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# Allocation rules for games with optimistic aspirations\*

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## Abstract

A game with optimistic aspirations specifies two values for each coalition of players: the first value is the worth that the players in the coalition can guarantee for themselves in the event that they coordinate their actions, and the second value is the amount that the players in the coalition aspire to get under reasonable but very optimistic assumptions about the demands of the players who are not included in the coalition. In this paper, in addition to presenting this model and justifying its relevance, we introduce allocation rules and extend the properties of efficiency, additivity, symmetry, and null player property to this setting. We demonstrate that these four properties are insufficient to find a unique allocation rule and define three properties involving null players and nullifying players that allow the identification of

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unique allocation rules. The allocation rules we identify are related to the Shapley value and the equal division rule.

**Keywords:** Optimistic aspirations, TU-games, Shapley value, equal division rule.

## 1 Introduction

In this paper we introduce games with optimistic aspirations and identify attractive allocation rules for such games through axiomatizations. A game with optimistic aspirations specifies two values for each coalition of players: the first value is the worth that the players in the coalition can guarantee for themselves in the event that they coordinate their actions (where the word guarantee implies a very conservative attitude), and the second value is the amount that the players in the coalition aspire to get under reasonable but very optimistic assumptions about the demands of the players who are not included in the coalition. We explain games with optimistic aspirations as well as our motivation for introducing such games by means of an example. Consider an interactive situation that can be described by the following 2-player strategic-form game

1,1	0,0
0,0	10,1

Also suppose that the two players involved recognize that they can benefit from cooperation and that they are trying to figure out what side-payments would be reasonable to use in order to give both of them the correct incentives to cooperate with each other. One approach to this question is to consider the TU-game  $(N, \underline{v})$  in which each coalition  $S$  is assigned the lower value of the zero-sum game between coalition  $S$  and coalition  $N \setminus S$  and some allocation, for example the Shapley value, of this game. This lower value approach is an adaptation of the classical one by von Neumann and Morgenstern (1944), which uses the value of the mixed extension of the zero-sum game between the coalition and its complement. The lower value represents a conservative view of the worth of a coalition because the coalition can guarantee that it will get a payoff equal to this value by sticking to an appropriate coordinated

strategy. We refer the reader to Carpente et al. (2005) for more extensive explanation as well as an axiomatization of the lower-value approach and suffice here by saying that this procedure applied to the situation in our example leads to the game  $(\{1, 2\}, \underline{v})$  with  $\underline{v}(1) = \underline{v}(2) = 0$  and  $\underline{v}(N) = 11$ .

A more optimistic perspective is to consider for each coalition  $S$  the upper value  $(\bar{v})$  of the zero-sum game between coalition  $S$  and coalition  $N \setminus S$ . This upper value assigns to each coalition the value that it can obtain under circumstances where it reacts optimally to the strategies played by the players outside the coalition under the assumption that those players are choosing their strategies with the purpose of holding the coalition members' payoffs down. In our example, doing so would result in the game  $(\{1, 2\}, \bar{v})$  with  $\bar{v}(1) = \bar{v}(2) = 1$  and  $\bar{v}(N) = 11$ . In Carpente et al. (2008) we incorporate this more optimistic view by considering interval games that associate with every coalition  $S$  the interval whose extremes are, respectively, the lower value and the upper value of the zero-sum game between coalition  $S$  and coalition  $N \setminus S$ .

However, as we clearly see in our example, neither the lower value nor the upper value reflect all possible asymmetries that may exist between the players and coalitions. In many situations, it is reasonable and customary to take into account what we will refer to as coalitions' optimistic aspirations - the value that the players in the coalition aspire to get under reasonable but very optimistic assumptions about the demands of the players who are not included in the coalition. For example, Bergantiños and Vidal-Puga (2007) consider optimistic TU-games in minimal cost spanning tree problems, and Maniquet (2003) considers an optimistic estimate of the costs of coalitions in queueing problems. Following ideas similar to those behind these two optimistic games, we define optimistic aspirations of coalitions in our example as  $o(1) = 10$ ,  $o(2) = 1$ , and  $o(N) = 11$ , because these values are the maximum values that each of the coalitions can obtain in any reasonable play of the game (note that player 1 can obtain 10 without any negative effect on player 2's payoff). Clearly, the game defined by the optimistic aspirations reflects the asymmetry that exists between the two players in our example. We think it desirable to consider a model that takes this sort of optimistic information into account, while at the same time recognizing that the players have no

strategies that guarantee them these optimistic payoffs. To this end, we introduce games with optimistic aspirations, which consist of a TU-game  $p$  (for “pessimistic”) that gives for each coalition the value that the players in the coalition can guarantee themselves through the use of some appropriate coordinated strategy, and a TU-game  $o$  (for “optimistic”) that gives for each coalition its optimistic aspiration - the value that the players in  $S$  could use as an aim in negotiations over payoffs in the grand coalition. The difference between games with optimistic aspirations and games with upper bounds as introduced in Carpenete et al. (2010) is subtle, but important: In a game with upper bounds, there is some external bound on the maximum payoff that a coalition can possibly get and any proposed allocation has to respect these bounds. In a game with optimistic aspirations, however, the optimistic aspirations model goals that coalitions have in mind and use in negotiations, but they do not constitute bounds on possible agreements that can be reached in the grand coalition.

We are interested in allocation rules for games with optimistic aspirations. To find reasonable allocation rules, we use for inspiration extensions of the familiar axioms that characterize the Shapley value for TU-games to the setting of games with optimistic aspirations.

The paper is organized as follows. In Section 2 we formally introduce games with optimistic aspirations and identify a way to decompose such games into basic games that are inspired by unanimity games. In Section 3 we introduce allocation rules and extend the properties efficiency, additivity, symmetry, and null player property to the setting of games with optimistic aspirations. We demonstrate that the 4 properties obtained are insufficient to find a unique allocation rule and we define 3 properties related to null players and nullifying players that allow the identification of unique allocation rules. The allocation rules we identify are related to the Shapley value and the equal division rule.

## 2 Games with optimistic aspirations

For every finite set  $N$ ,  $G(N)$  denotes the class of TU-games with set of players  $N$  and  $O(N)$  the class of games with optimistic aspirations and set of players

$N$ . Formally,

- $G(N) = \{v : 2^N \rightarrow \mathbb{R} \mid v(\emptyset) = 0\}$ .
- $O(N) = \{(p, o) \mid p, o \in G(N), p(S) \leq o(S) \forall S \subset N, \text{ and } p(N) = o(N)\}$ .

$G$  and  $O$  denote, respectively, the class of TU-games with a finite set of players and the class of games with optimistic aspirations and a finite set of players. It is well known that  $G(N)$  is a vector space of dimension<sup>1</sup>  $2^n - 1$ . The basis of this space that is most commonly used in axiomatic characterizations of the Shapley value is that of unanimity games  $u^S \in G(N)$ ,  $S \subset N$ , which are defined by  $u^S(T) = 1$  if  $T \subset N$  with  $S \subset T$  and  $u^S(T) = 0$  if  $T \subset N$  with  $S \not\subset T$ . Thus, it is tempting to use<sup>2</sup>  $\{(u^S, 0) \mid S \in 2^N \setminus \emptyset\} \cup \{(0, u^S) \mid S \in 2^N \setminus \emptyset\}$  as a basis for  $O(N)$ , which is after all a subset of  $G(N) \times G(N)$ . However,  $(u^S, 0) \notin O(N)$ , because it violates the condition  $p(S) \leq o(S)$  for all  $S \subset N$  that we have imposed on  $(p, o) \in O(N)$ . In fact,  $O(N)$  is not a vector space because the condition  $p(S) \leq o(S)$  for all  $S \subset N$  implies that if  $(p, o) \in O(N)$  such that  $p \neq o$ , then  $-(p, o) \notin O(N)$ .

In the following theorem we identify a basis of  $O(N)$  by establishing that every game  $(p, o) \in O(N)$  can be written as a linear combination of games with optimistic aspirations  $(u^S, u^S)$  and  $(u^N, u^S)$  in a unique way.

**Theorem 1** *Every game with optimistic aspirations can be written as a linear combination of games with optimistic aspirations in the family  $\{(u^S, u^S) \mid S \in 2^N \setminus \emptyset\} \cup \{(u^N, u^S) \mid S \in 2^N \setminus \emptyset\} \subset O(N)$  in a unique way.*

**Proof.** Let  $(p, o) \in O(N)$ . It is well known that every  $v \in G(N)$  can uniquely be written as a linear combination of unanimity games;  $v = \sum_{S \in 2^N \setminus \emptyset} a_S^v u^S$  with unanimity coefficients  $a_S^v \in \mathbb{R}$  for each  $S$ . Using this, we derive that

$$\begin{aligned} (p, o) &= \sum_{S \in 2^N \setminus \emptyset} a_S^p (u^S, 0) + \sum_{S \in 2^N \setminus \emptyset} a_S^o (0, u^S) \\ &= \sum_{S \in 2^N \setminus \emptyset} a_S^p (u^S, u^S) + \sum_{S \in 2^N \setminus \emptyset} (a_S^o - a_S^p) (0, u^S). \end{aligned} \quad (1)$$

<sup>1</sup>We adopt the common notation in which  $n$  denotes the cardinality of  $N$ ,  $s$  denotes the cardinality of a set  $S$ , and so on.

<sup>2</sup>The symbol 0 denotes both the value zero and the TU-game in which each coalition's value equals 0.

Notice that  $(0, u^S)$  is not in  $O(N)$  for any  $S \in 2^N \setminus \emptyset$  (because  $u^S(N) = 1 \neq 0 = 0(N)$ ). However, in expression (1) we can replace the game 0 by the game  $u^N$ , as we demonstrate below.

If  $T \subset N$ ,  $T \neq N$ , then  $u^N(T) = 0$ , so that

$$\sum_{S \in 2^N \setminus \emptyset} (a_S^o - a_S^p) u^N(T) = 0. \quad (2)$$

If  $T = N$ , then

$$\begin{aligned} \sum_{S \in 2^N \setminus \emptyset} (a_S^o - a_S^p) u^N(T) &= \sum_{S \in 2^N \setminus \emptyset} (a_S^o - a_S^p) \\ &= \sum_{S \in 2^N \setminus \emptyset} (a_S^o - a_S^p) u^S(N) \\ &= o(N) - p(N) = 0. \end{aligned} \quad (3)$$

Using (2) and (3) in (1), we derive

$$\begin{aligned} (p, o) &= \sum_{S \in 2^N \setminus \emptyset} a_S^p(u^S, u^S) + \sum_{S \in 2^N \setminus \emptyset} (a_S^o - a_S^p) (u^N, u^S) \\ &= \sum_{S \in 2^N \setminus \{\emptyset, N\}} a_S^p(u^S, u^S) + \sum_{S \in 2^N \setminus \{\emptyset, N\}} (a_S^o - a_S^p) (u^N, u^S) \\ &\quad + a_N^o(u^N, u^N), \end{aligned} \quad (4)$$

where we take the term  $(u^N, u^N)$  outside the summation signs because it appears in both the first and the second summation.

We now turn to demonstrating that the decomposition in (4) is unique. Let  $\alpha_S$ ,  $S \in 2^N \setminus \{\emptyset, N\}$ ,  $\beta_S$ ,  $S \in 2^N \setminus \{\emptyset, N\}$ , and  $\gamma_N$  be generic coefficients such that

$$(p, o) = \sum_{S \in 2^N \setminus \{\emptyset, N\}} \alpha_S(u^S, u^S) + \sum_{S \in 2^N \setminus \{\emptyset, N\}} \beta_S(u^N, u^S) + \gamma_N(u^N, u^N). \quad (5)$$

Then, obviously,

$$\begin{aligned} p &= \sum_{S \in 2^N \setminus \{\emptyset, N\}} \alpha_S u^S + \sum_{S \in 2^N \setminus \{\emptyset, N\}} \beta_S u^N + \gamma_N u^N \\ &= \sum_{S \in 2^N \setminus \{\emptyset, N\}} \alpha_S u^S + \left( \sum_{S \in 2^N \setminus \{\emptyset, N\}} \beta_S + \gamma_N \right) u^N \end{aligned} \quad (6)$$

and

$$o = \sum_{S \in 2^N \setminus \{\emptyset, N\}} (\alpha_S + \beta_S) u^S + \gamma_N u^N. \quad (7)$$

It thus follows from (6) and the uniqueness of the unanimity coefficients  $a_S^p$  that

$$\alpha_S = a_S^p \text{ for all } S \in 2^N \setminus \{\emptyset, N\} \quad (8)$$

and

$$\sum_{S \in 2^N \setminus \{\emptyset, N\}} \beta_S + \gamma_N = a_N^p \quad (9)$$

and it follows from (7) and the uniqueness of the unanimity coefficients  $a_S^o$  that

$$\alpha_S + \beta_S = a_S^o \text{ for all } S \in 2^N \setminus \{\emptyset, N\} \quad (10)$$

and

$$\gamma_N = a_N^o. \quad (11)$$

Combining (10) and (8), we obtain

$$\beta_S = a_S^o - a_S^p \text{ for all } S \in 2^N \setminus \{\emptyset, N\}. \quad (12)$$

Equalities (8), (11), and (12) demonstrate that the decomposition in (5) is necessarily the same as that in (4).<sup>3</sup>  $\square$

### 3 Allocation Rules

The objective of this paper is to find reasonable allocation rules for the class of games with optimistic aspirations  $O$ .

**Definition 2** *An allocation rule  $\psi$  on  $O$  is a map that associates a vector  $\psi(p, o) \in \mathbb{R}^N$  with every  $(p, o) \in O(N) \subset O$ .*

We have in mind to find an extension of the Shapley value and therefore we start by looking for allocation rules that satisfy axioms similar to those that axiomatize the Shapley value on the class of games  $G$ . Below, we extend

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<sup>3</sup>Note that (9), (11), and (12) are mutually consistent, as is demonstrated in the sequence of equalities  $a_N^p = \sum_{S \in 2^N \setminus \{\emptyset, N\}} \beta_S + \gamma_N = \sum_{S \in 2^N \setminus \{\emptyset, N\}} (a_S^o - a_S^p) + a_N^o = o(N) - p(N) + a_N^p = a_N^p$ .



the familiar properties that axiomatize the Shapley value on  $G$  to allocation rules for games with optimistic aspirations.<sup>4</sup>

**Efficiency (EFF).** Allocation rule  $\psi$  satisfies EFF if, for all  $(p, o) \in O(N)$

$$\sum_{i \in N} \psi_i(p, o) = p(N) (= o(N)).$$

**Additivity (ADD).** Allocation rule  $\psi$  satisfies ADD if, for every  $(p, o), (\bar{p}, \bar{o}) \in O(N)$

$$\psi\left((p, o) + (\bar{p}, \bar{o})\right) = \psi(p, o) + \psi(\bar{p}, \bar{o}).$$

Two players  $i, j \in N$  are said to be *symmetric* in  $(p, o)$  if  $p(S \cup \{i\}) = p(S \cup \{j\})$  and  $o(S \cup \{i\}) = o(S \cup \{j\})$  for every coalition  $S \subset N \setminus \{i, j\}$ .

**Symmetry (SYM).** Allocation rule  $\psi$  satisfies SYM if, for every  $(p, o) \in O(N)$  and players  $i, j \in N$  who are symmetric in  $(p, o)$

$$\psi_i(p, o) = \psi_j(p, o).$$

A player  $i \in N$  is said to be a *null player* in  $(p, o)$  if  $p(S \cup \{i\}) = p(S)$  and  $o(S \cup \{i\}) = o(S)$  for every coalition  $S \subset N$ .

**Null Player Property (NPP).** Allocation rule  $\psi$  satisfies NPP if, for every  $(p, o) \in O(N)$  and player  $i \in N$  who is a null player in  $(p, o)$

$$\psi_i(p, o) = 0.$$

In the following example, we demonstrate that the 4 properties defined above do not determine a unique allocation rule for games with optimistic aspirations.

**Example 3** *We consider convex combinations of the Shapley values  $\phi(p)$  and  $\phi(o)$ . For each  $\lambda \in [0, 1]$ , we define an allocation rule  $\phi^\lambda$  by*

$$\phi^\lambda(p, o) = \lambda\phi(p) + (1 - \lambda)\phi(o).$$

*It follows easily from the fact that the Shapley value satisfies the appropriate efficiency, additivity, symmetry, and null player properties, that  $\phi^\lambda$  satisfies*

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<sup>4</sup>A reader not familiar with the standard axiomatization of the Shapley value is referred to Shapley (1953) or, more widely available, Winter (2002).

*EFF, ADD, SYM, and NPP for each  $\lambda \in [0, 1]$ . However, varying  $\lambda$  leads to different allocations for games with optimistic aspirations  $(p, o)$  in which  $\phi(p) \neq \phi(o)$ . For an example of such a game, consider the player set  $N = \{1, 2\}$  and define the game with optimistic aspirations  $(p, o)$  by  $p(2) = 1$ ,  $p(1) = p(N) = 2$ , and  $o(1) = o(2) = o(N) = 2$ . Then  $\phi(p) = (\frac{3}{2}, \frac{1}{2})$  and  $\phi(o) = (1, 1)$ .*

### 3.1 Null players

It is clear from Example 3 that we need to augment the set of axioms that we obtained as straightforward extensions of the familiar ones for the Shapley value in order to pinpoint a unique allocation rule for games with optimistic aspirations.

A property that, in combination with the 4 previously defined axioms, allows us to pinpoint a unique allocation rule is the following.

**Strong Null Player Property (SNPP).** Allocation rule  $\psi$  satisfies SNPP if, for every  $(p, o) \in O(N)$  and  $i \in N$  it holds that (a) if  $i$  is a null player in  $p$ , then  $\psi_i(p, o) = \frac{1}{2}\psi_i(o, o)$ , and (b) if  $i$  is a null player in  $o$ , then  $\psi_i(p, o) = \frac{1}{2}\psi_i(p, p)$ .

The SNPP implies the NPP. To see this, consider a game with optimistic aspirations  $(p, o)$  and a player  $i$  who is a null player in both  $p$  and  $o$ . Because  $i$  is a null player in  $o$ , it follows from application of SNPP of  $\psi$  to the game with optimistic aspirations  $(o, o)$  that  $\psi_i(o, o) = \frac{1}{2}\psi_i(o, o)$ . From this it follows that  $\psi_i(o, o) = 0$  has to hold. Now, we can derive from SNPP and the fact that  $i$  is a null player in  $p$  that  $\psi_i(p, o) = \frac{1}{2}\psi_i(o, o) = 0$ .

We show in the following theorem that the SNPP leads us to the allocation rule  $\phi^\lambda$  with  $\lambda = \frac{1}{2}$ , as defined in Example 3.

**Theorem 4** *The allocation rule  $\phi^{1/2}$  is the unique allocation rule for  $O$  satisfying EFF, ADD, SYM, and SNPP.*

**Proof.** It follows easily from the fact that the Shapley value satisfies the appropriate efficiency, additivity, symmetry, and null player properties, that  $\phi^{1/2}$  satisfies EFF, ADD, SYM, and SNPP. Let  $\psi$  be an allocation rule on  $O$

that satisfies the four properties. It suffices to demonstrate that  $\psi$  is uniquely determined.

The fact that  $O(N)$  is not a vector space necessitates some caution when subtracting games. Suppose that  $(p, o)$  and  $(\bar{p}, \bar{o})$  are two games in  $O(N)$  such that  $(p, o) - (\bar{p}, \bar{o}) \in O(N)$  as well. Then ADD of  $\psi$  implies that

$$\psi(p, o) = \psi\left((p, o) - (\bar{p}, \bar{o}) + (\bar{p}, \bar{o})\right) = \psi\left((p, o) - (\bar{p}, \bar{o})\right) + \psi(\bar{p}, \bar{o}),$$

so that

$$\psi\left((p, o) - (\bar{p}, \bar{o})\right) = \psi(p, o) - \psi(\bar{p}, \bar{o}). \quad (13)$$

Now, let  $(p, o) \in O(N)$ . We demonstrated in Theorem 1 (see (4)) that

$$(p, o) = \sum_{S \in 2^N \setminus \{\emptyset, N\}} a_S^p(u^S, u^S) + \sum_{S \in 2^N \setminus \{\emptyset, N\}} (a_S^o - a_S^p)(u^N, u^S) + a_N^o(u^N, u^N),$$

which we re-write as

$$\begin{aligned} (p, o) &= \sum_{S \in 2^N \setminus \{\emptyset, N\}} a_S^p(u^S, u^S) + a_N^o(u^N, u^N) \\ &+ \sum_{S \in 2^N \setminus \{\emptyset, N\} \mid (a_S^o - a_S^p) \geq 0} (a_S^o - a_S^p)(u^N, u^S) \\ &- \sum_{S \in 2^N \setminus \{\emptyset, N\} \mid (a_S^o - a_S^p) < 0} (a_S^p - a_S^o)(u^N, u^S). \end{aligned} \quad (14)$$

It is easy to show that by first one-by-one adding the games in the first two lines of this expression and then one-by-one subtracting the games in the last line, we can obtain  $(p, o)$  through a chain of games that all are in  $O(N)$ .<sup>5</sup> Thus, we can use ADD and (13) to derive from (14) that

$$\begin{aligned} \psi(p, o) &= \sum_{S \in 2^N \setminus \{\emptyset, N\}} \psi\left(a_S^p(u^S, u^S)\right) + \psi\left(a_N^o(u^N, u^N)\right) \\ &+ \sum_{S \in 2^N \setminus \{\emptyset, N\} \mid (a_S^o - a_S^p) \geq 0} \psi\left((a_S^o - a_S^p)(u^N, u^S)\right) \\ &- \sum_{S \in 2^N \setminus \{\emptyset, N\} \mid (a_S^o - a_S^p) < 0} \psi\left((a_S^p - a_S^o)(u^N, u^S)\right). \end{aligned} \quad (15)$$

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<sup>5</sup>The details are available from the authors on request.

Thus, it remains to prove that  $\psi$  is uniquely determined on games with optimistic aspirations of the form appearing in (15).

**Case 1** Let  $S \in 2^N \setminus \{\emptyset\}$  and  $a \in \mathbb{R}$ . Consider the game  $(\tilde{p}, \tilde{o}) := a(u^S, u^S)$ . Clearly, all players in  $N \setminus S$  are null players in  $(\tilde{p}, \tilde{o})$  and thus by NPP<sup>6</sup>  $\psi_i(\tilde{p}, \tilde{o}) = 0$  for all  $i \in N \setminus S$ . In addition, by EFF it must hold that  $\sum_{i \in N} \psi_i(\tilde{p}, \tilde{o}) = \tilde{p}(N) = a$ , so that we obtain  $\sum_{i \in S} \psi_i(\tilde{p}, \tilde{o}) = a$ . Since all players in  $S$  are symmetric in  $(\tilde{p}, \tilde{o})$ , it follows from SYM that  $\psi_i(\tilde{p}, \tilde{o}) = \psi_j(\tilde{p}, \tilde{o})$  for all  $i, j \in S$ , so that we find that  $\psi_i(\tilde{p}, \tilde{o}) = \frac{a}{s}$  for all  $i \in S$  must hold.

**Case 2** Let  $S \in 2^N \setminus \{\emptyset, N\}$  and  $a \in \mathbb{R}$ ,  $a \geq 0$ . Consider the game  $(\tilde{p}, \tilde{o}) := a(u^N, u^S)$ . All players in  $N \setminus S$  are null players in  $\tilde{o} = au^S$ , so that it follows from SNPP that  $\psi_i(\tilde{p}, \tilde{o}) = \frac{1}{2}\psi_i(\tilde{p}, \tilde{p})$  for all  $i \in N \setminus S$ . It follows from Case 1 above that  $\psi_i(\tilde{p}, \tilde{p}) = \frac{a}{n}$  for all  $i \in N$ . By EFF it must hold that  $\sum_{i \in N} \psi_i(\tilde{p}, \tilde{o}) = \tilde{p}(N) = a$ , so that we obtain  $\sum_{i \in S} \psi_i(\tilde{p}, \tilde{o}) = a - \sum_{i \in N \setminus S} \psi_i(\tilde{p}, \tilde{o}) = a \left(1 - \frac{n-s}{2n}\right) = \frac{a}{2} \left(1 + \frac{s}{n}\right)$ . Since all players in  $S$  are symmetric in  $(\tilde{p}, \tilde{o})$ , it follows from SYM that  $\psi_i(\tilde{p}, \tilde{o}) = \psi_j(\tilde{p}, \tilde{o})$  for all  $i, j \in S$ , so that we find that  $\psi_i(\tilde{p}, \tilde{o}) = \frac{a}{2s} \left(1 + \frac{s}{n}\right) = \frac{a}{2s} + \frac{a}{2n}$  for all  $i \in S$  must hold.

Cases 1 and 2 above demonstrate that  $\psi$  is uniquely determined (and equal to  $\phi^{1/2}$ ) for all games with optimistic aspirations that appear in (15).

□

In principle, we can change the weights  $\frac{1}{2}$  on  $\psi(p, p)$  and  $\psi(o, o)$  in the SNPP. If, for some  $\lambda \in (0, 1)$ , we change these weights to  $\lambda$  and  $1 - \lambda$ , respectively, then we would obtain the convex combination  $\phi^\lambda$  of the Shapley values of  $p$  and  $o$  as the allocation rule satisfying EFF, ADD, SYM, and SNPP.<sup>7</sup> However, we see no motivation to treat the games  $p$  and  $o$  differently and thus we use the weights  $\frac{1}{2}$ .

Another option one may consider is to take the point of view that if a player  $i$  is a null player in the game  $o$ , then  $\psi_i(p, o) = \psi_i(p, p)$ , the motivation for which could be that if player  $i$  has no influence on the optimistic aspirations of coalitions, then player  $i$ 's allocation should be determined by his influence in  $p$  solely. However, such a property, combined with ADD, EFF,

<sup>6</sup>Note that we can use NPP because it is implied by SNPP.

<sup>7</sup>This is easily verified by going through the proof of Theorem 4 and making the appropriate adjustments in Case 2, which is the only place where adjustments are needed.

SYM, and NPP, leads to the conclusion that  $\psi(p, o) = \psi(p, p)$  for all games with optimistic aspirations  $(p, o) \in O(N)$  and thus the optimistic aspirations are not taken into account for any game.<sup>8</sup>

### 3.2 Nullifying players

Instead of concentrating on null players, we can also concentrate on nullifying players (cf. van den Brink (2007)). A nullifying player is one whose presence in a coalition causes the worth of that coalition to be equal to zero and thus such a player's presence prevents others from obtaining a positive worth. Therefore, the other players may argue that such a player deserves no positive payoff. On the other hand, the nullifying player himself can argue that he deserves no negative payoff either since he can guarantee himself zero by not joining any others. The nullifying player property states that a nullifying player gets a payoff 0. Van den Brink (2007) uses this property to axiomatize the equal division allocation rule for TU games. We extend the nullifying player property to the setting of games with optimistic aspirations and investigate if replacing NPP with the new property determines an allocation rule.

A player  $i \in N$  is said to be a *nullifying player* in  $(p, o)$  if  $p(S) = o(S) = 0$  for every coalition  $S \subset N$  with  $i \in S$ .

**Nullifying Player Property (NFPP).** Allocation rule  $\psi$  satisfies NFPP if, for every  $(p, o) \in O(N)$  and player  $i \in N$  who is a nullifying player in  $(p, o)$

$$\psi_i(p, o) = 0.$$

We show in the following theorem that the NFPP leads us to the *equal division allocation rule ED* defined by

$$ED_i(p, o) = \frac{p(N)}{n}.$$

**Theorem 5** *The allocation rule ED is the unique allocation rule for O satisfying EFF, ADD, SYM, and NFPP.*

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<sup>8</sup>A proof is available from the authors upon request.

**Proof.** It follows easily and straightforwardly that  $ED$  satisfies EFF, ADD, SYM, and NFPP. Let  $\psi$  be an allocation rule on  $O$  that satisfies the four properties. It suffices to demonstrate that  $\psi$  is uniquely determined. To do so, we use the canonical basis of  $G(N)$ , which consists of the games  $e^S \in G(N)$ ,  $S \subset N$ , defined by  $e^S(T) = 1$  if  $T = S$  and  $e^S(T) = 0$  if  $T \neq S$ . Every  $v \in G(N)$  can uniquely be written as a combination of canonical games as follows:  $v = \sum_{S \in 2^N \setminus \emptyset} v(S)e^S$ .

Now, let  $(p, o) \in O(N)$ . We easily derive that

$$\begin{aligned} (p, o) &= \sum_{S \in 2^N \setminus \emptyset} p(S)(e^S, 0) + \sum_{S \in 2^N \setminus \emptyset} o(S)(0, e^S) \\ &= \sum_{S \in 2^N \setminus \emptyset} p(S)(e^S, e^S) + \sum_{S \in 2^N \setminus \emptyset} (o(S) - p(S))(0, e^S). \end{aligned} \quad (16)$$

Unlike with the unanimity games (see Theorem 1), there is no problem with any of the games in (16) not being in  $O(N)$ . This holds because  $o(S) - p(S) \geq 0$  for all  $S \subset N$  and  $o(N) - p(N) = 0$ . Thus, like we did in the proof of Theorem 4, we can use ADD to derive that

$$\psi(p, o) = \sum_{S \in 2^N \setminus \emptyset} \psi\left(p(S)(e^S, e^S)\right) + \sum_{S \in 2^N \setminus \emptyset} \psi\left((o(S) - p(S))(0, e^S)\right). \quad (17)$$

Thus, it remains to prove that  $\psi$  is uniquely determined on games with optimistic aspirations of the form appearing in (17).

**Case 1** Let  $S \in 2^N \setminus \{\emptyset, N\}$  and  $a, b \in \mathbb{R}$ ,  $b \geq 0$ . Consider the games  $(\tilde{p}, \tilde{o}) := a(e^S, e^S)$  and  $(\hat{p}, \hat{o}) := b(0, e^S)$ . Clearly, all players in  $N \setminus S$  are nullifying players in  $(\tilde{p}, \tilde{o})$  and also in  $(\hat{p}, \hat{o})$  and thus by NFPP  $\psi_i(\tilde{p}, \tilde{o}) = \psi_i(\hat{p}, \hat{o}) = 0$  for all  $i \in N \setminus S$ . In addition, by EFF it must hold that  $\sum_{i \in N} \psi_i(\tilde{p}, \tilde{o}) = \tilde{p}(N) = 0 = \hat{p}(N) = \sum_{i \in N} \psi_i(\hat{p}, \hat{o})$ , so that we obtain  $\sum_{i \in S} \psi_i(\tilde{p}, \tilde{o}) = \sum_{i \in S} \psi_i(\hat{p}, \hat{o}) = 0$ . Since all players in  $S$  are symmetric in  $(\tilde{p}, \tilde{o})$  and also in  $(\hat{p}, \hat{o})$ , it follows from SYM that  $\psi_i(\tilde{p}, \tilde{o}) = \psi_j(\tilde{p}, \tilde{o})$  and  $\psi_i(\hat{p}, \hat{o}) = \psi_j(\hat{p}, \hat{o})$  for all  $i, j \in S$ , so that we find that  $\psi_i(\tilde{p}, \tilde{o}) = \psi_i(\hat{p}, \hat{o}) = \frac{0}{s} = 0$  for all  $i \in S$  must hold.

**Case 2** Let  $a \in \mathbb{R}$  and consider the game  $(\tilde{p}, \tilde{o}) := a(e^N, e^N)$ . By EFF it must hold that  $\sum_{i \in N} \psi_i(\tilde{p}, \tilde{o}) = \tilde{p}(N) = a$ . Since all players in  $S$  are symmetric in  $(\tilde{p}, \tilde{o})$ , it follows from SYM that  $\psi_i(\tilde{p}, \tilde{o}) = \psi_j(\tilde{p}, \tilde{o})$  for all  $i, j \in N$ , so that we find that  $\psi_i(\tilde{p}, \tilde{o}) = \frac{a}{n}$  for all  $i \in N$  must hold.

Remember that  $o(N) - p(N) = 0$ , so that we do not have to consider the game  $(0, e^N)$ . Thus, cases 1 and 2 above demonstrate that  $\psi$  is uniquely determined (and equal to  $ED$ ) for all games with optimistic aspirations that appear in (17).  $\square$

Instead of concentrating on the worths of coalitions that include a nullifying player, we can also concentrate on what happens to the worths of coalitions when a nullifying player joins it. Hence, instead of concentrating on the fact that a nullifying player causes the worth of any coalition he is a member of to be 0, we look at the change in worth that he causes when he joins various coalitions. To reflect this change of focus, we give an alternative (but equivalent) description of nullifying players.

A player  $i \in N$  is said to be a *nullifying player* in  $(p, o)$  if  $p(S \cup \{i\}) = o(S \cup \{i\}) = 0$  for every coalition  $S \subset N$ .

When a nullifying player joins a coalition  $S$  of players, he destroys the worth  $p(S)$  that coalition  $S$  could guarantee itself and also causes the optimistic aspiration to change from  $o(S)$  to 0. The destroyer player property (see below) states that a nullifying player should get a payoff that is equal to the value that he destroys in expectation by joining a coalition, assuming that first the cardinality of a coalition is selected at random (from the range 0 to  $n - 1$ ), then a coalition of that size is selected at random, and, finally, a random choice determines whether we consider the effect on  $p$  or  $o$ .

**Destroyer Player Property (DPP).** Allocation rule  $\psi$  satisfies DPP if, for every  $(p, o) \in O(N)$  and player  $i \in N$  who is a nullifying player in  $(p, o)$

$$\psi_i(p, o) = - \sum_{S \subset N \setminus \{i\}} \frac{1}{2n \binom{n-1}{s}} p(S) - \sum_{S \subset N \setminus \{i\}} \frac{1}{2n \binom{n-1}{s}} o(S).$$

It turns out that DPP together with EFF, ADD, and SYM, determines a unique allocation rule and that it is the rule  $\phi^{1/2}$ , which we already encountered in Theorem 4.

**Theorem 6** *The allocation rule  $\phi^{1/2}$  is the unique allocation rule for  $O$  satisfying EFF, ADD, SYM, and DPP.*

**Proof.** With regard to existence, it remains to demonstrate that  $\phi^{1/2}$  satisfies DPP. Let  $(p, o) \in O(N)$  be a game with optimistic aspirations with a nullifying player  $i \in N$ . Then

$$\begin{aligned}
\phi_i^{1/2}(p, o) &= \frac{1}{2}\phi_i(p) + \frac{1}{2}\phi_i(o) \\
&= \frac{1}{2} \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (p(S \cup \{i\}) - p(S)) \\
&\quad + \frac{1}{2} \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (o(S \cup \{i\}) - o(S)) \\
&= - \sum_{S \subset N \setminus \{i\}} \frac{1}{2n \binom{n-1}{s}} p(S) - \sum_{S \subset N \setminus \{i\}} \frac{1}{2n \binom{n-1}{s}} o(S),
\end{aligned}$$

where the second equality uses the definition of the Shapley value, and the third equality the fact that player  $i$  is nullifying in  $(p, o)$ .

To prove uniqueness, let  $\psi$  be an allocation rule on  $O$  that satisfies EFF, ADD, SYM, and DPP. It suffices to demonstrate that  $\psi$  is uniquely determined. As in the proof of Theorem 5, we derive using ADD of  $\psi$  that

$$\psi(p, o) = \sum_{S \in 2^N \setminus \emptyset} \psi\left(p(S)(e^S, e^S)\right) + \sum_{S \in 2^N \setminus \emptyset} \psi\left((o(S) - p(S))(0, e^S)\right).$$

Remember that  $o(N) - p(N) = 0$ , so that we do not have to consider the game  $(0, e^N)$ . For games  $(\tilde{p}, \tilde{o})$  defined either by  $(\tilde{p}, \tilde{o}) = a(e^S, e^S)$  for some  $S \in 2^N \setminus \{\emptyset\}$  and  $a \in \mathbb{R}$ , or by  $(\tilde{p}, \tilde{o}) = a(0, e^S)$  for some  $S \in 2^N \setminus \{\emptyset, N\}$  and  $a \in \mathbb{R}$  with  $a \geq 0$ , the following reasoning holds: All players in  $N \setminus S$  are nullifying players in  $(\tilde{p}, \tilde{o})$  and thus by DPP

$$\psi_i(\tilde{p}, \tilde{o}) = - \sum_{S \subset N \setminus \{i\}} \frac{1}{2n \binom{n-1}{s}} \tilde{p}(S) - \sum_{S \subset N \setminus \{i\}} \frac{1}{2n \binom{n-1}{s}} \tilde{o}(S)$$

for all  $i \in N \setminus S$ , which means that these values are uniquely determined. In addition, by EFF it must hold that  $\sum_{i \in N} \psi_i(\tilde{p}, \tilde{o}) = \tilde{p}(N)$ , so that we obtain  $\sum_{i \in S} \psi_i(\tilde{p}, \tilde{o}) = \tilde{p}(N) - \sum_{i \in N \setminus S} \psi_i(\tilde{p}, \tilde{o})$ . Since all players in  $S$  are symmetric in  $(\tilde{p}, \tilde{o})$ , it follows from SYM that  $\psi_i(\tilde{p}, \tilde{o}) = \psi_j(\tilde{p}, \tilde{o})$  for all  $i, j \in S$ , and thus we find that  $\psi_i(\tilde{p}, \tilde{o}) = \frac{\tilde{p}(N) - \sum_{i \in N \setminus S} \psi_i(\tilde{p}, \tilde{o})}{s}$  for all  $i \in S$  must hold. This uniquely determines  $\psi_i(\tilde{p}, \tilde{o})$  for all  $i \in S$ .  $\square$



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