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FINITELY REPEATED GAMES: A GENERALIZED NASH FOLK THEOREM

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Abstract: This paper characterizes the feasible, individually rational payoffs of finitely repeated games that can be approximated arbitrarily closely by Nash equilibria.

1 Introduction

Over the past twenty years, necessary and sufficient conditions have been published for numerous "folk theorems", assuring that the individually rational feasible payoffs of finitely or infinitely repeated games with complete information can be achieved by Nash or subgame perfect equilibria[1]. Results of this kind concerning Nash equilibrium have been obtained by Fudenberg and Maskin (1986) for infinite games and by Benoit and Krishna (1987) for finite games, and results concerning subgame perfect equilibrium in infinite games by Abreu (1988), Abreu et al. (1994) and Wen (1994), who used an "effective minimax" payoff concept. Smith (1995) obtained a necessary and sufficient condition for the arbitrarily close approximation of strictly rational feasible payoffs by subgame perfect equilibria in finite games: that the game have "recursively distinct Nash payoffs", a premise

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[1]The survey by Benoit and Krishna (1996) includes many of these results.

that relaxes the assumption in Benoit and Krishna (1985) that each player have multiple Nash payoffs in the stage game.

Smith claimed that this condition was also necessary for approximation of the individually rational feasible payoffs of finitely repeated games by Nash equilibria. In this paper I show that this is not so by establishing a similar but distinct sufficient condition that is weaker than both Smith's condition and the assumptions made by Benoit and Krishna (1987), and which really is necessary. Essentially, the difference between the subgame perfect and Nash cases hinges on the weakness of the Nash solution concept: in the Nash case it is not necessary for threats of punitive action against players who deviate from the equilibrium not to involve substantial loss to the punishing players themselves, i.e. threats need not be credible. All that is required is for the action sequence ρ_i played by each player i in the equilibrium approximating the desired payoff to finish with a series S_i of rounds in which player i cannot unilaterally improve his stage payoff by deviation from ρ_i, and for this terminal phase to start with a series S_i^0 of rounds in which the other players, regardless of the cost to themselves, can punish him effectively for any prior deviation by imposing a loss that wipes out any gains he may have made in deviating.

All the results mentioned above, except for that in Wen (1994), concern the approximability of the entire set of individually rational feasible payoffs. The theorem proved in this paper is more general in that, for any game, it characterizes the set of feasible payoffs that are approximable. Notation and concepts are introduced in Section 2 below, and in Section 3 the theorem is stated and proved.

2 Basic Notation, Definitions and an Example

2.1 *The Stage Game*

A game in strategic form is a triplet $G =< N, A, \varphi >$, where:

- $N = \{1, \ldots, n\}$ is the set of players,
- $A = \prod_{i \in N} A_i$ and A_i denotes the set of player i's strategies,
- $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $\varphi_i : A \rightarrow \mathbb{R}$ is the utility function of player i.

Let a_{-i} denote a strategy profile for players in $N \backslash \{i\}$, and with A_{-i} the set of such profiles; it is assumed that $\mu(a_{-i}) = \max_{a_i \in A_i}\{\varphi_i(a_{-i}, a_i)\}$ exists for each $i \in N$ and $a_{-i} \in A_{-i}$, and that $v_i = \min_{a_{-i} \in A_{-i}}\{\mu(a_{-i})\}$ exists for each $i \in N$. The vector $v = \{v_1, \ldots, v_n\}$ is the *minimax payoff vector.* The set F of all feasible and individually rational payoffs is the convex hull of the set $\{\varphi(a) : a \in A,\ \varphi(a) \geq v\}$.

To avoid confusion with the strategies of the repeated game, in what follows the strategies $a_i \in A_i$ and strategy profiles $a \in A$ of the stage game will be called actions and action profiles, respectively.

2.2 *The Repeated Game*

Let $G(\delta, T)$ denote the game consisting in the T-fold repetition of G with payoff discount parameter $\delta \in (0, 1]$. In this repeated game each player can decide his action in the current round in the light of all actions taken by all players in all previous rounds. Let $\alpha_i = \{\alpha_i^1, \ldots, \alpha_i^T\}$ denote the action sequence of player i, and $\varphi_i^t(\alpha)$ the stage payoff of player i at stage t when all agents play in accordance with α; then player i's payoff $\psi_i(\alpha)$ in $G(\delta, T)$ when α is played is defined to be his average discounted stage payoff: $\psi_i(\alpha) = ((1-\delta)/(1-\delta^T)) \sum_{t=1}^{T} \delta^{t-1} \varphi_i^t(\alpha)$.[2]

[2]Or, $\psi_i(\alpha) = (1/T) \sum_{t=1}^{T} \varphi_i^t(\alpha)$ if there are no discounts ($\delta = 1$).

2.3 *Minimax-Bettering Ladders*

For any m-player subset M of N, let $A_M = \prod_{i \in M} A_i$ and let $G(a_M)$ be the game induced for the $n - m$ players in $N \backslash M$ when the actions of the members of M constitute the fixed profile $a_M \in A_M$. By abuse of language, if $i \in N \backslash M$, $a_M \in A_M$ and $\sigma \in A_{N \backslash M}$ we write $\varphi_i(\sigma)$ for i's payoff at σ in $G(a_M)$. A minimax-bettering ladder belonging to a game G is defined to be a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$ where $\mathcal{N}$ is a strictly increasing sequence $\{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ of $h+1$ subsets of N $(h \geq 1)$, $\mathcal{A}$ is a sequence of action profiles $\{a_{N_1} \in A_{N_1}, \ldots, a_{N_{h-1}} \in A_{N_{h-1}}\}$ and Σ is a sequence $\{\sigma^1, \ldots, \sigma^h\}$ of Nash equilibria of $G = G(a_{N_0}), G(a_{N_1}), \ldots, G(a_{N_{h-1}})$, respectively, such that at σ^g the players of $G(a_{N_{g-1}})$ receiving payoffs strictly greater than their minimax payoff are exactly the members of $N_g \backslash N_{g-1}$: $\varphi_i(\sigma^g) > v_i \quad \forall i \in N_g \backslash N_{g-1}$ $\varphi_i(\sigma^g) \leq v_i \quad \forall i \in N \backslash N_g$. In algorithmic terms, if the first $g-1$ rungs of the ladder have been constructed, then for the g-th rung to exist the current game $G(a_{N_{g-1}})$ must have an equilibrium σ^g such that there exist members i of $N \backslash N_{g-1}$ for whom $\varphi_i(\sigma^g) > v_i$; $N_g \backslash N_{g-1}$ is defined as precisely this subset of players of $G(a_{N_{g-1}})$; and the game played in the next step is defined by some action profile a_{N_g}. The set N_h will be called the top rung of the ladder, a ladder with top rung N_h is said to be maximal if there is no ladder with top rung $N_{h'}$ such that N_h is a proper subset of $N_{h'}$, and a game G is said to be *decomposable as a complete minimax-bettering ladder* if it has a minimax-bettering ladder with N as its top rung. It is shown below that being decomposable as a complete minimax-bettering ladder is a necessary and sufficient condition for it to be possible to approximate all payoff vectors in F by Nash equilibria of $G(\delta, T)$ for some δ and T. Clearly, being decomposable as complete a minimax-bettering ladder is a weaker property than the requirement in Smith (1995), that at each step $g-1$ of a similar kind of ladder there be action profiles $a_{N_{g-1}}$, $b_{N_{g-1}}$ such that the games $G(a_{N_{g-1}})$ and $G(b_{N_{g-1}})$ have Nash equilibria σ_a^g and σ_b^g with $\varphi_i(\sigma_a^g) \neq \varphi_i(\sigma_b^g)$ for a non-empty set of players (the members of $N_g \backslash N_{g-1}$).

2.4 *An example*

In the equilibrium strategy profile constructed in Theorem 1 below, the action profile sequence in the terminal phase S_i referred to in the Introduction, consists of repetitions of $(a_{N_{g_i-1}}, \sigma^{g_i}), (a_{N_{g_i-2}}, \sigma^{g_i-1}), \ldots, (a_{N_2}, \sigma^2)$ and σ, where g_i is the unique integer such that $i \in N_{g_i} \backslash N_{g_i-1}$; and the σ^j are Nash equilibria of the corresponding games $G(a_{N_{j-1}})$. Since player i is a player in all these games, he can indeed gain nothing by unilateral deviation during this phase. In the potentially punishing series of rounds S_i^0, the action profile sequence consists of repetitions of $(a_{N_{g_i-1}}, \sigma^{g_i})$, in which i obtains more than his minimax payoff, with the accompanying threat of punishing prior unilateral deviation by i (or other members of $N_{g_i} \backslash N_{g_i-1}$) by minimaxing him instead.

L

	l	m	r
T	0,0,3	0,-1,0	0,-1,0
M	-1,0,0	0,-1,0	0,-1,0
B	-1,0,0	0,-1,0	0,-1,0

R

	l	m	r
T	0,3,-1	0,-1,-1	1,-1,-1
M	-1,0,-1	-1,-1,-1	0,-1,-1
B	-1,0,-1	-1,-1,-1	0-1,-1

Figure 1: A game that is decomposable as a complete minimax-bettering ladder

As an illustration of the above ideas, consider the three player game G shown in Figure 1. Its minimax payoff vector is $(0, 0, 0)$, and its only Nash equilibrium is the action profile $\sigma^1 = (T, l, L)$, with payoff vector $(0, 0, 3)$. Thus $N_1 = \{3\}$; player 3 can be punished by 1 and 2 by playing to one of his minimax profiles instead of playing $(T, l, \cdot)$. If player 3 now plays R ($a_{N_1} = R$), the resulting game $G(a_{N_1}) = G(R)$ has an equilibrium $\sigma^2 = (T, l)$ with payoff vector $(0, 3)$. Thus $N_2 = \{2, 3\}$ and player 2 can be punished by 1 and 3 by playing to one of his minimax profiles instead of playing $(T, \cdot, R)$. Finally if players 2 and 3 now play r and R ($a_{N_2} = (r, R)$), the resulting game $G(a_{N_2}) = G(r, R)$ has the trivial equilibrium $\sigma^3 = (T)$ with payoff 1 for player 1, who can therefore be punished by 2 and 3 if they play to one of his minimax profiles instead of playing $(\cdot, r, R)$.

2.5 Further Preliminaries

Proposition 1. *Given a game G, all maximal ladders have the same top rung.*

Proof. Suppose there are maximal ladders $L = \{\mathcal{N}, \mathcal{A}, \Sigma\}$, $L' = \{\mathcal{N}', \mathcal{A}', \Sigma'\}$ with $\mathcal{N} = \{\mathcal{N}_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ and $\mathcal{N}' = \{\mathcal{N}'_0 \subsetneq N'_1 \subsetneq \cdots \subsetneq N'_k\}$ such that, $N_h \neq N'_k$ and suppose without loss of generality that $N'_k \backslash N_h$ is nonempty. For each $j \in N'_k$ let g_j be the unique integer such that $j \in N'_{g_j} \backslash N'_{g_j - 1}$, and consider $i \in N'_k \backslash N_h$ such that $g_i = \min_{j \in N'_k \backslash N_h} g_j$. Then $N'_{g_i - 1} \subseteq N_h$, and we can define an action profile a_{N_h} by:

$$(a_{N_h})_j = \begin{cases} (a'_{N'_{g_i-1}})_j & j \in N'_{g_i - 1} \\ (\sigma'^{g_i})_j & j \in N_h \backslash N'_{g_i - 1} \end{cases}$$

where $\sigma'^{g_i} \in \Sigma'$ is an equilibrium of the game $G(a'_{N'_{g_i-1}})$ defined by the action profile $a'_{N'_{g_i-1}} \in \mathcal{A}'$.

Now the restriction of σ'^{g_i} to $N \backslash N_h$, which we denote by σ^{h+1}, is an equilibrium of $G(a_{N_h})$ (since σ'^{g_i} is an equilibrium of $G(a'_{N'_{g_i-1}})$, and $N \backslash N_h \subset N \backslash N'_{g_i - 1}$), and the subset of members j of $N \backslash N_h$ for whom $\varphi_j(\sigma^{h+1}) > v_j$ is precisely the set $N'_{g_i} \backslash N_h$. Since this set, which we denote by N_{h+1}, is nonempty (it contains i), the triplet $L'' = \{\mathcal{N}'', \mathcal{A}'', \Sigma''\}$ defined by

- $\mathcal{N}'' = \{\mathcal{N}_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h \subsetneq N_{h+1}\}$,
- $\mathcal{A}'' = \{a_{N_1}, \ldots, a_{N_{h-1}}, a_{N_h}\}$,
- $\Sigma'' = \{\sigma^1, \ldots, \sigma^h, \sigma^{h+1}\}$

is a ladder, the top rung of which properly contains that of L. Thus L is not maximal, which proves the Proposition. □

Consider a game G with maximal ladders with top rung N_{max}. Given $\hat{a} \in A_{N_{max}}$, let $\Lambda(\hat{a}) = \{\lambda = (\hat{a}, \sigma) \in A \mid \sigma \text{ Nash Equilibrium of } G(\hat{a})\}$, and let $\Lambda = \bigcup_{\hat{a} \in A_{N_{max}}} \Lambda(\hat{a})$. The set S of all N_{max}*-feasible payoffs* of G is defined to be the intersection of F with the convex hull of the set $\{\varphi(\lambda) \mid \lambda \in \Lambda\}$. If $u \in S$, then by the definition of N_{max}, $u_i = v_i$ for all $i \in N \backslash N_{max}$. Besides, when $N_{max} = N$ we have $\Lambda = A$ and $S = F$.

The promised result concerning the approximability of all payoffs in F by Nash equilibrium payoffs is obtained below as an immediate corollary of a more general theorem concerning the approximability of all payoffs in S. In this more general case, the collaboration of the members of N_{max} is secured by a strategy analogous to that sketched in the Example of Section 2.4, while the collaboration of the members of $N \backslash N_{max}$ is also ensured because none of them is able to obtain any advantage by unilateral deviation from any action profile in Λ.

3 The Theorem

In the theorem that follow, the set of action profiles A may consist either of pure or mixed action profiles; in the latter case, perfect monitoring is assumed, i.e. all players are cognizant not only of the pure actions actually put into effect at each stage, but also of the mixed actions of which they are realizations. Also, public randomization is assumed: at each stage of the repeated game, players can let their actions depend on the realization of an exogenous continuous random variable.[3] Public randomization is not crucial, but facilitates the proofs.

[3]The assumption of public randomization is almost without loss of generality, Fudenberg and Maskin (1991) having shown that, any correlated mixed action can be approximated by alternating pure actions with the appropriate frequencies.

Theorem 1. *Given a game G with a maximal minimax-bettering ladder with top rung N_{max}, and given a feasible payoff $u \in F$, then a necessary and sufficient condition for there to exist for each $\varepsilon > 0$, an integer $T_0 < \infty$ and a positive real number $\delta_0 < 1$ such that for all $T \geq T_0$ and $\delta \in [\delta_0, 1]$, $G(\delta, T)$ has a Nash equilibrium payoff w such that $\|w - u\| < \varepsilon$ is that u be $N_{max} - feasible$ (i.e. $u \in S$).*

Proof. $\overset{\text{suffic}}{\Longleftarrow}$ Let $a \in \Lambda$ be an action profile of G such that $\varphi(a) = u$, and let $L = \{\mathcal{N}, \mathcal{A}, \Sigma\}$ be a maximal minimax-bettering ladder of G. As noted above, the members of $N \backslash N_{max}$ have no incentive for unilateral deviation from a (by the definition of Λ). In order to calculate how many repetitions of $G(a_{N_{g_i-1}})$ will be necessary for the members of $N \backslash \{i\}$ to be able to punish a player $i \in N_{max}$ for prior deviation, let us define $\bar{\mu}(a) = \mu(a_{-i}) - \varphi_i(a)$ (the maximum "illicit" profit that player i can obtain by unilateral deviation from a), $\bar{\mu}_i = \max\{\bar{\mu}(a_{-i}), \bar{\mu}((a_{N_{h-1}}, \sigma^h)_{-i}), \ldots, \bar{\mu}(\sigma^1_{-i})\}$ and $r_i := \min\{r \in \mathbb{N} \,|\, r(\varphi_i(\sigma^{g_i}) - v_i) > \bar{\mu}_i\}$. Clearly, there exists $\delta_i \in (0,1)$ such that $\bar{\mu}_i - \sum_{k=1}^{r_i} \delta_i^k (\varphi_i(\sigma^{g_i}) - v_i) < 0$, so if the discount parameter δ is at least δ_i, r_i repetitions of $G(a_{N_{g_i-1}})$ suffice to allow player i to be punished. Further, if $\delta \geq \delta_0 = \max_{i \in N} \delta_i$ and, for all $g \in \{1, \ldots, h\}$, $q_g := \max_{i \in N_g \backslash N_{g-1}} r_i$, then q_g repetitions of $G(a_{N_{g_i-1}})$ suffice to allow any player in $N_g \backslash N_{g-1}$ to be punished. Given $\varepsilon > 0$, we therefore define the action profile sequence

$$\rho := \{\underbrace{a, \ldots, a}_{T-T_0+q_0}, \underbrace{\lambda^h, \ldots, \lambda^h}_{q_h}, \underbrace{\lambda^{h-1}, \ldots, \lambda^{h-1}}_{q_{h-1}}, \ldots, \underbrace{\lambda^1, \ldots, \lambda^1}_{q_1}\}$$

where $\lambda^k = (a_{N_{k-1}}, \sigma^k)$ (with $a_{N_{k-1}} \in \mathcal{A}$ and $\sigma^k \in \Sigma$), and q_0 is the smallest integer such that:

$$\left\| \frac{q_0\, \varphi(a) + q_h\, \varphi(\lambda^h) + \cdots + q_1\, \varphi(\lambda^1)}{q_0 + q_h + \cdots + q_1} - \varphi(a) \right\| < \varepsilon \qquad (1)$$

and $T_0 = q_0 + q_1 + \cdots + q_h$; and for $T \geq T_0$ and $\delta \in [\delta_0, 1]$ we prescribe for

$G(\delta, T)$ the strategy profile in which all players play ρ unless and until there is a unilateral deviation, in which case the deviating player is minimaxed by all the others. It is straightforward to check that this profile is a Nash equilibrium of $G(\delta, T)$, and since its payoff vector w differs from u by less than $T_0\varepsilon/T$ if $\delta = 1$ (by inequality 1), the same is certainly true if $\delta < 1$, in which case the earlier stage payoffs ($\varphi(a)$) receive greater weight than those of the endgame.

$\overset{\text{necess}}{\Longrightarrow}$ Consider a payoff $u \notin S$. Since S is a closed set, there exists ε such that $\|w - u\| < \varepsilon \Rightarrow w \notin S$. Therefore if there exists a strategy profile σ such that $\|\varphi(\sigma) - u\| < \varepsilon$, then $\varphi(\sigma) \notin S$ and there must be at least one stage of $G(\delta, T)$ in which σ prescribes an action profile not belonging to Λ (by the definition of S). Let q be the last such stage, and let $\bar{a} = (\bar{a}_{N_{max}}, \bar{a}_{N \backslash N_{max}})$ denote the corresponding action profile. By the definition of S, $\bar{a}_{N \backslash N_{max}}$ cannot be a Nash equilibrium of $G(\bar{a}_{N_{max}})$. Hence there is at least one member j of $N \backslash N_{max}$ who can increase his gains in round q by deviating unilaterally from $\bar{a}$; and since this behavior cannot subsequently be punished (by the definition of q, σ assigns j a stage payoff of v_j in all subsequent rounds), σ is not an equilibrium of $G(\delta, T)$. □

Corollary 1. *If the game G is decomposable as a complete minimax-bettering ladder, then for all $u \in F$ and for all $\varepsilon > 0$, there exist $T_0 < \infty$ and $\delta_0 < 1$ such that for all $T \geq T_0$ and $\delta \in [\delta_0, 1]$ there exists a Nash Equilibrium payoff w of $G(\delta, T)$, with $\|w - u\| < \varepsilon$.*

Proof. Since $N = N_{max} \Rightarrow F = S$, this result is an immediate consequence of Theorem 1. □

Corollary 2. *If the game G is not decomposable as a complete minimax-bettering ladder, then for all $T < \infty$ and $\delta \in (0, 1]$ the players i of $G(\delta, T)$ in $N \backslash N_{max}$ receive their minimax stage payoffs v_i at all Nash equilibria of $G(\delta, T)$.*

Proof. For all $u \in S$, $u_i = v_i$ for all $i \in N \backslash N_{max}$, so this result follows by an argument paralleling the proof of necessity in Theorem 1. □

Remarks

- **Remark 1.** Theorem 1 requires no use of the concept of effective minimax payoff[4], because non-equivalent utilities[4] are irrelevant to the approximation of N_{max} feasible payoffs by Nash equilibria, in which there is no need for threats to be credible.

- **Remark 2.** Theorem 1 assumes neither that the set of action profiles is finite, nor that it is compact and that the φ_i are continuous. It requires only that the minimax payoff v exist.[5]

- **Remark 3.** Corollary 1 holds for a wider class of games than the result obtained by Benoit and Krishna (1987).

- **Remark 4.** Theorem 1 raises the question of whether a similarly general result on the approximability of payoffs by equilibria also holds for subgame perfect equilibria. The main problem is to determine the subgame perfect equilibrium payoffs of players with "recursively distinct Nash payoffs" (Smith (1995)) when the game is not completely decomposable.

- **Remark 5.** The results of this paper can be easily extended to the case in which each player has a different discount δ.

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[4]See Abreu et al. (1994) and Wen (1994) for details of these concepts.

[5]And if it does does not exist, but the infsup payoff vector does, then a similar theorem holds for the latter. I thank Vijay Krishna for pointing this out to me.

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