Under the supposition that the first and second derivatives of $\mu(r_{t_i},\theta)$ are bounded, the process

$$\begin{aligned} W_n(x) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}_{\{r_{t_i}\le x\}}\left(\frac{Y_{t_i}}{\Delta}-\mu(r_{t_i},\hat\theta)+\mu(r_{t_i},\theta)-\mu(r_{t_i},\theta)\right) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}_{\{r_{t_i}\le x\}}\left[\left(\frac{Y_{t_i}}{\Delta}-\mu(r_{t_i},\theta)\right)+\left(\mu(r_{t_i},\theta)-\mu(r_{t_i},\hat\theta)\right)\right]. \end{aligned}$$

Considering the expansion in Taylor series of

$$\left(\mu(r_{t_i},\theta)-\mu(r_{t_i},\hat\theta)\right)$$

in the setting of $\hat\theta$, the process $W_n(x)$ is asymptotically equivalent to the process

$$W_n(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}_{\{r_{t_i}\le x\}}\left(\frac{Y_{t_i}}{\Delta}-\mu(r_{t_i},\theta)\right)+\eta'(x,\hat\theta)n^{1/2}(\hat\theta-\theta).$$

with

$$\eta(x;\theta) = -\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{r_{t_i}\le x\}}\frac{\partial\mu_i(r_{t_i},\theta)}{\partial\theta}$$

a function of the drift function gradient evaluated in each observation r_{t_i} and under the regularity conditions over the estimator of the parameter $\hat\theta$,

$$n^{1/2}(\hat\theta-\theta) = \Omega^{-1}n^{-1/2}U(r_{t_i},\theta)+o_p(1)$$

where

$$\ell_n(\theta) = \sum_{i=1}^{n}\log\left(p(r_{t_i}|r_{t_i-1};\theta)\right),$$

$$U(r_{t_i};\theta) = \frac{\partial\ell_n(\theta)}{\partial\theta} = \sum_{i=1}^{n}U_i(r_{t_i},\theta),$$

$$\Omega = \lim_{n\to\infty}\mathcal{I}(\theta), \quad \text{and} \quad \mathcal{I}(\theta) = -\frac{1}{n}\frac{\partial^2\ell_n(\theta)}{\partial\theta\partial\theta'}.$$

Finally, the process $W_n(x)$ is approximated by the process of cumulative sums $\hat W_n(x)$ whose first term corresponds to the estimation of the original process and the second term corresponds to the covariance structures generated from the estimation of the model considered,

$$\hat W_n(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}_{\{r_{t_i}\le x\}}\left\{\left(\frac{Y_{t_i}}{\Delta}-\mu(r_{t_i},\hat\theta)\right)+\eta'(x,\hat\theta)\mathcal{I}^{-1}(\hat\theta)U_i(r_{t_i},\hat\theta)\right\}G_i \tag{35}$$

where $G_1, G_2, \ldots, G_n$ are independent normal standard random variables. Same as with the tests previously presented in Section (2.1), the statistic is defined using

Table 5: Rate of rejections, $\hat{\alpha}_{KS}$ and $\hat{\alpha}_{CvM}$, estimated from model (25), for different values of ρ_d and several sample sizes, n, when the test statistic S_n is applied to the drift function ; $\hat{p}_{KS}$ and $\hat{p}_{CvM}$ correspond to the p-estimated values. The index (*KS*) and (*CvM*) denote the criteria of Kolmogorov-Smirnov and Cramer von Misses, respectively.

				$\hat{\alpha}_{KS}$			$\hat{\alpha}_{CvM}$		
ρ_d	n	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	10%	5%	1%	10%	5%	1%
0	100	0.434	0.468	0.146	0.056	0.005	0.089	0.031	0.007
	500	0.491	0.484	0.114	0.054	0.010	0.101	0.052	0.005
	1000	0.497	0.483	0.092	0.050	0.009	0.094	0.048	0.006
0.05	100	0.24	0.24	0.387	0.237	0.039	0.408	0.271	0.075
	500	0.243	0.247	0.398	0.227	0.074	0.40	0.285	0.09
	1000	0.255	0.267	0.348	0.218	0.059	0.355	0.216	0.06
0.07	100	0.138	0.125	0.615	0.442	0.123	0.651	0.499	0.183
	500	0.13	0.116	0.615	0.434	0.17	0.672	0.513	0.235
	1000	0.145	0.136	0.566	0.373	0.138	0.6	0.438	0.149
0.1	100	0.07	0.062	0.794	0.64	0.227	0.827	0.695	0.305
	500	0.077	0.057	0.781	0.609	0.295	0.843	0.727	0.414
	1000	0.085	0.07	0.75	0.583	0.231	0.785	0.661	0.325

$$S_n^{KS} = \sup_x |\hat{W}_n(x)|, \text{ or } S_n^{CvM} = \int_{\mathbf{R}} \left(\hat{W}_n(x)\right)^2 F_n(dx)$$

where F_n is the empirical distribution of $\{r_{t_i}\}_i$. Depending on the case, let us denote by $S_n = S_n^{KS}$ or $S_n = S_n^{CVM}$ to indicate if one or another statistic is applied. Then, H_0 is rejected if $S_n > s^*_{1-\alpha}$. The critical point $s^*_{1-\alpha}$ is estimated using MonteCarlo by

$$s^*_{1-\alpha} = S_n^{*\lceil B(1-\alpha)\rceil}.$$

The estimator of the *p-value* is given as:

$$\frac{\sharp\{S_n^{*j} > S_n\}}{B},$$

the proportion of bootstrap replicas S_n^* exceeding S_n. The advantage of this approximation is that it reduces the computational cost originated by the approximation process R_n^d, since for each iteration, a new estimation of the parameters is required for each bootstrap resampling. Evidently, this advantage is restricted by a good approximation of the terms $\eta'(x, \hat{\theta})$, $\mathcal{I}^{-1}(\hat{\theta})$, and $U_i(r_{t_i}, \hat{\theta})$. Table (5) presents the results of when this alternative procedure is applied to the simulation data from model (25). As observed, for $\rho_d = 0$, that is, model CKLS, the estimated values approach the test level. So, when the value of ρ_d is increased, the estimated values of the power are also increased, similar to the results obtained using the bootstrap approximation.

Table 6: Table comparing the running times between approximations $\hat{W}_n$ and R_n^{d*}, of the distribution of the process R_n^d; t_{iter}, running time for each approximation of D_n and S_n corresponding to a sample size n; t_{sim}, running time until obtaining $\hat{\alpha}_{KS}$.

	D_n				S_n			
n	$\hat{p}_{KS}$	$\hat{\alpha}_{KS}$	t_{iter}(sec)	t_{sim}(min)	$\hat{p}_{KS}$	$\hat{\alpha}_{KS}$	t_{iter}(sec)	t_{sim}(min)
100	0.477	0.081	10.78	180	0.434	0.146	0.136	2.26
500	0.499	0.110	34.92	582	0.491	0.114	0.362	6.04
1000	0.507	0.107	62.52	1042	0.497	0.092	0.621	10.35

Note that in table (5), the effect produced by increasing the sample size in the simulation study and estimating the power becomes more pronounced. It is evident that in this case the estimations of the parameters of the process as well as the approximation of the distribution of the process involved in the test are certainly more sensitive to any change in the simulation process. The increase in sample size, as mentioned before, is not considered like an increase in the number of observations in the observation horizon, but rather, this generates a different process in another observation horizon, which neither generates a direct profit in the estimation of the parameters nor of the test statistic.

Table (6) shows the running times for both approximations, D_n and S_n for the particular case, $\rho = 0$ (null hypothesis), and with significance level $\alpha = 0.10$. The differences between the running times are significant from one to another approximation, these becoming more pronounced when the size of the series is increased.

5 Applications to EURIBOR data

This Section presents the most important characteristics of an EURIBOR (Euro Interbank Offered Rate) interest rate series for different periods of time. The temporal period goes from October 15, 2001 to March 30, 2006, with daily frequency. The periods considered are $1, 2$ and 3 weeks and $1, 2, \ldots, 12$ months. The source for the data is the, Euribor Historical Data (EURIBOR-EBF). The figure (2) represents the daily level of the EURIBOR interest rates series for the different time periods considered. The most notable traits of the short-term series are: (a) it is persistent, spends long periods of time above and below the long-term value, (b) abrupt changes are observed in the level of the interest rates that are smoothed as the period of time becomes longer. The goodness of fit test suggested in the previous Sections will be applied to the EURIBOR interest rates series mentioned before, to test the goodness of fit of the models of Vasicek (1977) (VAS), Chan et al. (1992) (CKLS), in terms of the parametric form of the drift function and of the volatility function.

Table (7) shows the estimated values of the p-values, when the test statistic D_n and V_n are applied to EURIBOR series. The estimated *p-value* was significant for the long-term series, lasting

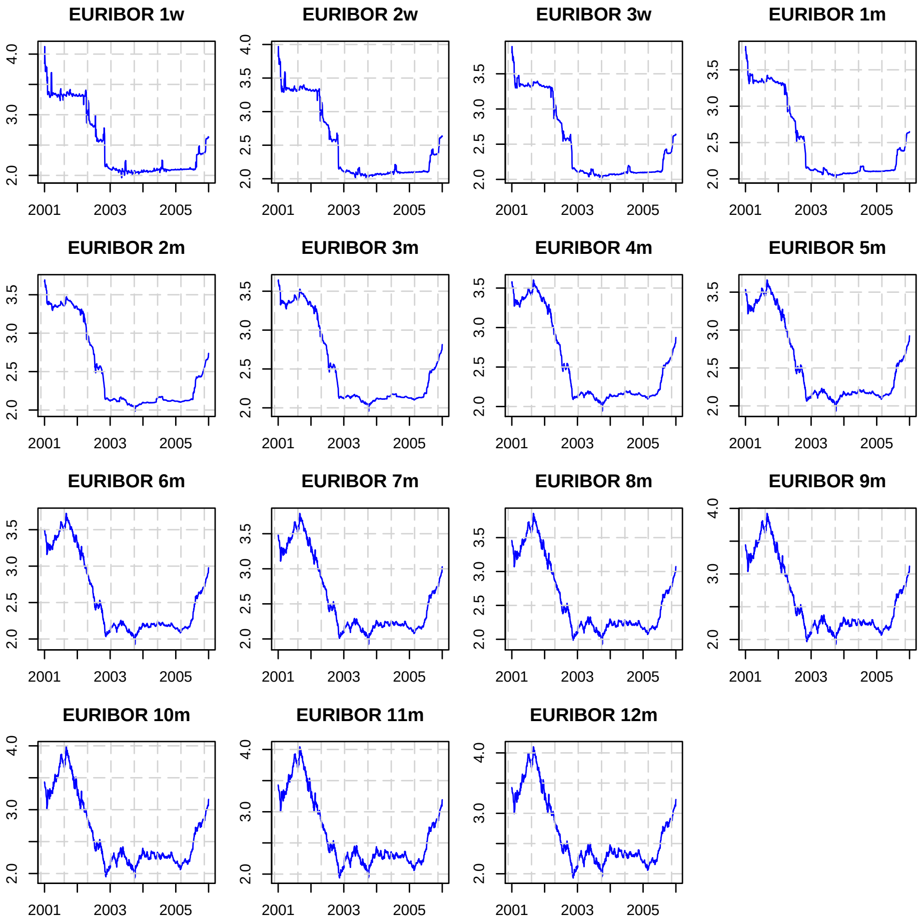

Figure 2: Euribor Series.

Table 7: P-values for testing the forms of the drift and volatility function for the Vasicek model (VAS) and the CKLS model. The index (*KS*) and (*CvM*) denote the criteria of Kolmogorov-Smirnov and Cramer von Misses, respectively.

Maturity	Drift function				Volatility function			
	Vasicek		CKLS		Vasicek		CKLS	
	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	$\hat{p}_{KS}$	$\hat{p}_{CvM}$	$\hat{p}_{KS}$	$\hat{p}_{CvM}$
1 week	0.033	0.011	0.053	0.138	0	0	0	0
2 weeks	0.212	0.029	0.405	0.239	0	0	0	0
3 weeks	0.109	0.030	0.251	0.168	0	0	0	0
1 month	0.105	0.018	0.254	0.122	0	0	0.002	0.0002
2 months	0.199	0.044	0.265	0.111	0	0	0	0
3 months	0.409	0.169	0.454	0.275	0	0	0	0
4 months	0.710	0.653	0.693	0.717	0	0	0	0.0002
5 months	0.999	0.977	0.999	0.975	0	0	0	0.0004
6 months	0.963	0.985	0.989	0.976	0	0	0	0
7 months	0.928	0.937	0.945	0.913	0	0	0.002	0
8 months	0.923	0.860	0.942	0.834	0	0	0.004	0
9 months	0.897	0.803	0.957	0.768	0	0	0	0
10 months	0.930	0.736	0.966	0.691	0	0	0.001	0.0004
11 months	0.885	0.666	0.909	0.618	0	0	0.001	0
12 months	0.804	0.598	0.918	0.558	0	0	0.001	0

Table 8: Comparative of the bootstrap approximations of the statistics D_n and S_n of the goodness of fit test for the drift function, (*KS*) corresponding to the criteria of Kolmogorov-Smirnov, critical values $\hat{c}^{*KS}_{0.10}$ (10% significance), *p-values* and t_{iter} running times by iteration, for the CKLS model.

	R_n^{d*}				$\hat{W}_n$			
Maturity	$\hat{p}_{KS}$	$c^{KS}_{0.10}$	D^{KS}_n	t_{iter}(min)	$\hat{p}_{KS}$	$c^{KS}_{0.10}$	S^{KS}_n	t_{iter}(min)
1 week	0.044	0.028	0.032	1.489	0.072	0.033	0.035	0.094
3 months	0.422	0.010	0.007	1.102	0.501	0.010	0.007	0.091
6 months	0.964	0.013	0.006	1.308	0.990	0.014	0.005	0.090
9 months	0.976	0.018	0.008	1.189	0.968	0.019	0.009	0.090
12 months	0.888	0.022	0.011	1.136	0.926	0.023	0.011	0.090

over three months. This suggests that the model can be adequate to explain the dynamics of the EURIBOR series, at least for the drift function.

However, the *p-value* in the case of the volatility function shows that the model is inadequate to explain the volatility of the series. In the case of series for periods less than three months the estimates of the *p-value*, for the goodness of fit test of the drift function, show relatively low values. Table (8) shows the results obtained when the S_n statistic presented in observation (1) was applied to an EURIBOR series and a CKLS model was considered to explain the dynamics of the interest rates. The Table also shows the results obtained when the T_n statistic, described in Section (2), was applied. As observed, the values obtained in each case present the minor discrepancies in terms of the values of the critical points, test statistic and *p-values*, so, the conclusions reached by one or another method do not imply, in most cases, differences. One major difference between the two methods is the running time, which is significant.

6 Conclusions

The results obtained clearly show that a goodness of fit test based on empirical processes is a tool that is able to discriminate among interest rate models. The test presented also shows satisfactory performance, reflected in its size or level of significance and its power. Then, the implementation of the test and the bootstrap resampling scheme is quite simple compared to the methods based on smoothing techniques. With respect to the results obtained when the test is applied to EURIBOR series, the test is able to discriminate between the models that are able to explain the dynamics of these series, for example, for the case of series with maturity above three months, the test determines that the CKLS models, with linear drift and with mean-reverse, tend to be adequate while for shorter periods of time the models are considered not fit to explain the behavior of the interest rate models. This last point is corroborated by the economic theory, which expresses that models with brief maturity periods tend to be more susceptible to the decisions of the regulatory organizations of the market.

References

Aït-Sahalia, Y. (1996). Testing continuous-time models of the spot interest rate. *The Review of Financial Studies*, 9(2):385–426.

Aït-Sahalia, Y. (1999). Transition densities for interest rate and other nonlinear diffusions. *Journal Finance*, 54(4):1361–1395.

Aït-Sahalia, Y. (2002). Maximum likelihood estimation of discretely sampled diffusions: A closed-form approximation approach. *Econometrica*, 70(1):223–262.

Arapis, M. and Gao, J. (2006). Empirical comparisons in short-term interest rate models using nonparametric methods. *Journal of Financial Econometrics*, 4(2):310–345.

Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications*. John Wiley and Sons, New York.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–54.

Brennan, M. and Schwartz, E. (1979). A continuous time approach to the pricing of bonds. *Journal of Banking and Finance*, 3(2):133–155.

Chan, K. C., Karolyi, G. A., Longstaff, F. A., and Sanders, A. B. (1992). An empirical comparison of alternative models of the short-term interest rate. *Journal of Finance*, 47(3):1209–27.

Chen, S. X., Gao, J., and Tang, C. Y. (2008). A test for model specification of diffussion processes. *Annals of Statistics*, 36(1):167–198.

Corradi, V. and White, H. (1999). Specification tests for the variance of a diffussion. *Journal of Time Series Analysis*, 20(3):253–270.

Cox, J., Ingersoll, J. E., and Ross, S. A. (1980). An analysis of variable rate loan contracts. *Journal of Finance*, 35(2):389–403.

Cox, J., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

Das, S. (2002). The surprise element: jumps in interest rates. *Journal of Econometrics*, 106(1):27–65.

Dette, H. and von Lieres und Wilkau, C. (2003). On a test for a parametric form of volatility in continuous time financial models. *Finance and Stochastics*, 7(3):363–384.

Fan, J., Jiang, J., Zhang, C., and Zhou, Z. (2003). Time-dependent diffussion models for term structure dynamics. *Statistica Sinica*, 13(4):965–992.

Fan, J. and Zhang, C. (2003). A reexamination of diffusion estimators with applications to financial model validation. *Journal of the American Statistical Association*, 98(461):118–134.

Gao, J. and Casas, I. (2008). Specification testing in discretized diffusion models: Theory and practice. *Journal of Econometrics*, 147(1):131–140.

Gao, J. and King, M. (2004). Adaptative testing in continuous-time diffusion models. *Econometric Theory*, 20(5):844–882.

Hong, Y. and Li, H. (2005). Nonparametric specification testing for continuous-time models with applications to term structure of interest rates. *Review of Financial Studies*, 18(1):37–84.

Iacus, S. M. (2008). *Simulation and Inference for Stochastics Differential Equations*. Springer.

Johanes, M. (2004). The statistical and economic role of jumps in continuous-time interest rate models. *Journal of Finance*, 59(1):227–260.

Koul, H. and Stute, W. (1999). Nonparametric model checks for time series. *Annals of Statistic*, 27(1):204–236.

Lee, S. and Wee, I.-S. (2008). Residual empirical process for diffusion processes. *Journal of the Korean Mathematical Society*, 45(3):683–693.

Li, F. (2007). Testing the parametric specification of the diffusion function in a diffusion process. *Econometric Theory*, 23:221–250.

Merton, R. C. (1973). The theory of rational option pricing. *Bell Journal of Economics*, 4:141–183.

Negri, I. and Nishiyama, Y. (2009). Goodness of fit test for ergodic diffusion processes. *Annals of the Institute of Statistical Mathematics*, 61(4):167–198.

Pan, Z. and Lin, D. Y. (2005). Goodness-of-fit methods for generalized linear mixed models. *Biometrics*, 61(4):1000–1009.

Prakasa-Rao, B. (1985). Estimation of the drift for diffusion process. *Statistic*, 16:263–275.

Prakasa-Rao, B. (1999). *Statistical inference for diffusion type processes*. Oxford University Press, New York.

Stanton, R. (1997). A nonparametric model of term structure dynamics and the market price of interest rate risk. *Journal of Finance*, 52(5):1973–2002.

Stute, W. (1997). Nonparametric model checks for regression. *Annals of Statistics*, 25(2):613–641.

Stute, W., González-Manteiga, W., and Quindimil, M. P. (1998a). Bootstrap approximations in model checks for regression. *Journal of the American Statistical Association*, 93(441):141–149.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5:177–188.

Reports in Statistics and Operations Research

2005

05-01 SiZer Map for Evaluating a Bootstrap Local Bandwidth Selector in Nonparametric Additive Models. *M. D. Martínez-Miranda, R. Raya-Miranda, W. González-Manteiga and A. González-Carmona.*

05-02 The Role of Commitment in Repeated Games. *I. García Jurado, Julio González Díaz.*

05-03 Project Games. *A. Estévez Fernández, P. Borm, H. Hamers*

05-04 Semiparametric Inference in Generalized Mixed Effects Models. *M. J. Lombardía, S. Sperlich*

2006

06-01 A unifying model for contests: effort-prize games. J. González Díaz

06-02 The Harsanyi paradox and the "right to talk" in bargaining among coalitions. J. J. Vidal Puga

06-03 A functional analysis of NOx levels: location and scale estimation and outlier detection. M. Febrero, P. Galeano, W. González-Manteiga

06-04 Comparing spatial dependence structures. R. M. Crujeiras, R. Fernández-Casal, W. González-Manteiga

06-05 On the spectral simulation of spatial dependence structures. R. M. Crujeiras, R. Fernández-Casal

06-06 An L_2-test for comparing spatial spectral densities. R. M. Crujeiras, R. Fernández-Casal, W. González-Manteiga.

2007

07-01 Goodness-of-fit tests for the spatial spectral density. R. M. Crujeiras, R. Fernández-Casal, W. González-Manteiga.

07-02 Presmothed estimation with left truncated and right censores data. M. A. Jácome, M. C. Iglesias-Pérez

07-03 Robust nonparametric estimation with missing data. G. Boente, W. González-Manteiga, A. Pérez-González

07-04 k-Sample test based on the common area of kernel density estimators, P. Martínez-Camblor, J. de Uña Álvarez, N. Corral-Blanco

07-05 A bootstrap based model checking for selection-biased data, J. L. Ojeda, W. González-Manteiga, J . A. Cristobal

07-06 The Gaussian mixture dynamic conditional correlation model: Bayesian estimation, value at risk calculation and portfolio selection, P. Galeano, M. C. Ausín

2008

08-01 ROC curves in nonparametric location-scale regression models, W. González-Manteiga, J. C. Pardo Fernández, I. Van Keilegom

08-02 On the estimation of α-convex sets, B. Pateiro-López, A. Rodríguez-Casal.

2009

09-01 Lasso Logistic Regression, GSoft and the Cyclyc Coordinate Descent Algorithm. Application to Gene Expression Data

2010

10-01 Asymptotic behaviour of robust estimators in partially linear models with missing responses: The effect of estimating the missing probability on simplified marginal estimators

10-02 First-Price Winner-Takes-All Contents

10-03 Goodness of Fit Test for Interest Rate Models: an approach based on Empirical Process

Previous issues (2001 – 2004):
http://eio.usc.es/pub/reports.html