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Balanced contributions for multi-issue allocation situations*

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Abstract

In this paper we introduce a property of balanced contributions in the context of multi-issue allocation situations. Using this property, we characterise the run-to-the-bank rule for multi-issue allocation situations.
Keywords: Multi-issue allocation situations, run-to-the-bank rule, balanced contributions.
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1 Introduction

Multi-issue allocation situations were introduced by Calleja et al. (2005) as an extension to bankruptcy situations (*cf* O'Neill (1982)). In a multi-issue allocation situation, the players do not have a single claim on the estate, but multiple claims. This multidimensionality of claims is not the result of some exogenously given difference in status or priority (*eg*, Kaminski (2000)). Rather, the various claims result from different issues, which all have the same status.

Calleja et al. (2005) generalise O'Neill's run-to-the-bank (recursive completion) rule to the class of multi-issue allocation situations and show that this coincides with the Shapley value (*cf* Shapley (1953)) of the corresponding multi-issue allocation game. In fact, they define two such rules and games, based on the so-called proportional approach and the queue approach. As is done in González-Alcón et al. (2003), who introduce and characterise an alternative run-to-the-bank rule for multi-issue allocation situations, we focus on the more

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pessimistic queue approach. Similar results can be obtained for the proportional approach.

Following O'Neill's characterisation of the run-to-the-bank rule for bankruptcy situations by a property he calls consistency, Calleja et al. (2005) characterise their rule for multi-issue allocation situations in a similar fashion. Because the underlying idea of a player "leaving" the game is not easily implemented in multi-issue allocation situations, they have to extend their domain to the class of so-called multi-issue allocation situations with awards.

This paper provides an alternative characterisation of the run-to-the-bank rule by Calleja et al., in terms of balanced contributions. This is based on the principle of reciprocity, as introduced by Myerson (1980), which is often used in the literature on the Shapley value. Myerson's property of balanced contributions asserts that for any two players the gain or loss to each player when the other "leaves" the game should be equal. In order to formulate this property in the context of multi-issue allocation situations, we follow Calleja et al. in using the framework with awards.

2 Preliminaries

This section contains some preliminary definitions concerning bankruptcy situations, multi-issue allocation situations and the run-to-the-bank rule.

A *TU game* is a pair (N, v) where N is the finite set of players and v is the characteristic function, which assigns a real number $v(S)$ to every coalition $S \subset N$. We assume that $v(\emptyset) = 0$.

A *bankruptcy problem* (O'Neill (1982)) is a triple (N, E, c) where N is the finite set of players, $c \in \mathbb{R}_+^N$ is the vector of claims, and E, with $0 \leq E \leq \sum_{i \in N} c_i$, represents the estate, which is the available amount to satisfy the players' claims.

A *multi-issue allocation (MIA) situation* (Calleja et al. (2005)) is a triple (N, E, C) where N is the finite set of players, $C \in \mathbb{R}_+^{R \times N}$ is the matrix of claims, and E is the estate to be divided. Each element c_{ki} represents the amount claimed by a player $i \in N$ in an issue $k \in R$, with R being the finite set of issues. We assume that $0 \leq E \leq \sum_{k \in R} \sum_{i \in N} c_{ki}$. Note that a bankruptcy problem can be interpreted as a MIA situation with $|R| = 1$.

The idea behind balanced contributions is to compare reduced situations, in which one of the players has been "sent away" with a particular payoff. In the framework of MIA, however, one cannot "send away" a player with a payoff, since it is unclear what the claims matrix in the reduced situation should be. Simply removing this player from the claims matrix does not work, because this ignores the interdependence between the issues. In order to accommodate the idea of balanced contributions, we use the same extension of the domain that Calleja et al. (2005) use for their characterisation of the run-to-the-bank rule using consistency.

A *multi-issue allocation (MIA) situation with awards* (Calleja et al. (2005))

is a 4-tuple (N, E, C, μ) where (N, E, C) is a MIA situation and $\mu \in \mathbb{R}^F$ represents an award vector related to the coalition $F \subset N$. The idea is that all players are still part of the game, but any solution must give the players in F their predetermined award μ. Hence, we assume this award vector μ to satisfy $\sum_{i \in F} \mu_i \leq E$ and $\sum_{i \in F} \mu_i = E$ if $F = N$.

Note that a MIA situation is a MIA situation with awards with $F = \emptyset$. So indeed, introducing awards extends the domain and any characterisation of a rule on the class of MIA situations with awards uniquely determines the restriction of this rule on the class of MIA situations without awards.

A *MIA solution with awards* Ψ is a function which associates with every MIA situation with awards (N, E, C, μ) a vector $\Psi(N, E, C, \mu) \in \mathbb{R}^N$ such that[1]

- $\Psi_F(N, E, C, \mu) = \mu$,
- $\sum_{i \in N} \Psi_i(N, E, C, \mu) = E$.

Let (N, E, C) be a MIA situation and consider an order[2] on the issues $\tau \in \Pi(R)$. We denote by $c_k = \sum_{i \in N} c_{ki}$ the total of claims according to issue k and by $c_{k,S} = \sum_{i \in S} c_{ki}$ the total of claims of coalition $S \subset N$ according to issue $k \in R$.

Suppose that only the first t issues in the order τ can be fully satisfied where $t = \max\{t' \mid \sum_{s=1}^{t'} c_{\tau(s)} \leq E\}$ and let $E' = E - \sum_{s=1}^{t} c_{\tau(s)}$ be the remaining estate. Next, suppose that E' is distributed among the players for the issue $\tau(t+1)$ according to the order $\sigma \in \Pi(N)$. Thus, only the first q agents in this order σ obtain their total claim, where $q = \max\{q' \mid \sum_{p=1}^{q'} c_{\tau(t+1)\sigma(p)} \leq E'\}$. Then the next function describes exactly the amount that the players in $S \subset N$ obtain according to issue $\tau(t+1)$ if the order on the players is $\sigma \in \Pi(N)$ and the remaining estate is E':

$$g(S, \tau(t+1), \sigma, E') = \begin{cases} E' - \sum\limits_{\substack{p=1 \\ \sigma(p) \in N \setminus S}}^{q} c_{\tau(t+1)\sigma(p)} & \text{if } \sigma(q+1) \in S, \\ \sum\limits_{\substack{p=1 \\ \sigma(p) \in S}}^{q} c_{\tau(t+1)\sigma(p)} & \text{if } \sigma(q+1) \notin S. \end{cases}$$

[1] Following Calleja et al. (2005), we do not include the condition of reasonability ($0 \leq \Psi_i(N, E, C) \leq \sum_{k \in R} c_{ki}$ for all $i \in N$) in the definition of multi-issue allocation solution with awards.

[2] $\Pi(R) = \{\tau \mid \tau : \{1, \ldots, |R|\} \to R\}$, so $\tau(k)$ denotes the issue placed at position k according to τ.

Given the orders τ and σ, the total payoff to coalition $S \subset N$ is given by

$$f_S(\sigma, \tau) = \sum_{s=1}^{t} c_{\tau(s),S} + g(S, \tau(t+1), \sigma, E').$$

The corresponding MIA game $(N, v_{(N,E,C)})$ is defined by

$$\begin{aligned} v_{(N,E,C)}(S) &= \min_{\tau \in \Pi(R)} \min_{\sigma \in \Pi(N)} f_S(\sigma, \tau) \\ &= E - \max_{\tau \in \Pi(R)} \max_{\sigma \in \Pi(N)} f_{N \setminus S}(\sigma, \tau) \\ &= E - \max_{\tau \in \Pi(R)} f_{N \setminus S}(\tau), \end{aligned}$$

where $f_{N \setminus S}(\tau) = f_{N \setminus S}(\hat{\sigma}, \tau) = \sum_{s=1}^{t} c_{\tau(s),N \setminus S} + \min\{c_{\tau(t+1),N \setminus S}, E'\}$
with $\hat{\sigma} \in \Pi(N)$ such that $\hat{\sigma}^{-1}(N \setminus S) = \{1, \ldots, |N \setminus S|\}$.

This game assigns to each coalition $S \subset N$ the quantity which is left after coalition $N \setminus S$ gets the maximal payoff by choosing an order in the issues and an order in the players. An optimal order for the players in $N \setminus S$ obviously puts them at the front of the queue.

Let (N, E, C, μ) a MIA situation with awards and take $\gamma \in \Pi(F)$. The *run-to-the-bank rule with awards (ρ)* is the MIA solution with awards defined by

$$\rho(\mu) = \frac{1}{|N \setminus F|!} \sum_{\sigma \in \Pi^{\gamma}(N)} \rho(\sigma, \mu),$$

with $\Pi^{\gamma}(N) = \{\sigma \in \Pi(N) \,|\, \forall_{q \in \{1,\ldots,|F|\}} : \sigma(q) = \gamma(q)\}$.

For all $\sigma \in \Pi^{\gamma}(N)$, $\rho(\sigma, \mu) \in \mathbb{R}^N$ is defined by $\rho_F(\sigma, \mu) = \mu$ and furthermore recursively by

$$\rho_{\sigma(p)}(\sigma, \mu) = \max_{\tau \in \Pi(R)} \left\{ f_{\sigma(p)}(\sigma, \tau) - \sum_{q=1}^{p-1} \Big[\rho_{\sigma(q)}(\sigma, \mu) - f_{\sigma(q)}(\sigma, \tau) \Big] \right\}$$

for all $p \in \{1, \ldots, |N|\}$ with $\sigma(p) \notin F$.

The vector $\rho_{\sigma(p)}(\sigma, \mu)$ is interpreted as follows. Firstly, all the players in F receive their awards and get a position at the front of the order σ. Then, each player in $N \setminus F$ receives the maximal payoff by choosing an order on the issues, keeping in mind that he has to compensate all the preceding players in the order σ for the difference between the payoff they have received and what they receive when the order on the issues is the order that the player chooses.

If $F = \emptyset$, then there are no fixed players to put at the front of the queue and the definition boils down to the run-to-the-bank rule for MIA situations. In the next section, we provide a characterisation of the rule on the wider domain of MIA situations with awards, which of course uniquely determines the run-to-the-bank rule for MIA situations without awards as well.

3 Balanced contributions

In this section we axiomatically characterise the run-to-the-bank rule with awards by means of the property of balanced contributions.

A MIA solution with awards Ψ satisfies *balanced contributions* if for all MIA situations with awards (N, E, C, μ) and for all $i, j \in N \setminus F$ we have that

$$\Psi_i(N, E, C, \mu) - \Psi_i(N, E, C, \mu^j) = \Psi_j(N, E, C, \mu) - \Psi_j(N, E, C, \mu^i),$$

where for all $\ell \in N \setminus F$, $\mu^\ell \in \mathbb{R}^{F \cup \{\ell\}}$ is such that $\mu^\ell_F = \mu$ and

$$\mu^\ell_\ell = \max_{\tau \in \Pi(R)} \Big[f_{F\cup\{\ell\}}(\sigma, \tau) - \sum_{k \in F} \mu_k \Big] = \max_{\tau \in \Pi(R)} f_{F\cup\{\ell\}}(\sigma, \tau) - \sum_{k \in F} \mu_k,$$

with $\sigma \in \Pi^\gamma(N)$ for arbitrary $\gamma \in \Pi(F)$ such that $\sigma(|F| + 1) = \ell$.

This property says that the loss or gain for player i when player j receives his maximal payoff and becomes a member in the coalition related to the awards vector is the same as the loss or gain for player j when player i receives his maximal payoff and becomes a member in the coalition related to the awards vector.

Theorem 3.1. *The run-to-the-bank rule with awards is the unique MIA solution with awards that satisfies balanced contributions.*

Proof:
Existence.
We first show that the rule satisfies balanced contributions. To this aim, let (N, E, C, μ) be a MIA situation with awards and let $\gamma \in \Pi(F)$. We define the TU game $(N \setminus F, w)$ by

$$w(S) = \begin{cases} v_{(N,E,C)}(S) & \text{if } S \subsetneq N \setminus F \\ E - \sum_{k \in F} \mu_k & \text{if } S = N \setminus F. \end{cases}$$

Myerson (1980) proves that in TU games the Shapley value[3] Φ satisfies a property of balanced contributions. Applying this result to w we obtain that for all $i, j \in N \setminus F$ we have[4]

$$\Phi_i(N \setminus F, w) - \Phi_i(N \setminus (F \cup \{j\}), w) = \Phi_j(N \setminus F, w) - \Phi_j(N \setminus (F \cup \{i\}), w). \quad (1)$$

We will show that

$$\rho_i(\mu) = \Phi_i(N \setminus F, w) \text{ for all } i \in N \setminus F. \quad (2)$$

For this purpose, let $i \in N \setminus F$ and let $\sigma \in \Pi^\gamma(N)$. Define the order $\alpha \in \Pi(N \setminus F)$ by $\alpha(p) = \sigma(|N| - p + 1)$ for all $p \in \{1, \ldots, |N \setminus F|\}$.

We distinguish between two cases:

[3] Given a TU game (N, v), the Shapley value of this game is defined by $\Phi_i(N, v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(v)$, where $m_i^\sigma(v) = v(\sigma(1), \ldots, i) - v(\sigma(1), \ldots, \sigma(\sigma^{-1}(i) - 1))$ is the marginal contribution of player i to the players in front of him according to σ.

[4] For convenience, we denote the restriction of the game w to $N \setminus (F \cup \{j\})$ also by w.

1. $i = \sigma(|F|+1)$.

$$\begin{aligned}
\rho_i(\sigma,\mu) &= \max_{\tau\in\Pi(R)} f_{\{\sigma(1),\ldots,i\}}(\sigma,\tau) - \sum_{k\in F}\mu_k \\
&= E - \min_{\tau\in\Pi(R)}\min_{\sigma'\in\Pi(N)} f_{\{\sigma(|F|+2),\ldots,\sigma(|N|)\}}(\sigma',\tau) - \sum_{k\in F}\mu_k \\
&= E - \sum_{k\in F}\mu_k - v_{(N,E,C)}(\{\sigma(|F|+2),\ldots,\sigma(|N|)\}) \\
&= w(N\setminus F) - w(\{\sigma(|F|+2),\ldots,\sigma(|N|)\}) \\
&= w(\{\alpha(1),\ldots,\alpha(|N\setminus F|)\}) - w(\{\alpha(1),\ldots,\alpha(|N\setminus F|-1)\}) \\
&= m^{\alpha}_{\alpha(|N\setminus F|)}(w) = m^{\alpha}_i(w).
\end{aligned}$$

2. $i \neq \sigma(|F|+1)$, *ie*, $\exists_{s\in\{2,\ldots,|N\setminus F|\}} : i = \sigma(|F|+s)$.

$$\begin{aligned}
\rho_i(\sigma,\mu) &= \max_{\tau\in\Pi(R)} f_{\{\sigma(1),\ldots,i\}}(\sigma,\tau) - \max_{\tau\in\Pi(R)} f_{\{\sigma(1),\ldots,\sigma(|F|+s-1)\}}(\sigma,\tau) \\
&= E - \min_{\tau\in\Pi(R)}\min_{\sigma'\in\Pi(N)} f_{\{\sigma(|F|+s+1),\ldots,\sigma(|N|)\}}(\sigma',\tau) \\
&\quad -E + \min_{\tau\in\Pi(R)}\min_{\sigma'\in\Pi(N)} f_{\{\sigma(|F|+s),\ldots,\sigma(|N|)\}}(\sigma',\tau) \\
&= v_{(N,E,C)}(\{\sigma(|F|+s),\ldots,\sigma(|N|)\}) \\
&\quad -v_{(N,E,C)}(\{\sigma(|F|+s+1),\ldots,\sigma(|N|)\}) \\
&= w(\{\sigma(|F|+s),\ldots,\sigma(|N|)\}) \\
&\quad -w(\{\sigma(|F|+s+1),\ldots,\sigma(|N|)\}) \\
&= w(\{\alpha(1),\ldots,\alpha(|N\setminus F|-s+1)\}) \\
&\quad -w(\{\alpha(1),\ldots,\alpha(|N\setminus F|-s)\}) \\
&= m^{\alpha}_{\alpha(|N\setminus F|-s+1)}(w) = m^{\alpha}_i(w).
\end{aligned}$$

Thus

$$\rho_i(\mu) = \frac{1}{|N\setminus F|!}\sum_{\sigma\in\Pi^{\gamma}(N)}\rho_i(\sigma,\mu) = \frac{1}{|N\setminus F|!}\sum_{\alpha\in\Pi(N\setminus F)} m^{\alpha}_i(w) = \Phi_i(N\setminus F, w).$$

In the same way, using $E - \sum_{k\in F\cup\{j\}}\mu^j_k = v_{(N,E,C)}(N\setminus(F\cup\{j\}))$ by the definition of μ^j, one can show

$$\rho_i(\mu^j) = \Phi_i(N\setminus(F\cup\{j\}), w) \text{ for all } i,\, j\in N\setminus F. \tag{3}$$

Finally, as a result of (1),(2), and (3), for all $i, j \in N\setminus F$ we have

$$\begin{aligned}
&\rho_i(\mu) - \rho_i(\mu^j) = \Phi_i(N\setminus F, w) - \Phi_i(N\setminus F\cup\{j\}, w) \\
&= \Phi_j(N\setminus F, w) - \Phi_j(N\setminus F\cup\{i\}, w) = \rho_j(\mu) - \rho_j(\mu^i).
\end{aligned}$$

Uniqueness.
We show uniqueness by induction on the size of F. Suppose that Ψ^1 and Ψ^2 are two MIA solutions with awards satisfying balanced contributions.

If $F = N$, by the definition of MIA solution with awards, $\Psi^1(N, E, C, \mu) = \mu = \Psi^2(N, E, C, \mu)$.

If $|F| = |N| - 1$, on account of the definition of MIA solution with awards, we know that $\Psi^1_k(N, E, C, \mu) = \mu_k = \Psi^2_k(N, E, C, \mu)$ for all $k \in F$. In view of the fact that any MIA solution with awards satisfies efficiency, we conclude that in this case $\Psi^1(N, E, C, \mu) = \Psi^2(N, E, C, \mu)$.

Let $t \in \{0, \ldots, |N| - 2\}$ and assume that $\Psi^1(N, E, C, \mu) = \Psi^2(N, E, C, \mu)$ for every MIA situation with awards (N, E, C, μ) with $|F| = t + 1$.

Let (N, E, C, μ) be such that $|F| = t$ and let $i, j \in N \backslash F$. Then by balanced contributions we have

$$\Psi^1_i(N, E, C, \mu) - \Psi^1_j(N, E, C, \mu) = \Psi^1_i(N, E, C, \mu^j) - \Psi^1_j(N, E, C, \mu^i)$$

$$= \Psi^2_i(N, E, C, \mu^j) - \Psi^2_j(N, E, C, \mu^i) = \Psi^2_i(N, E, C, \mu) - \Psi^2_j(N, E, C, \mu), \quad (4)$$

where the second equality follows from the induction hypothesis. Due to the definition of a MIA solution with awards,

$$\Psi^1_k(N, E, C, \mu) = \mu_k = \Psi^2_k(N, E, C, \mu) \text{ for all } k \in F. \quad (5)$$

Using (5) and efficiency

$$\sum_{k \in N \backslash F} \Psi^1_k(N, E, C, \mu) = E - \sum_{k \in F} \mu_k = \sum_{k \in N \backslash F} \Psi^2_k(N, E, C, \mu). \quad (6)$$

By (4) and (6) we obtain that

$$\Psi^1_k(N, E, C, \mu) = \Psi^2_k(N, E, C, \mu) \text{ for all } k \in N \setminus F.$$

This last expression and (5) give us uniqueness. □

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